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NUCLEATION FIELD OF THE INFINITE FERROMAGNETIC CIRCULAR CYLINDER AT HIGH TEMPERATURE

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Abstract—Near, but below Curie's temperature T_c, the magnetization increases with applied field above saturation. Therefore, when we approach T_c in a ferromagnet it becomes no longer possible to neglect the change in magnitude of the local magnetization due to magnetic fields. For the purpose of our problem the Brown's equations are extended by using a variational procedure. The equations derived are used to study the problem of the nucleation field of the infinite circular ferromagnetic cylinder. The regular part of the solution of the linearized equations is given in terms of Bessel functions and the resulting algebraic eigenvalue problem is solved numerically. The dependence of the nucleation field from the various parameters of the problem is discussed as well as the size of the single domain particle considered.

1. INTRODUCTION

The nucleation field for an infinite cylinder with the field applied along its axis has been rigourously investigated, exploring the whole eigenvalue spectrum of Brown's equations, by Aharoni and Shtrikman [1]. Aharoni and Shtrikman have shown that curling and buckling are the only modes of nucleation for any radius of the cylinder; the other modes giving higher nucleation fields. From the results obtained in [1] the conclusion of Frei *et al.* [2] is confirmed, that the hysteresis curve of an infinite cylinder with the field parallel to its axis, is symmetrical rectangular loop so that the nucleation field is identical with the coercive force. The size dependence of the nucleation field for various particle geometries has been investigated in [3–8]. The analysis was based on the classical theory of micromagnetics [9]. The purpose of Brown's theory is the description of ferromagnetic bodies by means of a vector field (local magnetization) with constant magnitude and with direction varying continuously with the position. This constraint can be easily accepted at temperatures significantly lower than the Curie temperature since in this range the susceptibility $\aleph$ of the ferromagnetic material can be disregarded in a first approximation and consequently the magnitude of the magnetization is determined only by the temperature. Near, but below Curie's temperature, the local magnetization increases its magnitude with applied field above saturation. That is, in addition to the spontaneous magnetization there is a significant susceptibility above saturation [10] and therefore it becomes no longer possible to neglect the change in magnitude of the local magnetization due to magnetic fields. For the study of rigid ferromagnets at temperatures close to the Curie temperature there are available phenomenological field equations due to Minnaja [11] and Maugin [12].

In the present work an attempt is made to study the nucleation field for an infinite cylinder [1] at high temperature. For the purpose of our analysis the fundamental equations that govern, in a phenomenological manner, the behaviour of rigid ferromagnets near the Curie point are derived by means of a variational procedure. The general regular solution for the nucleation field problem has been obtained in terms of Bessel functions. The mathematical analysis followed is analogous to that of [1] and the determination of the roots of the transcendental equation resulting from the

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX boundary conditions, of the problem under discussion, was carried out by following [13,14]. From the analysis it becomes clear that a detailed knowledge of the magnetic equation of state of the material is needed for the determination of the true nucleation field. For the case of the infinite cylinder we also discussed the nucleation field problem in the framework of Landau's theory of second order phase transitions [15]. From the results obtained it is obvious that "Curling" and "Buckling" modes are still the dominant ones. Departure from the above modes could appear in the case where the magnetocrystalline anisotropy constant, for cubic cryştals (K_1), is positive; this result is in agreement with that of Kondorsky [16]. Finally the dependence of the nucleation from the various parameters entering into the problem has been studied numerically as well as the "Exact Buckling" mode. An estimation of the particle size is also presented.

2. PROBLEM FORMULATION

The fundamental equations that govern, in a phenomenological manner, the behaviour of ferromagnetic rigid materials near the Currie point are derived by means of a variational procedure which minimizes the total energy [9]. This energy at temperature $T < T_c$ is assumed to consist of:

(i) The exchange energy

$$U_{ex} = \frac{1}{2} \int_V a_{ik}\, M_{j,i}\, M_{j,k}\, dV \tag{2.1}$$

(ii) The magnetostatic self-energy

$$U_m = -\frac{1}{2} \int_V M_i\, H_i'\, dV \tag{2.2}$$

(iii) The energy of interaction with the external field $\mathbf{H}^o$

$$U_H = -\int_V M_i\, H_i^o\, dV \tag{2.3}$$

(iv) The anisotropy energy

$$U_A = \int_V \bar{\omega}(M_i)\, dV \tag{2.4}$$

and

(v) The potential isotropic energy

$$V_I = \int_V f(M)\, dV, \qquad M = (M_i\, M_i)^{1/2}. \tag{2.5}$$

Here $M_i\,(i = 1, 2, 3)$ denotes the i^{th} Cartesian component of $\mathbf{M}$; $\phi_{,i} \equiv \frac{\partial \phi}{\partial x_i}$ (where ϕ is any function of the coordinates x_i) and summation over repeated subscripts is understood. The vector function $H_i' = H_i - H_i^o$ is the part of the magnetizing force due to magnetization M_i, a_{ik} is a symmetrical tensor (exchange tensor). This tensor in a cubic crystal reduces to $a_{ik} = a\delta_{ik}$, $a > 0$ and it will be, in a first approximation, dependent only upon the material. $\bar{\omega}(M_i)$ and $f(M)$ are the anisotropy energy and the potential energy which depends on M_i and the absolute value of $\mathbf{M}$, respectively, per unit volume. The equilibrium condition is given as

$$\delta U_{tot} = \delta(U_{ex} + U_m + U_H + U_A + U_I) = 0 \tag{2.6}$$

without any supplementary condition on M_i.

Assuming that there is no surface distribution of magnetic dipoles on the boundary ∂V (with unit outward normal $\mathbf{n}$) of a rigid ferromagnet that occupies the finite volume V, from the condition (2.6), for cubic ferromagnetic crystal, we find that M_i satisfy the equations

$$a\, \nabla^2 M_i - \frac{\partial \bar{\omega}}{\partial M_i} + H_i - \frac{df}{dM}\frac{M_i}{M} = 0, \qquad \text{in } V \tag{2.7}$$

and

$$\frac{\partial M_i}{\partial n} = 0, \qquad \text{on } \partial V \tag{2.8}$$

where ∇^2 denotes the Laplace operator.

A spatially uniform solution $M_i = M_i^o$, $\nabla \mathbf{M}^o = 0$ throughout V, yields

$$H_i^o = \left.\frac{\partial \bar{\omega}}{\partial M_i}\right|_o + \left.\frac{\partial f}{\partial M}\,\frac{M_i}{M}\right|_o + D_{ij} M_j^o,$$

where H_i^o is the externally applied field and D_{ij} is the demagnetization tensor of V. This equation shows the existence of a finite magnetic susceptibility in micromagnetics at high temperature, but the material in general does not have a linear magnetic behaviour [12].

The equations (2.7) and (2.8) are the modified Brown's equations at high temperature. We suppose now that the ferromagnetic body is under uniform magnetization (the demagnetization field is homogeneous-uniformly magnetized ellipsoid $\mathbf{M}^o$ which defines an axis of easy magnetization). In this case eqs. (2.7) and (2.8) are satisfied by all uniform vector fields parallel to the axis $\mathbf{M}^o/M$. Since the vector $\mathbf{M}^o$ defines an axis of easy magnetization we can suppose that $\bar{\omega}(M_i)$ reaches its minimum when $\mathbf{M}$ is parallel to the uniform solution $\mathbf{M}^o$, that is

$$\bar{\omega}(M_i) = \frac{\partial \bar{\omega}}{\partial M_i} = 0, \qquad \text{for } \mathbf{M} \times \mathbf{M}^o = 0. \tag{2.9}$$

Assuming that $M_i^o = M^o\,\mathbf{i}_3$ from eqs. (2.7) and (2.9) we obtain

$$H_3 = H_3^o + H_3' = \left.\frac{df}{dM}\right|_o . \tag{2.10}$$

To ensure the stability of the equilibrium $\mathbf{M}^o$, the second variation of the energy U_{tot} with respect to a small deviation $\mathbf{m}$ from $\mathbf{M}^o$, where m_i are functions of position, has to be positive.

