Streaming Model of Computation

A streaming algorithm processes a data stream S:

- Input is presented as a sequence of items and can be examined in only a few passes (typically just one).
- The algorithm has limited memory and cannot store the whole input sequence.
- The algorithm can spend limited processing time per item.
- In some problems we are satisfied with an approximate answer.
- Approximation algorithms can be based on sketches (summaries) of the data stream in memory.

Streaming Graph Algorithms

In many applications we deal with massive graphs. E.g. (vertices – edges):

- Web-pages – hyperlinks
- Neurons – synapses
- IP addresses – network flows
- People – friendships

Processing such graphs with a classic graph algorithm may be infeasible!

But it may be possible to use an algorithm developed for the **data stream model**.

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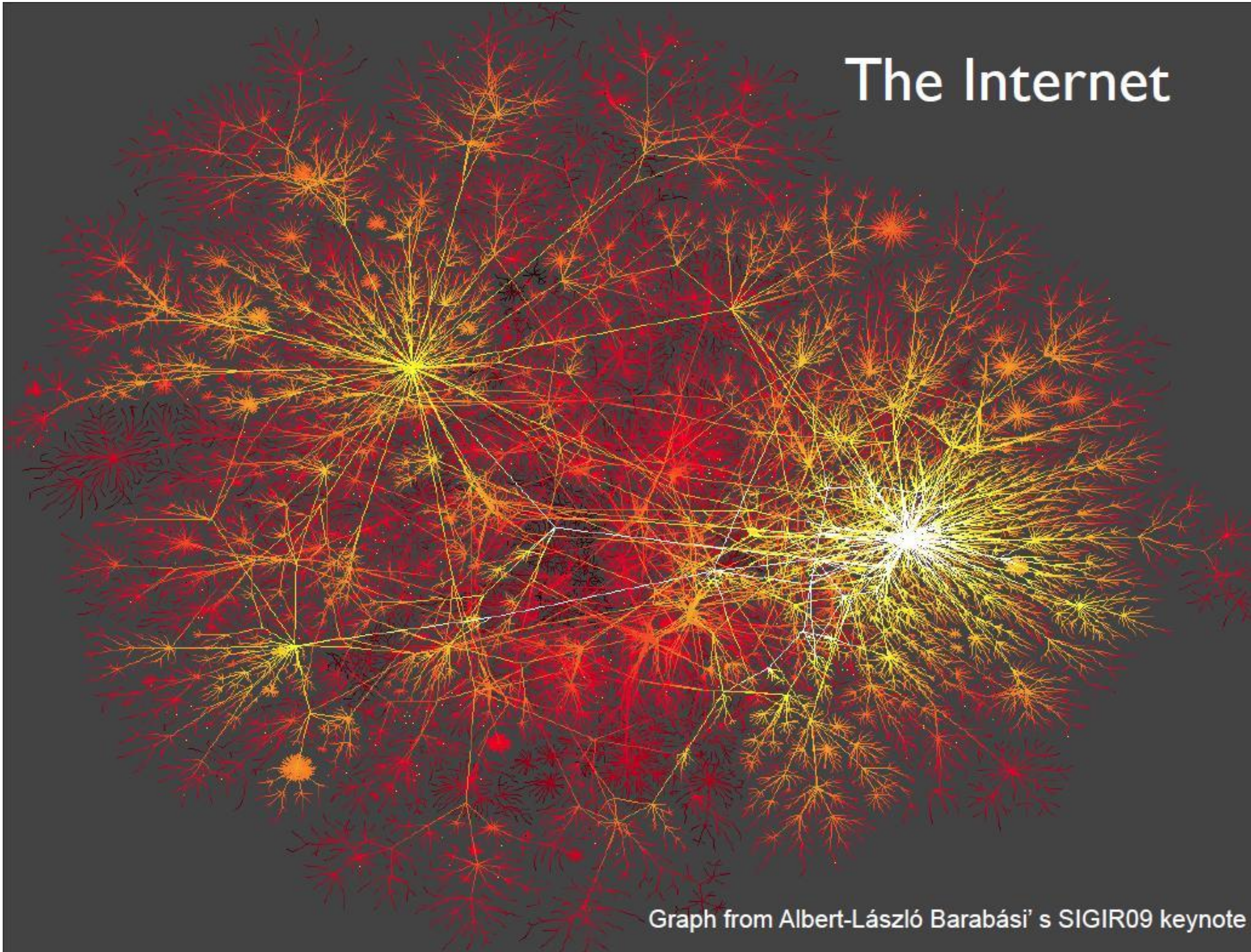
But it may be possible to use an algorithm developed for the **data stream model**.

Presentation based on:
A. McGregor "Graph Stream Algorithms: A Survey" [ACM SIGMOD Record 2014]


The Internet
Graph from Albert-László Barabási' s SIGIR09 keynote

Streaming Graph Algorithms

Data stream model

- The input is given by a stream of data. E.g., the stream could be the graph edges.
- The algorithm can use a limited amount of memory to process the stream.
- The input stream must be processed in the order it arrives.

Related goals:

- Real-time systems.
- I/O efficiency.
- Trade-off size and accuracy.

Streaming Graph Algorithms

Data stream model

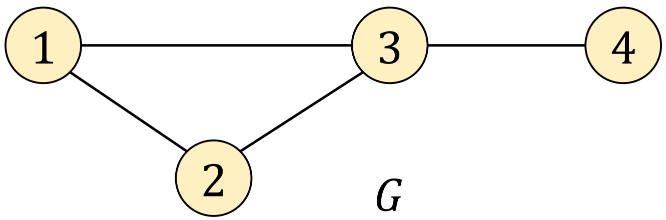
How much memory should our model allow in order to be able to process a graph with n vertices?

- Most problems are intractable if space is $< n$.
- We will work in the **semi-streaming model** that allows $O(n \log^k n)$ memory, for some constant k.
- Some algorithms will be randomized. We will say that an event E occurs **with high probability** if $\Pr[E] \geq 1 - 1/n$.

Streaming Graph Algorithms

Graph connectivity

Data stream S: Edges of a graph $G = (V, E)$ with $n = |V|$



E.g., data stream $S = (1,2), (2,3), (1,3), (3,4)$

Streaming Graph Algorithms

Graph connectivity

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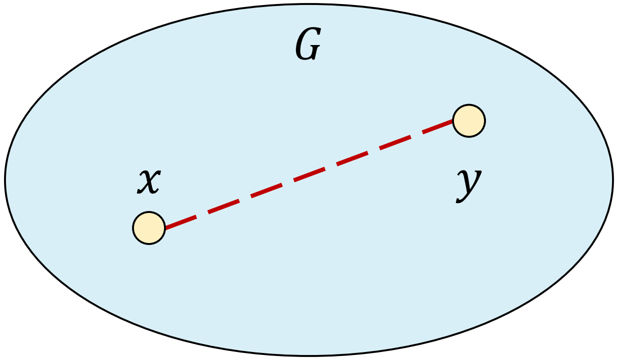
The goal is to test if G is connected, i.e., for any two vertices there is a path that connects them.