Following the standard procedure it is found that the second variation in the energy, $\delta^2 U_{tot}$, is given by the relation

$$\delta^2 U_{tot} = \int_V \left[\frac{a}{2}\left\{(\nabla m_1)^2 + (\nabla m_2)^2 + (\nabla m_3)^2\right\} + \frac{1}{2} m_i \frac{\partial}{\partial M_i}\left(m_k \frac{\partial \bar{\omega}}{\partial M_k}\right) + \frac{1}{2} m_i \frac{\partial}{\partial M_i}\left(m_k \frac{\partial f}{\partial M_k}\right) - \frac{1}{2} m_i h_i'\right] dV \tag{2.11}$$

where h_i' is due to the poles of m_i and the derivatives with respect to M_i and M have to be evaluated at the state $()^o$.

For carrying out the variation, the reciprocity relation [9]

$$\int_V \mathbf{M}^o \cdot \mathbf{h}'\, dV = \int_V \mathbf{m} \cdot \mathbf{H}^o\, dV \tag{2.12}$$

has been used as well as the relations (2.9) and (2.10).

Since

$$m_i \frac{\partial}{\partial M_i}\left(m_k \frac{\partial \bar{\omega}}{\partial M_k}\right) = \frac{\partial^2 \bar{\omega}}{\partial M_1^2} m_1^2 + 2\frac{\partial^2 \bar{\omega}}{\partial M_1\, \partial M_2} m_1 m_2 + \frac{\partial^2 \bar{\omega}}{\partial M_2^2} m_2^2 \tag{2.13}$$

and

$$m_i \frac{\partial}{\partial M_i}\left(m_k \frac{\partial f}{\partial M_k}\right) = \frac{(m \times \mathbf{M}^o)^2}{(M^o)^3}\frac{df}{dM} + m_3^2 \frac{d^2 f}{dM^2} = (m_1^2 + m_2^2)\left(\frac{H_3^o + H_3'}{M^o}\right) + m_3^2 \frac{d^2 f}{dM^2}, \tag{2.14}$$

the relation (2.11) is written as

$$\delta^2 U_{tot} = \int_V \left[\frac{a}{2}\left\{(\nabla m_1)^2 + (\nabla m_2)^2 + (\nabla m_3)^2\right\} + \frac{1}{2}\left\{(m_1^2 + m_2^2)\left(\frac{H_3^o + H_3'}{M_o}\right) + m_3^2 \frac{d^2 f}{dM^2} + \frac{\partial^2 \bar{\omega}}{\partial M_1^2} m_1^2 + 2\frac{\partial^2 \bar{\omega}}{\partial M_1\, \partial M_2} m_1 m_2 + \frac{\partial^2 \bar{\omega}}{\partial M_2^2} m_2^2\right\} - \frac{1}{2} m_i h_i'\right] dV. \tag{2.15}$$

Consider now a definite m_i and begin with H_3^o large enough to insure stability. If H_3^o is decreased it must reach a value $\tilde{H}$ at which for a given m_i, $\delta^2 U_{tot}$ vanishes. This value of $\tilde{H}$ has to satisfy the relation

$$-\left(\tilde{H}+H_3'\right)=\frac{P}{Q}=\frac{M^o}{\int_V (m_1^2+m_2^2)\,dV}\left[a\int_V \left\{(\nabla m_1)^2+(\nabla m_2)^2+(\nabla m_3)^2\right\}dV\right.$$
$$\left.+\int_V\left\{m_3^2\frac{d^2f}{dM^2}+\frac{\partial^2\bar{\omega}}{\partial M_1^2}m_1^2+\frac{\partial^2\bar{\omega}}{\partial M_1\,\partial M_2}m_1m_2+\frac{\partial^2\bar{\omega}}{\partial M_2^2}M_2^2\right\}-\int_V m_i\,h_i'\,dV\right] \tag{2.16}$$

if at least one m_1 and m_2 does not vanish everywhere; otherwise it must be true that the expression $[\ldots]$ in (2.16) is equal to zero. In that case the dependence on H_3^o is implicit in function f.

The highest value reached by $\tilde{H}$ among all values corresponding to the different vector fields m_i is the nucleation field. We therefore minimize P/Q with respect to the functions m_i, $i=1,2,3$. To find the necessary condition for such a minimum we set the first variation of P/Q equal to zero. This requires

$$\delta P+(\tilde{H}+H_3')\,\delta Q=0. \tag{2.17}$$

Following the standard procedure we obtain

$$\int_V\left\{M^o\left(-a\nabla^2 m_1+\frac{\partial^2\bar{\omega}}{\partial M_1^2}m_1+\frac{\partial^2\bar{\omega}}{\partial M_1\,\partial M_2}m_2-h_1'\right)+\left(\tilde{H}+H_3'\right)m_1\right\}\delta m_1\,dV$$
$$+\int_V\left\{M^o\left(-a\nabla^2 m_2+\frac{\partial^2\bar{\omega}}{\partial M_2^2}m_2+\frac{\partial^2\bar{\omega}}{\partial M_1\,\partial M_2}m_1-h_2'\right)+\left(\tilde{H}+H_3'\right)m_2\right\}\delta m_2\,dV$$
$$+\int_V\left\{M^o\left(-a\nabla^2 m_3+m_3\frac{d^2f}{dM^2}-h_3'\right)\right\}\delta m_3\,dV+\int_{\partial V}\left\{a\,M^o\frac{\partial m_i}{\partial n}\right\}\delta m_i\,dS=0. \tag{2.18}$$

From (2.18), setting the coefficients of δm_i, $\{\ \}$ equal to zero we get in V

$$M^o\left(a\nabla^2 m_1-\frac{\partial^2\bar{\omega}}{\partial M_1^2}m_1-\frac{\partial^2\bar{\omega}}{\partial M_1\,\partial M_2}m_2+h_1'\right)-\left(\tilde{H}+H_3'\right)m_1=0$$
$$M^o\left(a\nabla^2 m_2-\frac{\partial^2\bar{\omega}}{\partial M_2^2}m_2-\frac{\partial^2\bar{\omega}}{\partial M_1\,\partial M_2}m_1+h_2'\right)-\left(\tilde{H}+H_3'\right)m_2=0$$
$$M^o\left(a\nabla^2 m_3-m_3\frac{d^2f}{dM^2}+h_3'\right)=0 \tag{2.19}$$

and on ∂V

$$\frac{\partial m_i}{\partial n}=0,\qquad i=1,2,3,\qquad \left(\frac{\partial}{\partial \mathbf{n}}=\mathbf{n}\cdot\nabla\right). \tag{2.20}$$

Equations (2.19) and (2.20) may be written in compact form as [11],

$$\mathbf{M}^o\times\left\{a\nabla^2\mathbf{m}-\frac{\partial}{\partial\mathbf{M}}\left(\mathbf{m}\cdot\frac{\partial\bar{\omega}}{\partial\mathbf{M}}\right)+\mathbf{h}'\right\}+\mathbf{m}\times\left(\tilde{\mathbf{H}}+\mathbf{H}'\right)=0$$
$$\mathbf{M}^o\cdot\left\{a\nabla^2\mathbf{m}-\mathbf{m}\frac{d^2f}{d\mathbf{M}^2}+\mathbf{h}'\right\}=0 \tag{2.21}$$

and

$$\frac{\partial\mathbf{m}}{\partial n}=0, \tag{2.22}$$

respectively, where $\partial(\ \ )/\partial\mathbf{M}$ is a vector whose components are $\partial(\ \ )/\partial M_i$, $i=1,2,3$ and the derivatives with respect to M_i and M are evaluated at the state $(\ \ )^o$.

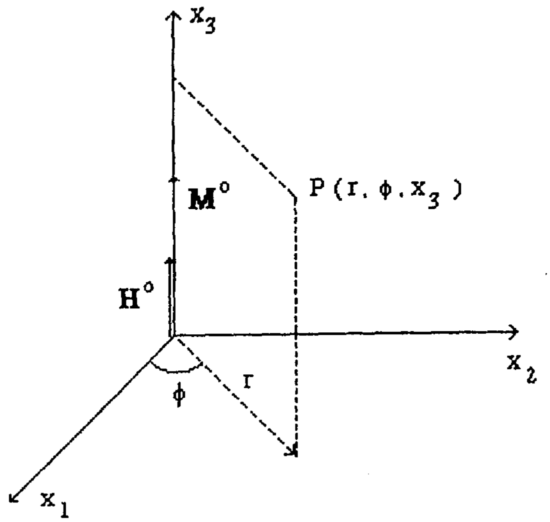

Figure 1.

The field $\mathbf{h}'$, being the field that arises from the free poles of $\mathbf{m}$, is calculated from

$$h'_i = -U_{,i}$$
$$\nabla^2 U = 4\pi\, m_{k,k} \tag{2.23}$$

inside V,

$$\nabla^2 U = 0, \tag{2.24}$$

outside V and

$$U_{\text{in}} = U_{\text{out}}$$
$$-\frac{\partial U_{\text{in}}}{\partial n} + 4\pi\, m_n = -\frac{\partial U_{\text{out}}}{\partial n}, \tag{2.25}$$

on the surface ∂V, where U is the scalar potential associated with the free poles of the magnetization $\mathbf{m}$ and m_n is the component of $\mathbf{m}$ in the direction of the outward normal on ∂V ($m_n = \mathbf{m}\cdot\mathbf{n}$).

From equation (2.21_2) we have

$$M_i^o m_i = \left(\frac{d^2 f}{dM^2}\right)^{-1} M_i^o \left\{a\nabla^2 m_i + h'_i\right\}. \tag{2.26}$$

In the limit as $\frac{d^2 f}{dM^2}$ goes to infinity, which corresponds to a susceptibility $\aleph$ that goes to zero, the equation (2.26) yields

$$M_i^o m_i = 0, \qquad \text{i.e.,} \qquad M = \text{const.} \tag{2.27}$$

which is the condition in the theory of micromagnetics at low temperature (Brown's theory). Under the condition (2.27) the present formulation reduces to the standard one in the case of vanishing susceptibility.

The determination of the nucleation field requires a complete discussion of the fundamental equations (2.21)–(2.25) and the knowledge of the equation of state for the material under discussion. The eigenvalue problem of equations (2.21)–(2.25) has nontrivial solution only for certain ranges of $\tilde{\mathbf{H}}$. In particular, there is no solution if $\tilde{\mathbf{H}}$ is very large. The largest value of $\tilde{\mathbf{H}}$ for which a nontrivial solution exists represents the field for which the distribution of the magnetization can first deviate from the uniform state.

3. THE INFINITE CIRCULAR CYLINDER

Consider an infinite circular cylinder of radius r_0 whose axis coincides with the homogeneous field $\mathbf{H}^o$ and with x_3-axis (Figure 1). Crystallographically the cylinder is assumed to be cubic with anisotropy energy given by [17]:

$$\bar{\omega} = R_1 \left(M_1^2 M_2^2 + M_2^2 M_3^2 + M_3^2 M_1^2\right) + R_2 M_1^2 M_2^2 M_3^2 \tag{3.1}$$

where

$$R_1 = \frac{K_1}{M^4}, \quad R_2 = \frac{K_2}{M^6}$$

and K_1, K_2 are the anisotropy coefficients.

The x_3-axis is supposed to be the easy magnetization axis, therefore we have

$$\mathbf{M}^o = M^o \mathbf{i}_3, \qquad \mathbf{H} = H_3 \mathbf{i}_3 = H_n \mathbf{i}_3$$

and

$$m_i \frac{\partial}{\partial M_i}\left(m_k \frac{\partial \bar{\omega}}{\partial M_k}\right) = \begin{cases} 2R_1 (M^o)^2, & \text{for } i = 1,2, \\ 0, & \text{for } i = 3. \end{cases}$$

The determination of the nucleation field in the case under discussion is reduced to the solution of the following eingenvalue problem:

$$\begin{aligned} & a\, M^o\, \nabla^2 m_i - (2R_1 (M^o)^3 + H_n) m_i - M^o\, U_{,i} = 0, \qquad i = 1, 2 \\ & a \nabla^2 m_3 - m_3 \frac{d^2 f}{dM^2} - M^o\, U_{,3} = 0 \\ & \nabla^2 U = 4\pi\, m_{k,k}, \end{aligned} \tag{3.2}$$

for $x_1^2 + x_2^2 \le \rho^2$

$$\nabla^2 U = 0, \tag{3.3}$$

for $x_1^2 + x_2^2 \ge \rho^2$ and

$$\begin{aligned} & \frac{\partial m_i}{\partial n} = 0, \qquad U_{\text{in}} = U_{\text{out}} \\ & -\frac{\partial U_{\text{in}}}{\partial n} + 4\pi\, m_n = -\frac{\partial U_{\text{out}}}{\partial n} \end{aligned} \tag{3.4}$$

for $x_1^2 + x_2^2 = \rho^2$.

Introducing the cylindrical coordinate system (r, ϕ, x_3) (Figure 1), the dimensionless quantities,

$$\begin{aligned} & t = \frac{r}{r_o}, \quad h = \frac{2R_1 (M^o)^3 + H_n}{\pi M^o}, \quad u = \frac{U}{\pi\, M^o\, \rho_o}, \quad \rho_o = \sqrt{a}, \quad p = \frac{x_3}{r_o} \\ & S = \frac{r_o}{\rho_o}, \quad \tilde{m}_i = \frac{m_i}{M^o}, \quad \lambda = S^2 \frac{d^2 f}{dM^2}, \qquad i = r,\, \phi,\, x_3 \end{aligned} \tag{3.5}$$

the operator

$$\nabla'^2() = ()_{,tt} + \frac{1}{t}()_{,t} + \frac{1}{t^2}()_{,\phi\phi} + ()_{,pp} \tag{3.6}$$

and following Aharoni and Shtrikman [1] the equations (3.2)–(3.4) are transformed as

$$\begin{aligned} & \nabla'^2 \tilde{m}_r - \frac{1}{t^2}\tilde{m}_r - \frac{2}{t^2}\tilde{m}_{\phi,\phi} - \pi S^2 h \tilde{m}_r - \pi S u_{,t} = 0 \\ & \nabla'^2 \tilde{m}_r - \frac{1}{t^2}\tilde{m}_\phi + \frac{2}{t^2}\tilde{m}_{r,\phi} - \pi S^2 h \tilde{m}_\phi - \frac{\pi S}{t} u_{,\phi} = 0 \\ & \nabla'^2 \tilde{m}_3 - \lambda \tilde{m}_3 - \pi S u_{,p} = 0 \\ & \nabla'^2 u = 4S \left\{ \tilde{m}_{r,t} + \frac{1}{t}(\tilde{m}_r + \tilde{m}_{\phi,\phi}) + \tilde{m}_{3,\phi} \right\} \end{aligned} \tag{3.7}$$

for $t \le 1$

$$\nabla'^2 u = 0, \tag{3.8}$$

for $t \geq 1$, and

$$\begin{aligned} &\tilde{m}_{r,t} = \tilde{m}_{\phi,t} = \tilde{m}_{3,t} = 0 \\ &u_{\text{in}} = u_{\text{out}} \\ &-\frac{\partial u_{\text{in}}}{\partial t} + 4S\tilde{m}_r = -\frac{\partial u_{\text{out}}}{\partial t} \end{aligned} \tag{3.9}$$

for $t = 1$.

The complete regular solution of equations (3.7) is a linear combination of functions of the type

$$\begin{aligned} \tilde{m}_r &= A_r(t)\cos(kp - p_o)\cos(n\phi - \phi_o), \\ \tilde{m}_\phi &= A_\phi(t)\cos(kp - p_o)\sin(n\phi - \phi_o), \\ \tilde{m}_3 &= A_3(t)\sin(kp - p_o)\cos(n\phi - \phi_o), \\ u &= \mathsf{U}_t(t)\cos(kp - p_o)\cos(n\phi - \phi_o), \end{aligned} \tag{3.10}$$

where k, p_o, ϕ_o are real constants and n is an integer to insure periodicity in the tangential direction.