Streaming Graph Algorithms

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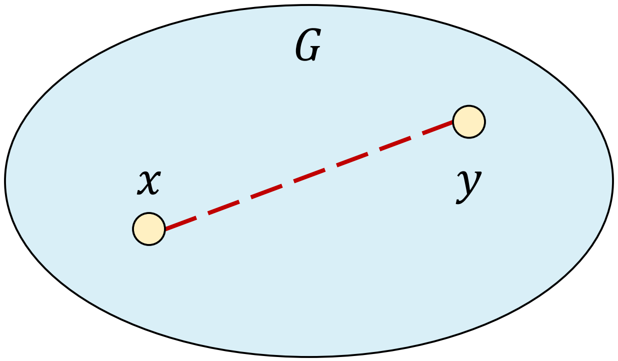


Simple algorithm: Maintain a set of edges H. When we read the next edge (u, v) from the stream, we add it to H if there is currently no path between u and v.

Streaming Graph Algorithms

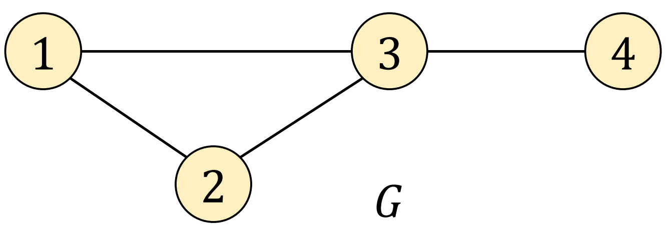
Spanners

a-spanner H of a graph $G = (V, E)$: subgraph of G such that for all pairs $u, v \in V$,

$$d_G(u,v) \leq d_H(u,v) \leq a \cdot d_G(u,v)$$

$d_G(u,v)$ = length of the shortest path between u and v in G

$d_H(u,v)$ = length of the shortest path between u and v in H



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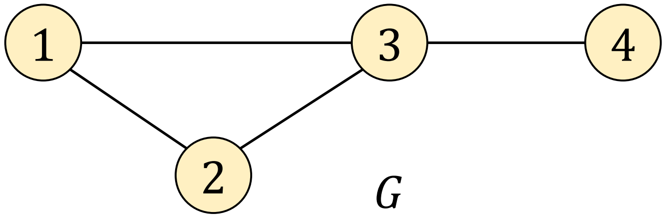
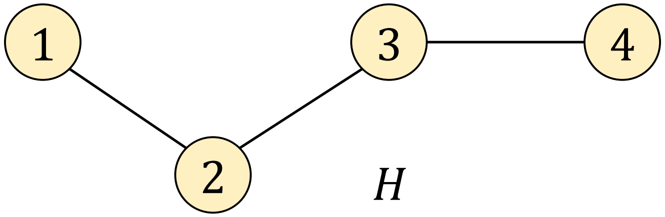


G

2-spanner



H

Streaming Graph Algorithms

Spanners

Construction of an a-spanner H: add next edge (u, v) if it does not create a short cycle in H

Greedy Spanner Algorithm

1. $H \leftarrow \emptyset$
2. **for** each edge $(u, v) \in S$ **do**
3. **if** $d_H(u, v) > a$ **then** add (u, v) to H
4. **return** H

- Does this work?
- What is the size (#edges) of the spanner?

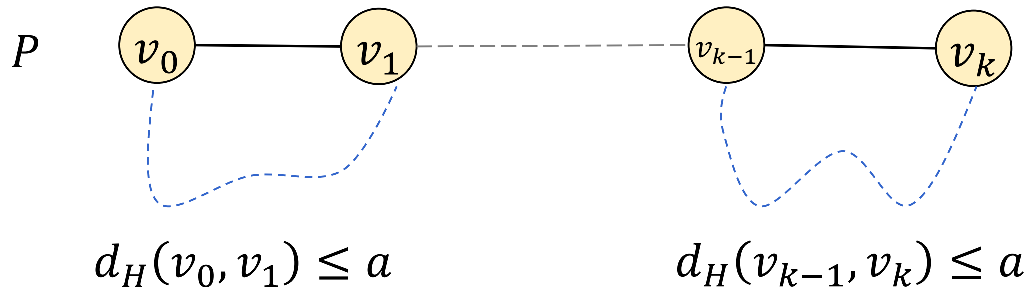
Streaming Graph Algorithms

Spanners

Proof that the Greedy Spanner Algorithm works:

For any edge (x, y) of G we have $d_H(x, y) \leq a$.

Consider a path $P = (v_0, v_1, .., v_{k-1}, v_k)$ in G



Length of P in $G = k = d_G(v_0, v_1) + d_G(v_1, v_2) + \cdots + d_G(v_{k-1}, v_k)$

Streaming Graph Algorithms

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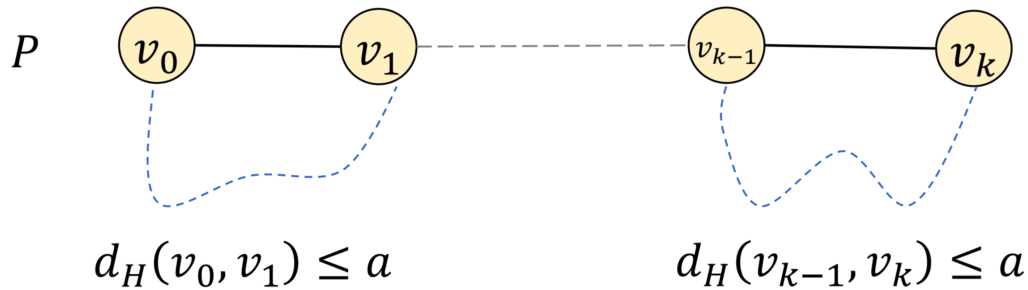

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Length in H $\leq d_H(v_0, v_1) + d_H(v_1, v_2) + \cdots + d_H(v_{k-1}, v_k)$

$$\leq a \cdot d_G(v_0, v_1) + a \cdot d_G(v_1, v_2) + \cdots + a \cdot d_G(v_{k-1}, v_k) = a \cdot k$$

Streaming Graph Algorithms

Spanners

How many edges are inserted into H?