Substituting the solution (3.10) into equations (3.7), adding and subtracting (3.7_1) and (3.7_2), respectively, we obtain

$$\begin{aligned} &\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \left(\frac{n+1}{t}\right)^2 - k^2 - \pi S^2 h\right\}(A_r + A_\phi) + \pi S\left(n\frac{\mathsf{U}_t}{t} - \frac{d\mathsf{U}_t}{dt}\right) = 0 \\ &\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \left(\frac{n-1}{t}\right)^2 - k^2 - \pi S^2 h\right\}(A_r - A_\phi) - \pi S\left(n\frac{\mathsf{U}_t}{t} + \frac{d\mathsf{U}_t}{dt}\right) = 0 \\ &\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \left(\frac{n}{t}\right)^2 - k^2 - \lambda\right\}A_3 + \pi S k \mathsf{U}_t = 0 \\ &\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \left(\frac{n}{t}\right)^2 - k^2\right\}\mathsf{U}_t - 2S\left\{(n+1)\frac{A_r + A_\phi}{t} - (n-1)\frac{A_r - A_\phi}{t}\right. \\ &\qquad\left. + \frac{d(A_r + A_\phi)}{dt} + \frac{d(A_r - A_\phi)}{dt} + 2kA_3\right\} = 0. \end{aligned} \tag{3.11}$$

A solution of (3.11) is

$$\begin{aligned} A_r - A_\phi &= \alpha_1 J_{n-1}(i\mu t) \\ \mathsf{U}_t &= \alpha_2 J_n(i\mu t) \\ A_r + A_\phi &= \alpha_3 J_{n+1}(i\mu t) \\ A_3 &= \alpha_4 J_n(i\mu t) \end{aligned} \tag{3.12}$$

where J_n is Bessel's function of the first kind of order n, if the following equations are satisfied

$$\begin{aligned} &i\mu\pi S\alpha_2 + (\mu^2 - k^2 - \pi S^2 h)\alpha_3 = 0 \\ &(\mu^2 - k^2 - \pi S^2 h)\alpha_1 - i\mu\pi S\alpha_2 = 0 \\ &\pi S k\alpha_2 + (\mu^2 - k^2 - \lambda)\alpha_4 = 0 \\ &2i\mu S\alpha_1 + (\mu^2 - k^2)\alpha_2 - 2i\mu S\alpha_3 - 4Sk\alpha_4 = 0. \end{aligned} \tag{3.13}$$

The existence of a nontrivial solution of (3.13) requires the determinant of the coefficients of α_i, $i = 1, 2, 3, 4$ to be zero, that is

$$\begin{aligned} (\mu^2 - k^2 - \pi S^2 h)\left\{4\pi S^2\mu^2(\mu^2 - k^2 - \lambda) - \left[4\pi S^2 k^2 + (\mu^2 - k^2)(\mu^2 - k^2 - \lambda)\right]\right. \\ \left. \times\ (\mu^2 - k^2 - \pi S^2 h)\right\} = 0. \end{aligned} \tag{3.14}$$

This equation implies for μ either the value

$$\mu_1 = \left(k^2 + \pi S^2 h\right)^{1/2} \tag{3.15}$$

or one of the three values

$$\mu_i = \left\{ y_i + \frac{1}{3}\left(\omega + 3k^2 + \lambda\right) \right\}^{1/2}, \qquad i = 2, 3, 4, \tag{3.16}$$

where

$$\omega = \pi S^2 (h+4), \quad p = \frac{1}{9}\left\{3\omega\lambda - (\omega+\lambda)^2\right\} < 0$$
$$q = -\frac{1}{27}(\omega^3 + \lambda^3) + \frac{1}{18}\omega\lambda(\omega^2 + \lambda^2) + 2\pi S^2 k^2 (\lambda - \omega + 4\pi S^2)$$
$$D = -p^3 - q^2, \quad r = \begin{cases} \sqrt{|p|}, & q > 0 \\ -\sqrt{|p|}, & q < 0 \end{cases}$$

and y_i are given as

$$y_2 = -2r\cos\frac{\theta}{3}, \quad y_{3,4} = 2r\cos\left(60^o \mp \frac{\theta}{3}\right),$$
$$\theta = \mathrm{Arc}\left(\cos\left(\frac{q}{r^3}\right)\right), \quad \theta \in [o^o, 90^o], \qquad \text{for } D \geq 0$$

and

$$y_2 = -2r\cosh\frac{\phi}{3}, \quad y_{3,4} = r\cosh\frac{\phi}{3} \pm i\sqrt{3}\, r\sinh\frac{\phi}{3},$$
$$\phi = \mathrm{Arc}\left(\cosh\left(\frac{q}{r^3}\right)\right), \quad \phi \in [0^o, 90^o], \qquad D < 0.$$

Since there are four values for μ, eqs. (3.12) represent the general regular solution of (3.11) while the other four solutions are the associated Neumann functions. For $\mu = \mu_1$ and $\pi S^2 h \neq \lambda$ from (3.13), we obtain

$$\alpha_2^{(1)} = \alpha_4^{(1)} = 0, \quad \alpha_1^{(1)} = \alpha_3^{(1)}, \tag{3.17}$$

while for μ_l, $l = 2, 3, 4$ we have

$$\begin{aligned} \alpha_2^{(l)} &= \frac{4iS\mu_l(\mu_l^2 - k^2 - \lambda)}{R_l}\alpha_1^{(l)} \\ \alpha_3^{(l)} &= -\alpha_1^{(l)} \\ \alpha_4^{(l)} &= \frac{4i\pi S^2 \mu_l}{R_l}\alpha_1^{(l)}, \qquad \text{for } R_l \neq 0, \end{aligned} \tag{3.18}$$

where $l = 2, 3, 4$ and $R_l = 4\pi S^2 k^2 + (\mu_l^2 - k^2)(\mu_l^2 - k^2 - \lambda)$.

The general regular solution of (3.11) is

$$\begin{aligned} A_r - A_\phi &= \sum_{l=1}^{4} \alpha_1^{(l)} J_{n-1}(i\mu_l t) \\ \mathrm{U}_t &= \sum_{l=2}^{4} \frac{-4iS\mu_l\left(\mu_l^2 - k^2 - \lambda\right)}{R_l}\alpha_1^{(l)} J_n(i\mu_l t) \\ A_r + A_\phi &= \sum_{l=2}^{4} \alpha_3^{(l)} J_{n+1}(i\mu_l t) \\ A_3 &= \sum_{l=1}^{4} \alpha_4^{(l)} J_n(i\mu_l t). \end{aligned} \tag{3.19}$$

In the case where μ_l are all different and nonzero the general solution (3.19) involves 4-arbitrary constants $\alpha_1^{(l)}$, $l = 1, 2, 3, 4$, while the other 4-independent solutions have coefficients $\alpha_l^{(l)}$ equal to zero since the Neumann functions are infinite at $t = 0$. The potential u for $t \geq 1$ is the solution of (3.8), namely

$$u = B H_n^{(1)}(i\,k\,t) \cos(k\,p - p_o) \cos(n\,\phi - \phi_o), \tag{3.20}$$

where $H_n^{(1)}$ is the Hankel function of the first kind and n^{th}-order. Applying the boundary condition (3.9_2), we obtain the coefficient B as

$$B = \frac{-4\,i\,S}{H_n^{(1)}(i\,k)} \sum_{l=2}^{4} \frac{\mu_l\,(\mu_l^2 - k^2 - \lambda)}{R_l}\, \alpha_1^{(l)}\, J_n(i\,\mu_l\,t). \tag{3.21}$$