- Let $a = 2t - 1$, for some integer t.
- Then H does not contain cycles of length $< 2t$.
- By a known result in Graph Theory, any such graph has at most

$$O\left(n^{1+1/t}\right)$$

edges.

Streaming Graph Algorithms

Minimum Spanning Tree

Data stream S: Edges of a weighted graph $G = (V, E, w)$ with $n = |V|$

Construction: if next edge (u, v) creates a cycle C in H, delete from H the maximum weight edge of C.

Greedy MST Algorithm

1. $H \leftarrow \emptyset$
2. **for** each edge $e = (u, v) \in S$ **do**
3.   **if** e creates a cycle C in H **then**
4.   find the maximum weight edge $f \in C$
5.   add e to H
6.   delete f from H
7. **return** H

Streaming Graph Algorithms

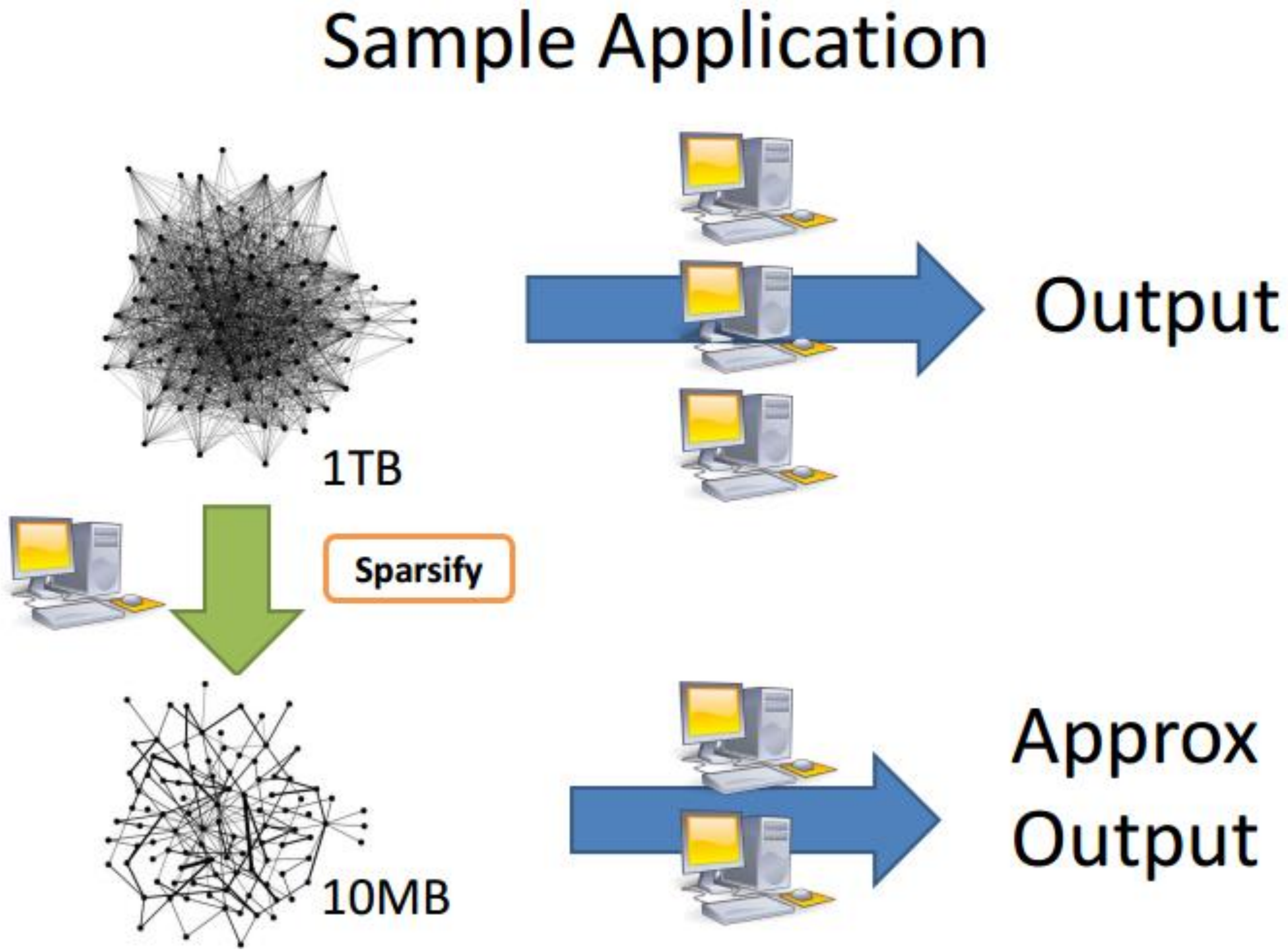
Graph Sparsification

Given a graph $G = (V, E)$ we want to construct a **weighted subgraph** $H = (V, E_H, w)$ of G that estimates various (connectivity) properties of G

E.g.:

- Cut sparsification [Benczur-Karger]
- Spectral sparsification [Spielman-Teng]

Streaming Graph Algorithms



Picture from https://simons.berkeley.edu/sites/default/files/docs/1768/slidessrivastava1.pdf

Streaming Graph Algorithms

Cuts in Graphs

Weighted graph $G = (V, E, w)$. Edge weights $w : E \to R$

A-cut: partition of V into two sets A and $V \backslash A$

$\delta_G(A)$ = set of edges in G crossing the A-cut. $\delta_G(A) = \{(u, v) \in E : u \in A, v \in V \backslash \mathrm{A}\}$

Size of A-cut in G: $\lambda_A(G) = \sum_{e \in \delta_G(A)} w(e)$

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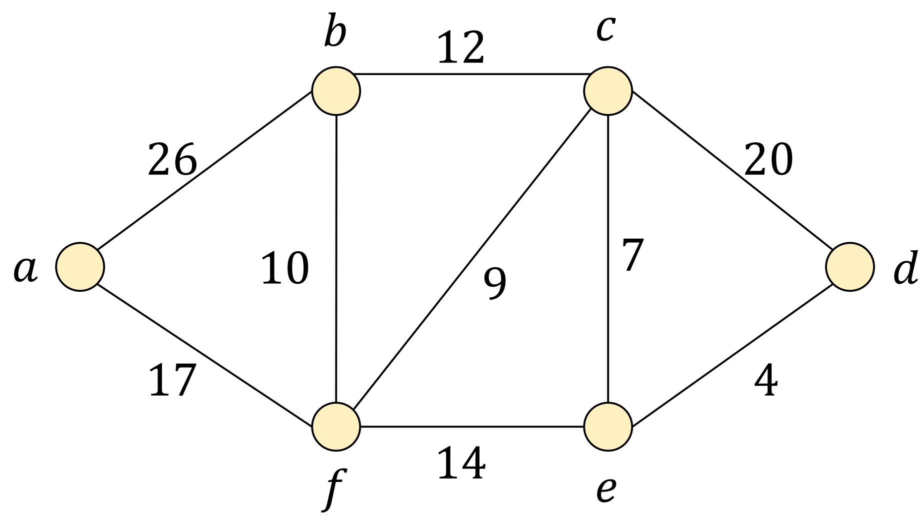