From the boundary conditions (3.9_3) and (3.9_1), we obtain

$$\begin{aligned} \frac{2\,n\,J_n(i\,\mu_1)}{i\,\mu_1}\,\alpha_1^{(l)} + \sum_{l=2}^{4} \frac{k^2}{R_l}\,\alpha_1^{(l)} \Big\{ &\left(4\,\pi\,S^2 - \mu_l^2 + k^2 + \lambda\right)\left(J_{n-1}(i\,\mu_l) - J_{n+1}(i\,\mu_l)\right) \\ &+ \frac{\mu_l}{k}\left(\mu_l^2 - k^2 - \lambda\right) \frac{J_n(i\,\mu_l)}{H_n^{(1)}(i\,k)} \left(H_{n-1}^{(1)}(i\,k) - H_{n+1}^{(1)}(i\,k)\right)\Big\} = 0 \end{aligned}$$

and

$$\begin{aligned} &i\,\mu_1\left(J_{n-2}(i\,\mu_1) - J_{n+2}(i\,\mu_1)\right)\alpha_1^{(1)} + \sum_{l=2}^{4} i\,\mu_l\,\alpha_1^{(l)} \left\{J_{n-2}(i\,\mu_l) + J_{n+2}(i\,\mu_l) - 2J_n(i\,\mu)\right\} = 0 \\ &i\,\mu_1\left(J_{n-2}(i\,\mu_1) + J_{n+2}(i\,\mu_1) - 2J_n(i\,\mu_1)\right)\alpha_1^{(1)} + \sum_{l=2}^{4} i\,\mu_l\,\alpha_1^{(l)} \left\{J_{n-2}(i\,\mu_l) - J_{n+2}(i\,\mu_l)\right\} = 0 \\ &\sum_{l=2}^{4} \frac{2\,\pi\,S^2\,k\,\mu_l^2}{R_l}\,\alpha_1^{(l)} \left\{J_{n-1}(i\,\mu_l) - J_{n+1}(i\,\mu_l)\right\} = 0, \end{aligned} \tag{3.22}$$

respectively.

Supposing that $\alpha_4^{(1)} = 0$ and $\alpha_1^{(1)} = \alpha_3^{(1)}$, which means $\pi\,S^2\,h \neq \lambda$, the system of equations (3.22) is an algebraic homogeneous system with unknown coefficients $\alpha_1^{(l)}$, $l = 1, 2, 3, 4$. A nontrivial solution of (3.22) implies that

$$\begin{vmatrix} \dfrac{4\,n\,J_n(i\,\mu_1)}{i\,\mu_1} & \gamma(\mu_2)\Omega_1(\mu_2) & \gamma(\mu_3)\Omega_1(\mu_3) & \gamma(\mu_4)\Omega_1(\mu_4) \\ \Omega_1(\mu_1) & \Omega_3(\mu_2) & \Omega_3(\mu_3) & \Omega_3(\mu_4) \\ \Omega_3(\mu_1) & \Omega_2(\mu_2) & \Omega_2(\mu_3) & \Omega_2(\mu_4) \\ 0 & \delta(\mu_2)\Omega_4(\mu_2) & \delta(\mu_3)\Omega_4(\mu_3) & \delta(\mu_4)\Omega_4(\mu_4) \end{vmatrix} = 0 \tag{3.23}$$

where

$$\begin{aligned} \gamma(\mu_l) &= \frac{2\,k^2}{R_l}, \quad \delta(\mu_l) = \frac{\pi\,S^2\,\mu_l^2}{R_l}\,\gamma(\mu_l), \\ \Omega_1(\mu_l) &= (4\,\pi\,S^2 - \mu_l^2 + k^2 + \lambda)\Omega_4(\mu_l) + \frac{\mu_l}{k}(\mu_l^2 - k^2 - \lambda)\frac{J_n(i\,\mu_l)}{H_n^{(1)}(ik)}\left(H_{n-1}^{(1)}(ik) - H_{n+1}^{(1)}(ik)\right) \\ \Omega_2(\mu_l) &= i\,\mu_l\left(J_{n-2}(i\,\mu_l) - J_{n+2}(i\,\mu_l)\right), \quad \Omega_2(\mu_l) = i\,\mu_l\left(J_{n-2}(i\,\mu_l) + J_{n+2}(i\,\mu_l) - 2\,J_n(i\,\mu_l)\right) \\ \Omega_4(\mu_l) &= \left(J_{n-1}(i\,\mu_l) - J_{n+1}(i\,\mu_l)\right), \qquad k \neq 0. \end{aligned} \tag{3.24}$$

The eigenvalue problem of equations (3.7–9) was reduced to the determination of the largest root of the equation (3.23), for every value of S, which has the functional form

$$F(h,\,k,\,\lambda) = 0, \qquad \text{for every } S. \tag{3.25}$$

The determination of the nucleation field requires also a knowledge of the dependence of f on M and T, where T is the absolute temperature. If $f = f(M)$ is known, from (2.10) and (3.5) we obtain

$$h = \frac{2\,R_1(M^o)^2}{\pi} + \frac{1}{\pi M^o}\left.\frac{df}{dM}\right|_o \tag{3.26}$$

and

$$\lambda = S^2 \left.\frac{d^2 f}{dM^2}\right|_o . \tag{3.27}$$

In the framework of Landau's theory [15] the isotropic function f is given as

$$f(M,T) \approx f_o + A\,M^2 + B\,M^4, \tag{3.28}$$

where f_o is a function only of temperature and pressure. The coefficients A and B are given as [18]

$$A = \lambda_w\left(1 - \frac{T_c}{T}\right), \quad B = \frac{3\,\lambda_w}{20}\,\frac{\{(2s+1)^2+1\}}{(s+1)^2} \tag{3.29}$$

where λ_w is an empirical constant, called by Weiss the molecular field-constant and conditioned by the exchange interaction [18] and s is the spin of an atom (for $s = 1/2 \Rightarrow B = \lambda_w/3$). Introducing the expression (3.28), taking into account (3.29), into (3.26) and (3.27), we obtain

$$h = \frac{2\,R_1(M^o)^2}{\pi} + \frac{2}{\pi}\,\lambda_w\left\{1 - \frac{T_c}{T} + \frac{2}{3}(M^o)^2\right\} \tag{3.30}$$

and

$$\lambda = S^2\,\lambda_w\left\{1 - \frac{T_c}{T} + 2(M^o)^2\right\}, \tag{3.31}$$

respectively. Finally, the relationship between h and λ is given as

$$h \approx \frac{2\,R_1(M^o)^2}{\pi} + \frac{4}{3\,\pi}\lambda_w\left(1 - \frac{T_c}{T}\right) + \frac{1}{3\,\pi\,S^2}\lambda. \tag{3.32}$$

Up to now the solution of the problem has been described in the most general manner. To obtain the largest eigenvalue h, it is convenient to treat the cases $n = 0$, $n > 1$ and $n = 1$ separately [1].

For $n = 0$ the system of equations (3.11) gives

$$\begin{aligned}
&\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{1}{t^2} - k^2 - \pi\,S^2\,h\right\}A_\phi = 0\\
&\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - \frac{1}{t^2} - k^2 - \pi\,S^2\,h\right\}A_r - \pi\,S\frac{d\mho_t}{dt} = 0\\
&\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - k^2 - \lambda\right\}A_3 + \pi\,S\,k\,\mho_t = 0\\
&\left\{\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - k^2\right\}\mho_t - 4\,S\left\{\frac{dA_r}{dt} + \frac{A_r}{t} + k\,A_3\right\} = 0.
\end{aligned} \tag{3.33}$$

The general solution of equation (3.33_1) is

$$A_\phi = B\,J_1(i\,\mu_1 t). \tag{3.34}$$

Substituting (3.34) in the boundary condition $\frac{dA_\phi}{dt}|_{t=1} = 0$, we obtain the eigenvalue equation

$$\frac{dJ_1(i\,\mu_1)}{d(i\,\mu_1)} = 0. \tag{3.35}$$

The smallest root of this equation is $i\mu_1 = 1.841$, therefore,

$$h(k,\, S) = -1.08\, S^{-2} - \frac{k^2}{\pi S^2}. \tag{3.36}$$

The maximum value of h is

$$h_n(S) = -1.08\, S^{-2}. \tag{3.37}$$

This is the same result as obtained earlier [1] for the magnetization curling ($A_r = A_3 = U_t = 0$). An underconstrained eigenvalue of the equations (3.33_2–3.33_4) is taken by neglecting the self-magnetostatic energy ($U_t = 0$) and discarding (3.33_4). The solution of (3.33_2) with $U_t = 0$ is identical to (3.33_1) while equation (3.33_3) becomes

$$\left(\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} - k^2 - \lambda\right) A_3 = 0. \tag{3.38}$$

The general regular solution of equation (3.38) is

$$A_3 = C\, J_o(i\mu_3 t), \quad \mu_3 = (k^2 + \lambda)^{1/2}. \tag{3.39}$$

From the boundary condition $\frac{dA_3}{dt}|_{t=1} = 0$ we get

$$\frac{dJ_o(i\mu_3)}{d(i\mu_3)} = -J_1(i\mu_3) = 0. \tag{3.40}$$

The smallest root of (3.40) is $i\mu_3 = 3.832$, thus

$$\lambda = -14.684 - k^2. \tag{3.41}$$

In the region of validity of (3.32), we have

$$h = \varepsilon - \tau + \frac{\lambda}{3\pi S^2} = -1.558\, S^{-2} - \tau + \varepsilon, \tag{3.42}$$

where

$$\varepsilon = \frac{2R_1(M^o)^2}{\pi}, \quad \tau = \frac{4}{3\pi}\lambda_w \left(\frac{T_c}{T} - 1\right) > 0$$

and k is taken equal to zero, since we are interesting in $h_n = \max h$.

In order to have

$$h_n \geq -1.08\, S^{-2},$$

the inequality

$$\varepsilon \geq \tau + 0.475\, S^{-2}, \qquad (K_1 > 0) \tag{3.43}$$

must be satisfied, results which is in agreement with that reported in [16].

For $n > 1$ an underconstained solution of equations (3.11) is obtained by neglecting the self-magnetostatic energy ($U_t = 0$) and discarding (3.11_4), namely

$$\begin{aligned} A_r - A_\phi &= \alpha_1 J_{n-1}(i\mu_1 t) \\ A_r + A_\phi &= \alpha_3 J_n(i\mu_1 t) \\ A_3 &= \alpha_4 J_n(i\mu_3 t). \end{aligned} \tag{3.44}$$

Introducing (3.44) into the boundary conditions (3.9), we obtain the following equations for the nucleation field

$$k^2 + \pi S^2 h_n = -x_{n-1}^2, \quad k^2 + \pi S^2 h_n = -x_{n+1}^2, \quad k^2 + \lambda_n = -x_n^2, \tag{3.45}$$

where x_n is the smallest root of $J_n'(x) = 0$. Since $x_n (n > 1)$ is a monotonous increasing function of n, equations (3.45_{1-2}) cannot give more positive nucleation field than for curling. From the last of equations (3.45) and the constitutive equation (3.32), we obtain

$$h = -\frac{x_n^2}{3\pi} S^{-2} + \varepsilon - \tau. \tag{3.46}$$

For $h_n \geq h_n^{\text{curl}}$ we must have

$$-\frac{x_n^2}{3\pi} S^{-2} + \varepsilon - \tau > -\frac{x_1^2}{3\pi} S^{-2}. \tag{3.47}$$

Since $-x_n^2 < -x_1^2$, the condition for $h_n \geq h_n^{\text{curl}}$ is

$$\varepsilon \geq \frac{x_n^2 - 3x_1^2}{3\pi S^2} + \tau. \tag{3.48}$$

For $n = 1$ the nucleation field will directly be calculated by solving numerically the equation (3.23). The general solution (3.19) has been obtained under the assumption that all roots μ_l of the equation (3.14) are different and nonzero. In what follows we will discuss the special cases for $n = 1$.

(i) $\mu_1 = 0$: In this case, from (3.14), we obtain

$$\mu_2 = 0, \quad \mu_{3,4} = \left\{ \left(\frac{\lambda}{2} - \pi S^2 (h+2)\right) \pm \left(\left(\frac{\lambda}{2} - 2\pi S^2\right)^2 + 4\pi^2 S^4 h\right)^{1/2} \right\}^{1/2} \tag{3.49}$$

and the general regular solution is given as

$$\begin{aligned}
A_r &= \frac{1}{2}\left\{ \left(\frac{2\pi S^2(\lambda - k^2)}{k^2+\lambda} - k^2\right) c_1 t^2 + c_2 + \sum_{l=3}^{4} c_l \left(J_2(i\mu_l t) - J_o(i\mu_l t)\right) \right\} \\
A_\phi &= -\frac{1}{2}\left\{ k^2 + 2\pi S^2 \frac{(5k^2+3)}{k^2+\lambda} \right\} c_1 t^2 - \frac{1}{2} c_2 + \sum_{l=3}^{4} c_l \frac{J_1(i\mu_l t)}{i\mu_l t} \\
U_t &= 8 S c_1 t + \sum_{l=3}^{4} c_l \frac{i\mu_l}{\pi S} J_1(i\mu_l t) \\
A_3 &= \frac{8\pi S^2 k}{k^2+\lambda} c_1 t - \sum_{l=3}^{4} c_l \frac{i\mu_l k}{\mu_l^2 - k^2 - \lambda} J_1(i\mu_l t).
\end{aligned} \tag{3.50}$$

Using the boundary conditions (3.9_1) and asking for a nontrivial solution in the case under discussion, we obtain

$$F^{(i)}(k,\lambda) = \begin{vmatrix} \gamma_1 & \Omega_1(\mu_3) & \Omega(\mu_4) \\ \gamma_2 & 4J_2(i\mu_3) & 4J_2(i\mu_4) \\ \gamma_3 & \delta(\mu_3)\Omega_2(\mu_3) & \delta(\mu_4)\Omega_2(\mu_4) \end{vmatrix} = 0, \tag{3.51}$$

where

$$\begin{aligned}
\gamma_1 &= 4\left\{ \frac{2\pi S^2(\lambda - k^2)}{k^2+\lambda} - k^2 \right\}, \quad \gamma_2 = 4\left\{ k^2 + 2\pi S^2 \frac{(5k^2+3\lambda)}{k^2+\lambda} \right\} \\
\gamma_3 &= \frac{16\pi S^2 k}{k^2+\lambda}, \quad \delta(\mu_l) = \frac{\mu_l^2 k}{\mu_l^2 - k^2 - \lambda} \\
\Omega_1(\mu_l) &= i\mu_l \left\{ 3J_1(i\mu_l) - J_3(i\mu_l) \right\}, \quad \Omega_2(\mu_l) = J_o(i\mu_l) - J_2(i\mu_l).
\end{aligned} \tag{3.52}$$

We note that on physical grounds we obtain $c_2 = 0$.

(ii) One of μ_l, $l = 2,3,4$ is equal to zero ($\pi S^2 h \neq \lambda$). From (3.14) it is seen that it will happen either for $k = 0$ or for $4\pi S^2 h + k^2 = -\lambda$. In the case $k = 0$ the roots of (3.14) are

$$\mu_1 = (\pi S^2 h)^{1/2}, \quad \mu_2 = 0, \quad \mu_3 = \lambda^{1/2}, \quad \mu_4 = \left(\pi S^2(h+4)\right)^{1/2} \tag{3.53}$$

and the regular solution is given as

$$\begin{aligned} A_r - A_\phi &= c_1 J_o(i\mu_1 t) + c_2 + c_4 J_o(i\mu_4 t) \\ \mathrm{U}_t &= -S h c_2 t - \frac{4 i S}{\mu_3} c_4 J_1(i\mu_4 t) \\ A_r + A_\phi &= c_1 J_2(i\mu_1 t) - c_4 J_2(i\mu_4 t), \quad A_3 = 0 \end{aligned} \tag{3.54}$$

which is identical to that obtained in [1] in the framework of Brown's equations for the nucleation field ($M_i M_i = M_s^2 = \text{const.}$).