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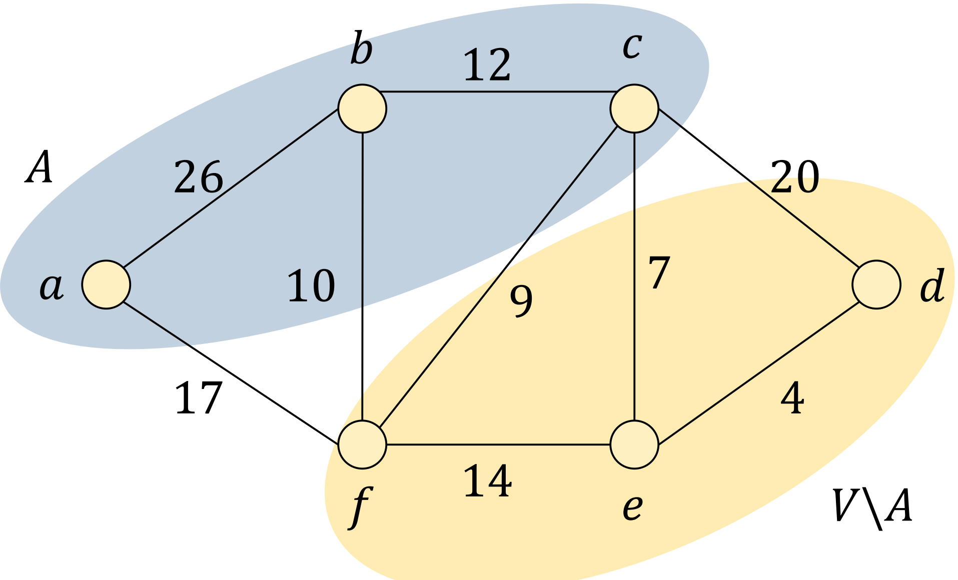

$\lambda_A(G) =$
$w(a, f) + w(b, f) + w(c, f) + w(c, e)$
$+ w(c, d) = 17 + 10 + 9 + 7 + 20 = 63$

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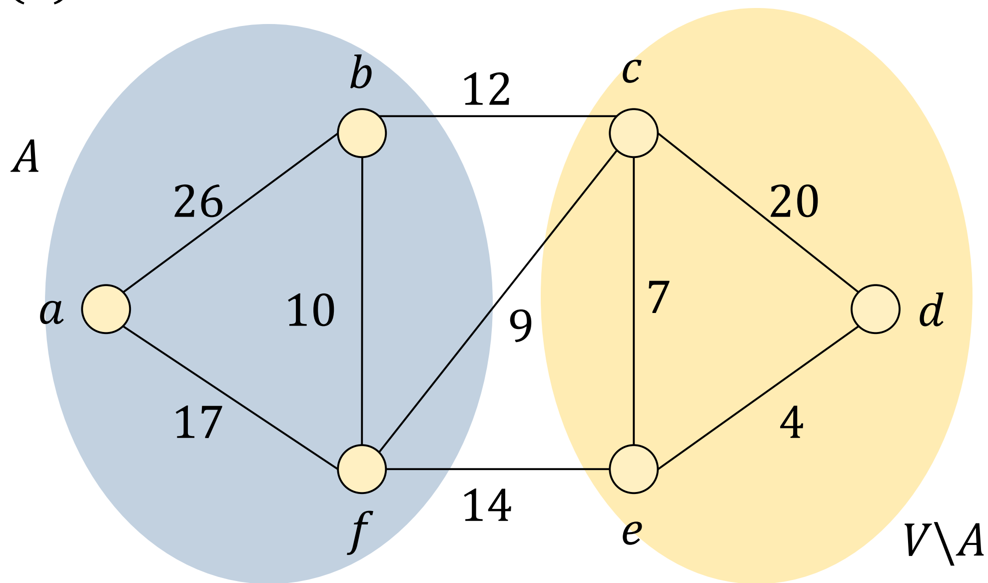

$\lambda_A(G) = w(b, c) + w(f, c) + w(f, e)$
$= 12 + 9 + 14 = 35$

Streaming Graph Algorithms

Cut Sparsification

Given a graph $G = (V, E)$ we want to construct a **weighted subgraph** $H = (V, E_H, w)$ of G that estimates the size of each cut of G

$(1+\varepsilon)$ cut sparsification

$$(1-\varepsilon) \cdot \lambda_A(G) \;\leq\; \lambda_A(H) \;\leq\; (1+\varepsilon) \cdot \lambda_A(G)$$

for all vertex subsets $A \subset V$

Streaming Graph Algorithms

Graph Laplacian

Weighted graph $G = (V, E, w)$. Edge weights $w : E \rightarrow R$

Laplacian of G: $n \times n$ real matrix L_G, $n = |V|$

$$L_G(i,j) = \begin{cases} \sum_{(i,k)\in E} w(i,k)\,, & i = j \\ -w(i,j), & i \neq j \end{cases}$$

where $w(i,j) = 0$ if $(i,j) \notin E$

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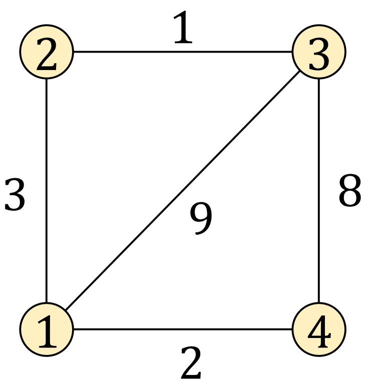
$$L_G = \begin{pmatrix} 14 & -3 & -9 & -2 \\ -3 & 4 & -1 & 0 \\ -9 & -1 & 18 & -8 \\ -2 & 0 & -8 & 10 \end{pmatrix}$$



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Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a real vector in $\mathbb{R}^n$. Recall that $x^T = (x_1 \quad \cdots \quad x_n)$

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Then

$$x^T L_G x = \sum_{(i,j)\in E} w(i,j)\left(x_i - x_j\right)^2$$

Streaming Graph Algorithms

Spectral Sparsification

Graph $G = (V, E)$

A **weighted subgraph** $H = (V, E_H, w)$ of G is a $(1 + \varepsilon)$ spectral sparsifier of G if

$$(1 - \varepsilon) \cdot x^T L_G x \ \leq x^T L_H x \leq \ (1 + \varepsilon) \cdot x^T L_G x$$

for all real vectors $x \in \mathbb{R}^n$

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A spectral sparsifier of G can approximate:

- Size of all cuts
- Eigenvalues
- Effective resistances (in the corresponding electrical network)
- Properties of random walks

Streaming Graph Algorithms

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Theorem [Spielman and Teng] A $(1+\varepsilon)$ spectral sparsifier with $O(n \log n / \varepsilon^2)$ edges can be constructed in $O(m \text{ polylog}(n)/\varepsilon^2)$, where n is the number of vertices and m is the number of edges of the input graph.