In the case where $4\pi S^2 h + k^2 = -\lambda$, the roots of equation (3.14) are

$$\begin{aligned} \mu_1 &= (k^2 + \pi S^2 h)^{1/2}, \quad \mu_2 = 0, \\ \mu_{3,4} &= \frac{1}{\sqrt{2}} \left\{ (2k^2 + \pi S^2 h) \pm \left[\pi^2 S^4 (h+8)^2 + 16\pi S^2 k^2\right]^{1/2} \right\}^{1/2} \end{aligned} \tag{3.55}$$

and the regular solution is given as

$$\begin{aligned} A_r &= \frac{c_1}{2}\{J_o(i\mu_1 t) + J_2(i\mu_1 t)\} + \frac{c_2}{2} + \sum_{l=3}^{4} \frac{c_l}{2}\{J_o(i\mu_l t) - J_2(i\mu_l t)\} \\ A_\phi &= \frac{c_1}{2}\{-J_o(i\mu_1 t) + J_2(i\mu_1 t)\} - \frac{c_2}{2} + \sum_{l=3}^{4} \frac{c_l}{2}\{J_o(i\mu_l t) - J_2(i\mu_l t)\} \\ \mathrm{U}_t &= -\frac{k^2 + \pi S^2 h}{2\pi S} c_2 t + \sum_{l=3}^{4} \frac{(-i)}{\pi S \mu_l} (\mu_l^2 - k^2 - \pi S^2 h) c_l J_1(i\mu_l t), \qquad t \le 1 \\ A_3 &= \frac{k(k^2 + \pi S^2 h)}{8\pi S^2} c_2 t + \sum_{l=3}^{4} \frac{i(\mu_l^2 - k^2 - \pi S^2 h)}{\mu_l(\mu_l^2 - k^2 - \lambda)} k c_l J_1(i\mu_l t) \\ \mathrm{U}_t &= -\frac{k^2 + \pi S^2 h}{2\pi S} t - \frac{1}{t} \sum_{l=3}^{4} \frac{-i}{\pi S \mu_l} (\mu_l^2 - k^2 - \pi S^2 h) c_l J_1(i\mu_l t), \qquad t \ge 1. \end{aligned} \tag{3.56}$$

Using the boundary conditions and asking for nontrivial solution, we obtain

$$F^{(ii)}(h, k, \lambda, S) = \begin{vmatrix} \frac{4SJ_1(i\mu_1)}{i\mu_1} & \frac{k^2 + \pi S^2(h+2)}{\pi S} & \Omega_1(\mu_3) & \Omega_1(\mu_4) \\ \Omega_2(\mu_1) & 0 & \Omega_2(\mu_3) & \Omega_3(\mu_4) \\ \Omega_3(\mu_1) & 0 & \Omega_2(\mu_3) & \Omega_2(\mu_4) \\ 0 & \frac{k^2(k^2+\pi S^2 h)}{4\pi S^2} & -\gamma(\mu_3)\Omega_4(\mu_3) & -\gamma(\mu_4)\Omega_4(\mu_4) \end{vmatrix} = 0, \tag{3.57}$$

where

$$\begin{aligned} \Omega_1(\mu_l) &= \frac{\pi S^2(2-h) + k^2 - \mu_l^2}{\pi S} \Omega_4(\mu_l) - \frac{(\mu_l^2 - k^2 - \pi S^2 h)}{\pi S} \frac{J_1(i\mu_l)}{i\mu_l} \\ \Omega_2(\mu_l) &= i\mu_l \{J_1(i\mu_l) + J_3(i\mu_l)\}, \quad \Omega_3(\mu_l) = i\mu_l \{3 J_1(i\mu_l) - J_3(i\mu_l)\} \\ \Omega_4(\mu_l) &= J_o(i\mu_l) - J_2(i\mu_l), \quad \gamma(\mu_l) = \frac{k(\mu_l^2 - k^2 - \pi S^2 h)}{\mu_l^2 - k^2 - \lambda}. \end{aligned}$$

Finally, the case $\mu_1 = \mu_2 = \mu_3 = \mu_4$ implies that $k^2 + \pi S^2 h = 0$ ($\mu_1 = 0$), which is of no interest.

4. NUMERICAL RESULTS AND DISCUSSION

It is obvious that the case $n = 1$ discussed in the previous paragraph is quite complicated. Although the general regular solution has been obtained in terms of Bessel functions, the secular equation resulting from the boundary conditions is a transcendental one and its roots have to be determined numerically. The method followed to solve equation (3.23), ($F(h, k, \lambda, S) = 0$) consists of minimizing its square. In particular, we used the optimization package [13,14] MERLIN-2.0, since it offers a convenient environment to work with. It allows the search area to be restricted to selected intervals, that in turn can be searched exhaustively. The minimisation algorithms used are the quasi-Newton ones known as BFGS and DFP, the conjugate gradient method of Polak and Ribiere, the non-linear Simplex method of Nelder and Mead, a Monte-Carlo search with occasional line searches and a modification of the alternating variables method, all of which are documented and referenced in [14,15] and the accompanying MERLIN user's manual. The proposed numerical procedure was first applied to calculate the nucleation field (buckling solution) of the classical problem treated in [1]. The results obtained are presented in [19] and it was found to be in agreement with those of [1].

The solution of equation (3.23) for each S and λ yields pairs (h, k) corresponding to nucleation. The results obtained by the proposed numerical method are cited in Table 1.

Table 1. Exact values of h bases on eq. (3.23).

S	$k(\times 10^{-7})$	$-\lambda$	$-h(\times 10^{-2})$	S	$k(\times 10^{-7})$	$-\lambda$	$-h(\times 10^{-2})$
0.0005	0.1	0	0	0.05	0.1	0	0
		0.1	0.00261987			0.1	0.00236652
		0.4	0.0766624			0.4	0.0693453
		0.6	0.207781			0.6	0.195681
		0.7	0.303799			0.7	0.295346
		0.8	0.605982			0.8	0.426090
		0.9	0.606168			0.9	0.593968
		1.0	0.821141			1.0	0.806014
		1.5	2.90554			1.5	2.87500
		2.0	8.32138			2.0	8.25967
		3.0	74.1730			3.0	73.8612
0.01	0.1	0	0	0.1	0.1	0	0
		0.1	0.00246045			0.1	0
		0.4	0.0733498			0.4	0.0584826
		0.6	0.202329			0.6	0.177567
		0.7	0.303535			0.7	0.272978
		0.8	0.435994			0.8	0.398963
		0.9	0.605787			0.9	0.561502
		1.0	0.819982			1.0	0.767544
		1.5	2.90459			1.5	2.79228
		2.0	8.31946			2.0	8.08977
		3.0	74.1618			3.0	72.9674

In Figure 2 it is shown the variation of the nucleation field with the size parameter S of the particle and $\lambda \propto 1/\aleph$.

In the range of S considered, $S \in [0.005, 0.12]$, the nucleation, $-h$, is increasing with decreasing particle size and with increasing $|\lambda|$. The corresponding exact buckling eigenfunctions are shown in figure 3. This result suggests that the buckling eigenfunctions should be approximated as $(A_r = -A_\phi = \text{const.}, A_3(t))$. We note that all the calculations here were carried out by neglecting the magnetocrystalline anisotropy energy. Below T_c the magnetic anisotropy begins to play an important role in the magnetic response. As T_c is approached the anisotropy for iron scales with T as $K_1 = 1.6 \times 10^4((T_c - T)/T_c)^{1.51}\, Joule/m^3$ [10].

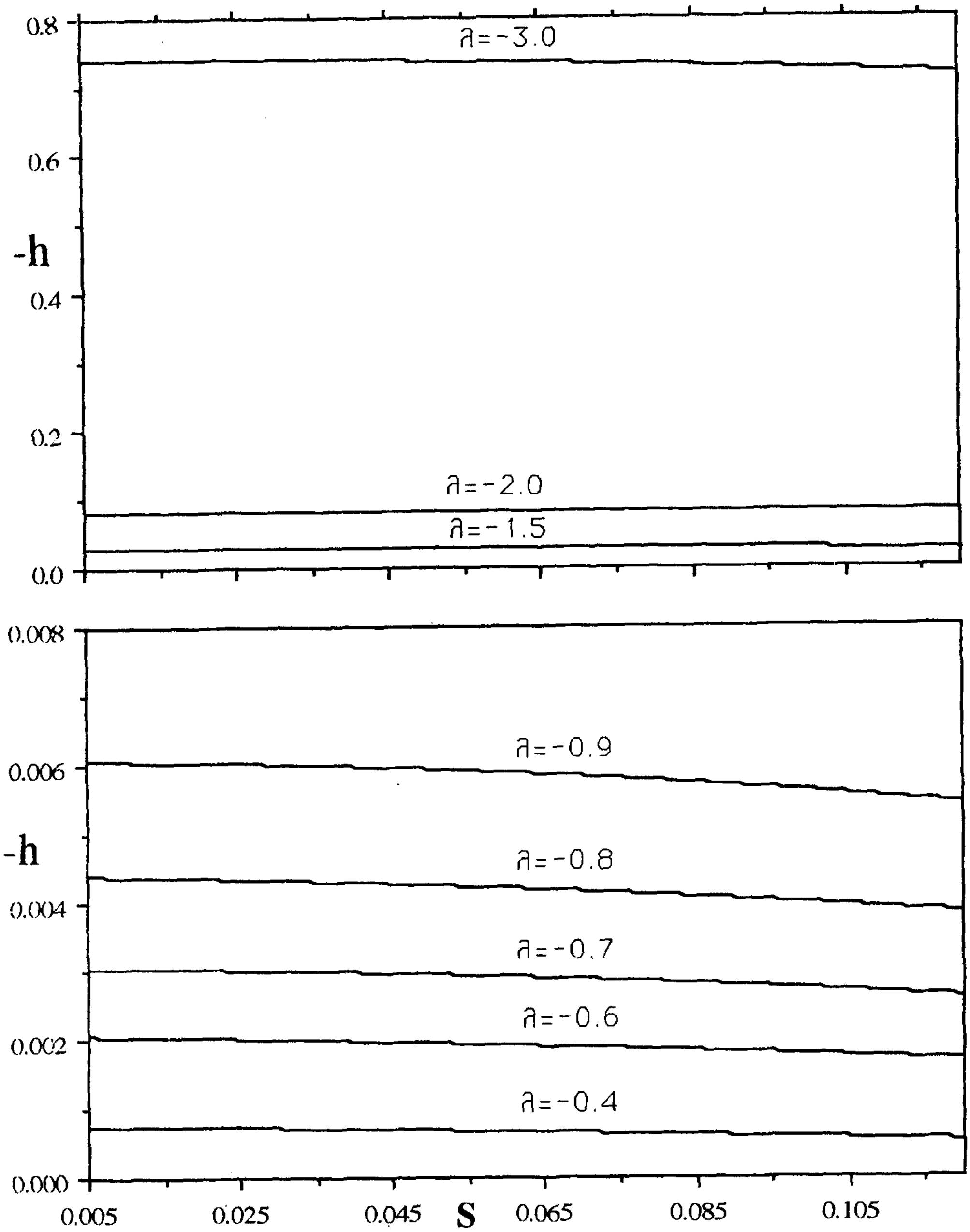

Figure 2. Variation of h versus S and λ.

From the present analysis we are led to the conclusion that "Curling" and "Buckling" are the main mechanisms which "ignite" reversion of magnetization of an infinite cylinder even in the neighbourhood of the Curie point. An easier mechanism than those already mentioned exists only if $K_1 > 0$ as was pointed out by Kondorsky [16] and in the present work as well. Assuming that $a = 2A(M^o)^{-2}$, where A is the exchange constant, we have

$$r_0 = \frac{S\sqrt{2\mu_o A}}{M^o}.$$

Below T_c the spontaneous magnetization is given by [10],

$$M^o = M_1\left(\frac{T_c - T}{T_1}\right)^\beta,$$

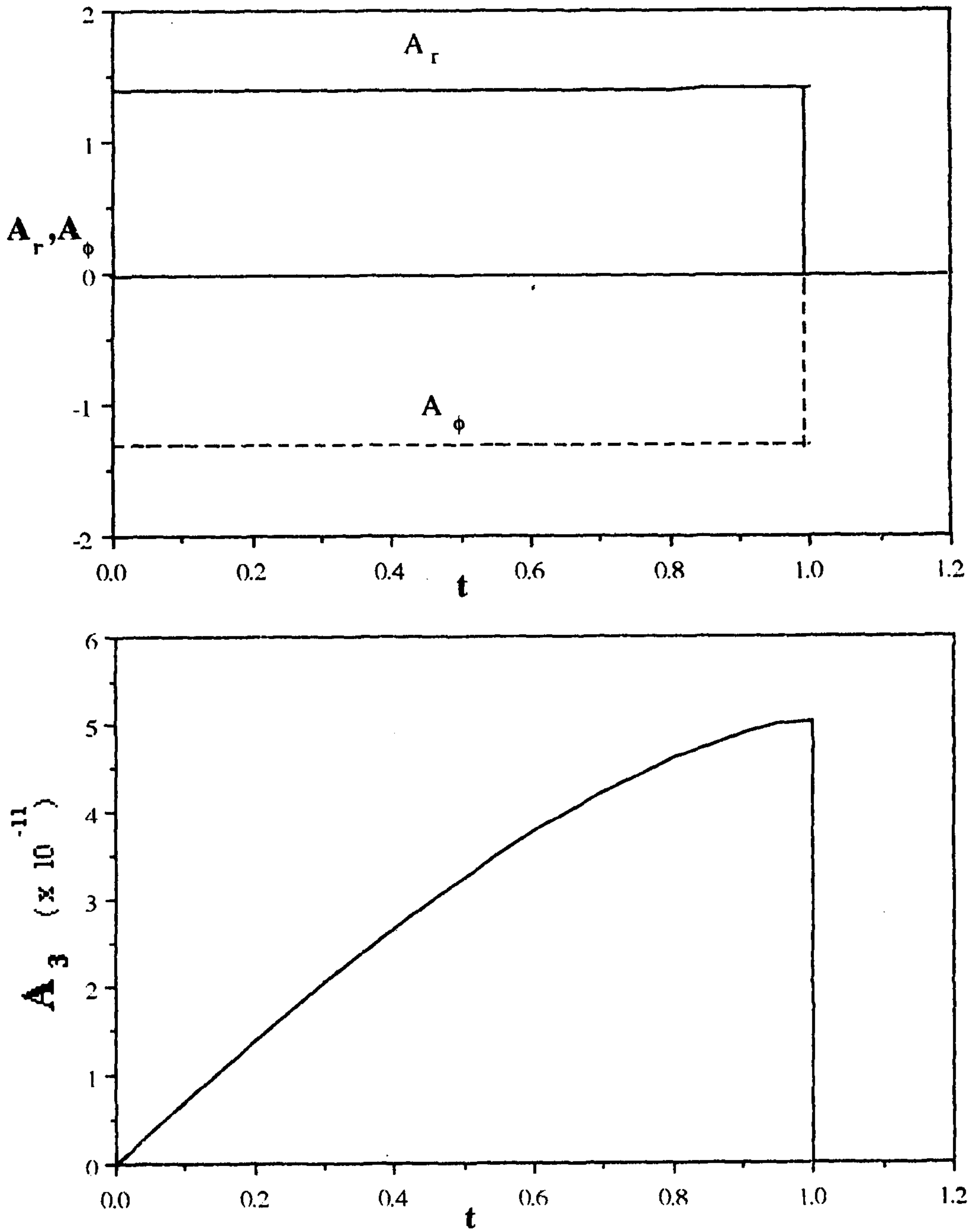

Figure 3. Reduced direction cosines A_r, A_ϕ and A_3 as functions of the reduced radial distance t for: ($h = -0.0279228$, $S = 0.1$, $\lambda = -1.5$, $k = 10^{-8}$).

Table 2. Particle radius based on Reference [10].

S	$r_o(nm)$		
	$(T_c - T)/T_c = 10^{-6}$	$(T_c - T)/T_c = 10^{-5}$	$(T_c - T)/T_c = 10^{-4}$
0.005	8.2	3.5	1.5
0.01	16.5	7.1	3.0
0.05	82.5	35.4	15.2
0.1	165.0	70.7	30.3

where β is the critical exponent for the spontaneous magnetization. For iron $\beta = 0.368$, $M_1 = 0.0244\, M_s$, $M_s = 2\, Tesla$ (M_s is the spontaneous magnetization at low temperature), $T_c = 770^o\, K$, $T_1 = 1.45^o\, K$, $A = 2 \times 10^{-11}\, Joule/m$ and $\mu_o = 1.26 \times 10^6\, Henry/m$ (μ_o is the vacuum permeability). In Table 2 are cited the radius of the single domain particle considered at temperatures near but below T_c.

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