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Theorem [Batson, Spielman and Srivastava] A graph with n vertices has a $(1+\varepsilon)$ spectral sparsifier with $O(n/\varepsilon^2)$ edges.

Streaming Graph Algorithms

Spectral Sparsification – Construction in the semi-streaming model

- Use as a black box any existing algorithm ALG that returns a $(1+\gamma)$ spectral sparsifier.
- ALG returns a spectral sparsifier with $size(\gamma) = \mathrm{O}(n/\gamma^2)$ number of edges.

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- ALG returns a spectral sparsifier with $size(\gamma) = \mathrm{O}(n/\gamma^2)$ number of edges.

We use the following properties of spectral sparsification

- **Mergeable:** Suppose H_1 and H_2 are β spectral sparsifiers of two graphs G_1 and G_2 on the same set of vertices. Then $H_1 \cup H_2$ is a β spectral sparsifier of $G_1 \cup G_2$.
- **Composable:** If H_3 is a β spectral sparsifier for H_2 and H_2 is a δ spectral sparsifier for H_1 then H_3 is a $\beta\delta$ spectral sparsifier for H_1.

Streaming Graph Algorithms

Spectral Sparsification – Construction in the semi-streaming model

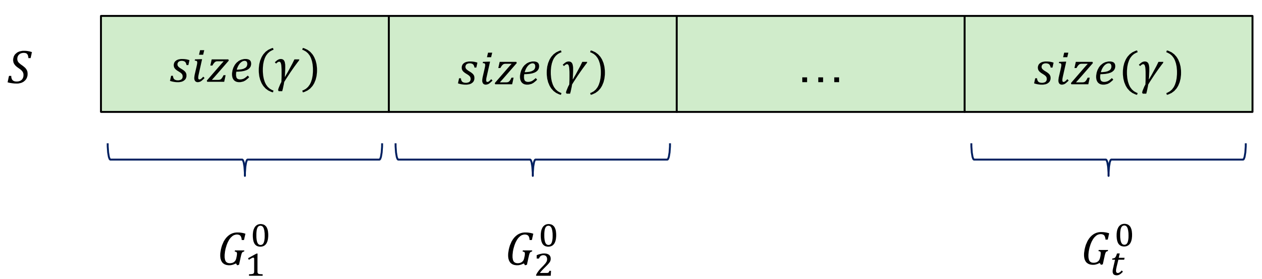
Let $G = (V, E)$ be the input graph with $n = |V|$ and $m = |E|$

Data stream S = the m edges of G

Set $t = m/size(\gamma)$. For simplicity assume that t is a power of 2

We divide S into t segments of $size(\gamma)$ edges

G_i^0 = graph that consists of the edges in the i-th segment



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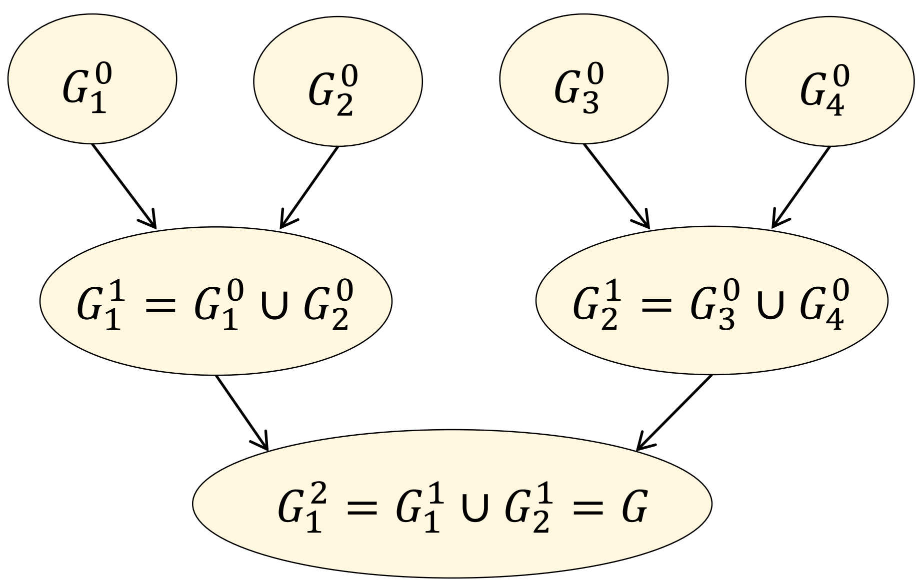
For $i = 1,2,\dots,\lg t$ and $j = 1,2,\dots,t/2^i$ define $G_i^j = G_{2i-1}^{j-1} \cup G_{2i}^{j-1}$

E.g., for $t = 4$



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For each G_i^j define a weighted subgraph H_i^j :

- $H_i^0 = G_i^0$
- $H_i^j = \mathrm{ALG}(H_{2i-1}^{j-1} \cup H_{2i}^{j-1}),\ \ j > 0$

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Spectral Sparsification – Construction in the semi-streaming model

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Set $\gamma = \varepsilon/(2\lg t) \Rightarrow (1+\gamma)^{\lg t} \sim (1+\varepsilon)$

Then $H_1^{\lg t}$ is a $(1+\varepsilon)$ sparsifier of G

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<u>Space required</u>

$$\begin{tikzcd}
H_{2i-1}^{j-1} \arrow[dr] & & H_{2i}^{j-1} \arrow[dl] \\
& H_i^j = \mathrm{ALG}\left(H_{2i-1}^{j-1} \cup H_{2i}^{j-1}\right) &
\end{tikzcd}$$

Delete H_{2i-1}^{j-1} and H_{2i}^{j-1}
as soon as H_i^j is computed

$\Rightarrow$

For each j we need to store H_i^j
only for two values of i

Streaming Graph Algorithms

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$\Rightarrow$
For each j we need to store H_i^j
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So at any given time we need to store $\leq 2 \cdot size(\gamma) \cdot \lg t = O(n \lg^3 n/\varepsilon^2)$

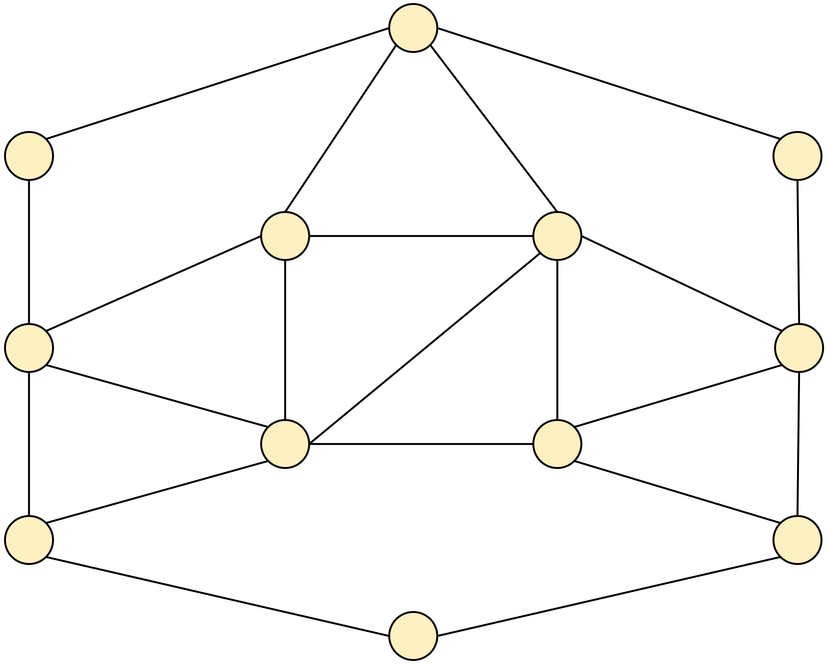
Streaming Graph Algorithms

Matchings

Graph $G = (V, E)$

Matching: Subset of edges $M \subseteq E$ such that each vertex is adjacent to at most one edge in M

Goal: Find a maximum cardinality matching M^*



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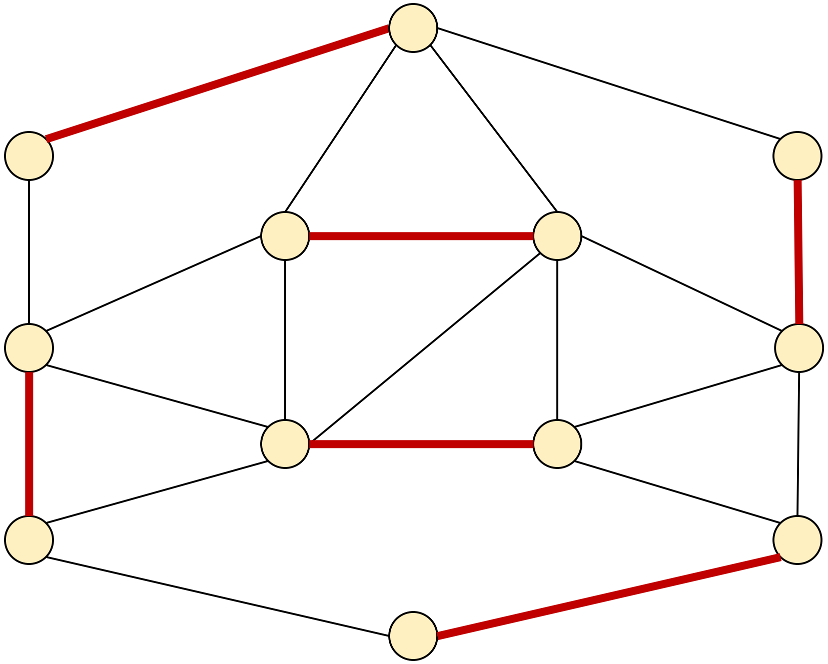

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Greedy Matching Algorithm

1. $M \leftarrow \emptyset$
2. **for** each edge $e \in S$ **do**
3. **if** $M \cup \{e\}$ is a matching **then** add e to M
4. **return** M

Streaming Graph Algorithms

Matchings

Graph $G = (V, E)$

Matching: Subset of edges $M \subseteq E$ such that each vertex is adjacent to at most one edge in M

Goal: Find a maximum cardinality matching M^*

The Greedy Matching Algorithm computes a matching M with cardinality $|M| \geq |M^*|/2$

Streaming Graph Algorithms

Matchings

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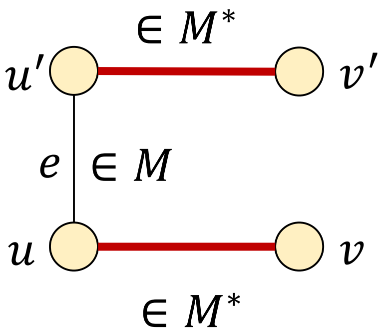
Goal: Find a maximum cardinality matching M^*

The Greedy Matching Algorithm computes a matching M with cardinality $|M| \geq |M^*|/2$

Consider an edge $(u, v) \in M^*$

If $(u, v) \notin M$ then M must contain at least one edge e adjacent to u or to v

e is adjacent to at most 2 edges of M^*



Streaming Graph Algorithms

Weighted Matchings

Weighted graph $G = (V, E, w)$. Edge weights $w : E \to \mathbb{R}^+$ ($w(e) > 0, \forall e \in E$)

Goal: Find a maximum weight matching M^*

As before, we process the edges of the stream S as they arrive and try to augment the current matching M

Streaming Graph Algorithms

Weighted Matchings

Weighted graph $G = (V, E, w)$. Edge weights $w : E \rightarrow \mathbb{R}^+$ ($w(e) > 0$, $\forall e \in E$)

Goal: Find a maximum weight matching M^*

As before, we process the edges of the stream S as they arrive and try to augment the current matching M

Let e be the next edge read from S. Let C be the edges of M that are in conflict with e : and edge in C and e are adjacent to a common vertex.

Streaming Graph Algorithms

Weighted Matchings

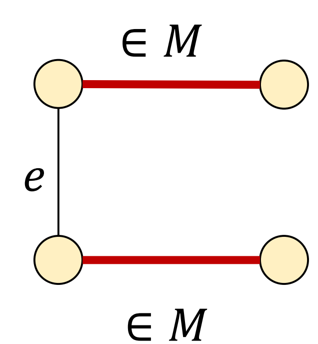
Weighted graph $G = (V, E, w)$. Edge weights $w : E \rightarrow \mathbb{R}^+$ ($w(e) > 0, \forall e \in E$)

Goal: Find a maximum weight matching M^*

As before, we process the edges of the stream S as they arrive and try to augment the current matching M

Let e be the next edge read from S. Let C be the edges of M that are in conflict with e : and edge in C and e are adjacent to a common vertex.

C has at most two edges. Let $w(C)$ be the total weight of the edges in C.

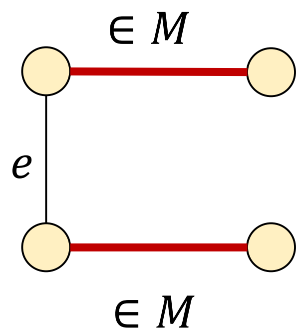


If $w(e) > w(C)$ then we increase the weight of M by including e and deleting the edges of C.

Streaming Graph Algorithms

Weighted Matchings

Let e be the next edge read from S. Let C be the edges of M that are in conflict with e : and edge in C and e are adjacent to a common vertex.



$w(C)$ = total weight of the edges in C.

If $w(e) > w(C)$ then we increase the weight of M by including e and deleting the edges of C.

Greedy Weighted Matching Algorithm

$M \leftarrow \emptyset$

for each edge $e \in S$ **do**

  let C be the set of edges that are in conflict with e

  if $w(e) > w(C)$ **then** add e to M and delete C from M

return M

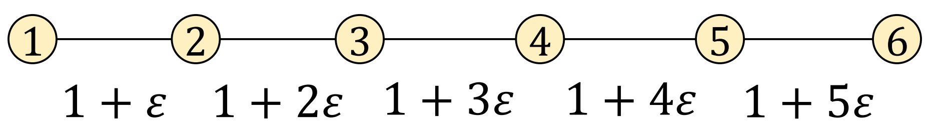
Streaming Graph Algorithms

Weighted Matchings

Consider the following scenario

$S = \ (1,2),(2,3),(3,4),\dots,(n,n-1)$

Edge $e_i = (i, i+1)$ has weight $w(e_i) = 1 + i\varepsilon$, for a small $\varepsilon > 0$



$M = \{\}$

Streaming Graph Algorithms

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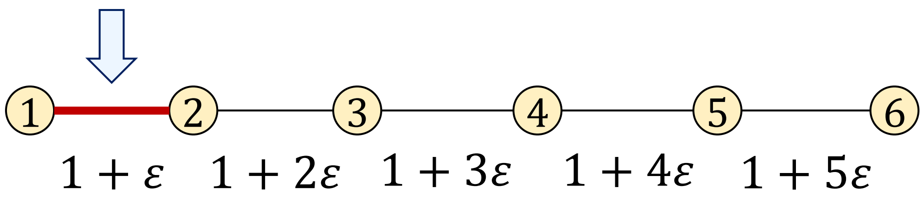

$M = \{(1,2)\}$

Streaming Graph Algorithms

Weighted Matchings

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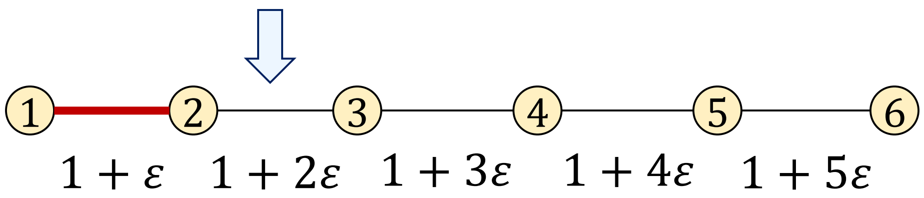

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Streaming Graph Algorithms

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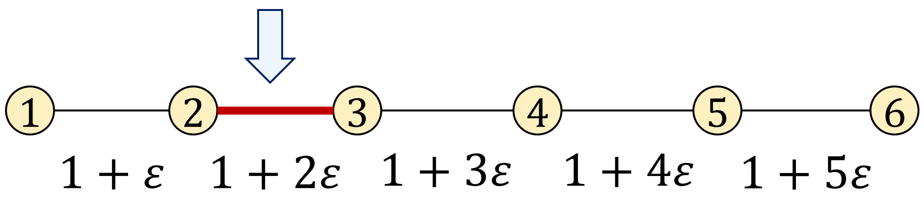

$M = \{(2,3)\}$

Streaming Graph Algorithms

Weighted Matchings

Consider the following scenario

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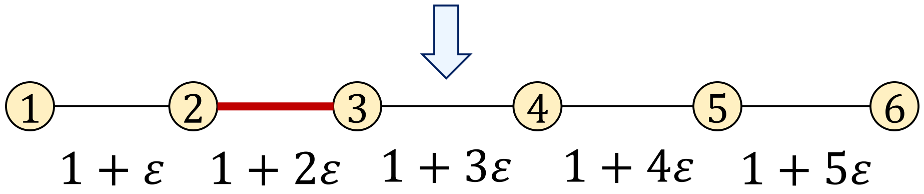

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Streaming Graph Algorithms

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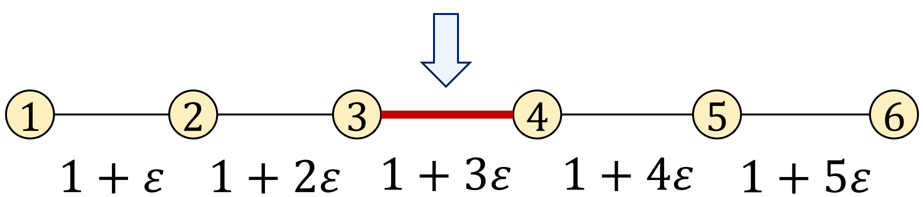

$M = \{(3,4)\}$

Streaming Graph Algorithms

Weighted Matchings

Consider the following scenario

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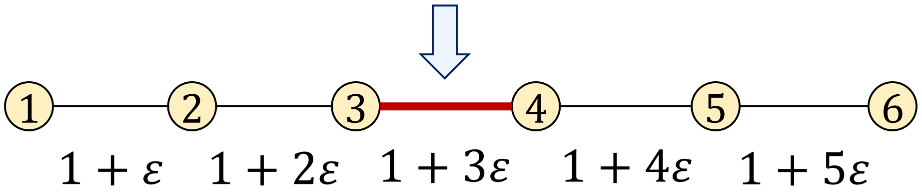

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Streaming Graph Algorithms

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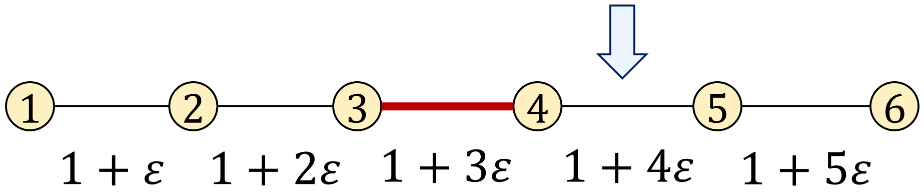

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Streaming Graph Algorithms

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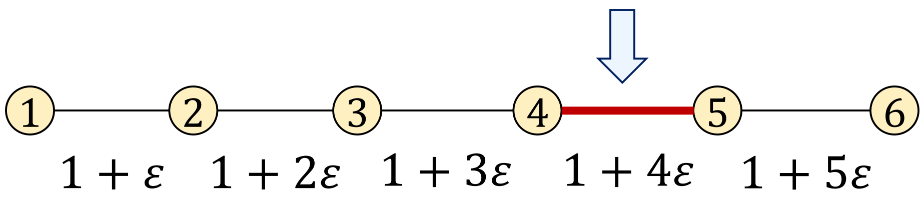

$M = \{(4,5)\}$

Streaming Graph Algorithms

Weighted Matchings

Consider the following scenario

$S = \ (1,2), (2,3), (3,4), \ldots, (n, n-1)$

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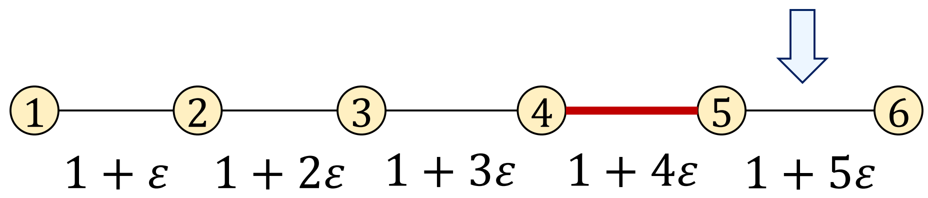

$M = \{(4,5)\}$

Streaming Graph Algorithms

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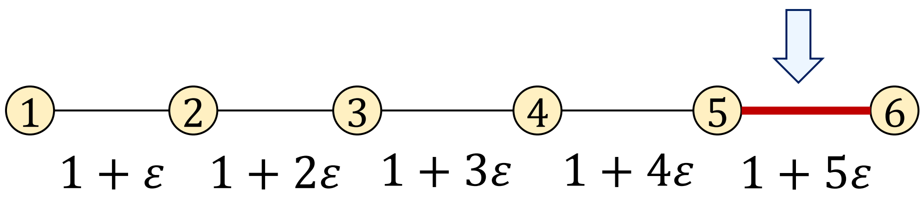


$M = \{(5,6)\} \qquad w(M) = 1 + 5\varepsilon$

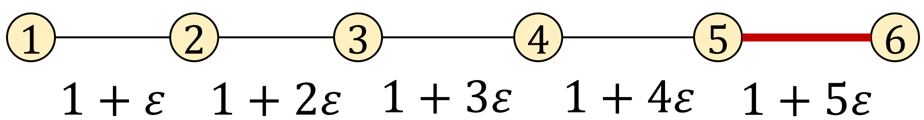
Streaming Graph Algorithms

Weighted Matchings

Consider the following scenario

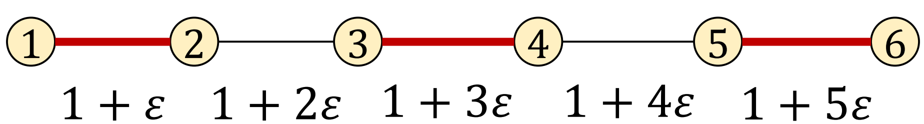
$S =\ (1,2),(2,3),(3,4),\dots,(n,n-1)$

Edge $e_i = (i, i+1)$ has weight $w(e_i) = 1 + i\varepsilon$, for a small $\varepsilon > 0$



$M = \{(5,6)\}$

$w(M) = 1 + 5\varepsilon$



$M^* = \{(1,2),(3,4),(5,6)\}$

$w(M^*) = 3 + 9\varepsilon$

Streaming Graph Algorithms

Weighted Matchings

Consider the following scenario

$S = \ (1,2), (2,3), (3,4), \dots, (n, n-1)$

Edge $e_i = (i, i+1)$ has weight $w(e_i) = 1 + i\varepsilon$, for a small $\varepsilon > 0$

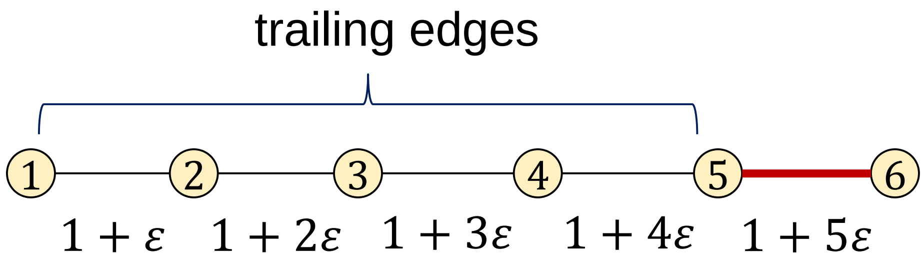
The computed matching M has weight $w(M) = 1 + (n-1)\varepsilon$

The optimal matching M has weight $w(M^*) = \sum_i (1 + (2i-1)\varepsilon) > (n-1)/2$

Hence, the approximation ratio is $\dfrac{w(M^*)}{w(M)} > \dfrac{(n-1)/2}{1 + (n-1)\varepsilon} \sim \dfrac{n}{2}$

Streaming Graph Algorithms

Weighted Matchings



$M = \{(5,6)\}$

$w(M) = 1 + 5\varepsilon$

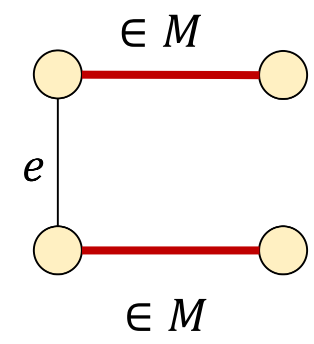
The problem is that the trailing edges of S that were once inserted into M but removed later may have much larger total weight than the edges added later.

Streaming Graph Algorithms

Weighted Matchings

Modified algorithm

We include e in M If $w(e) > \beta w(C)$ for some constant $\beta = (1+\gamma) > 1$.

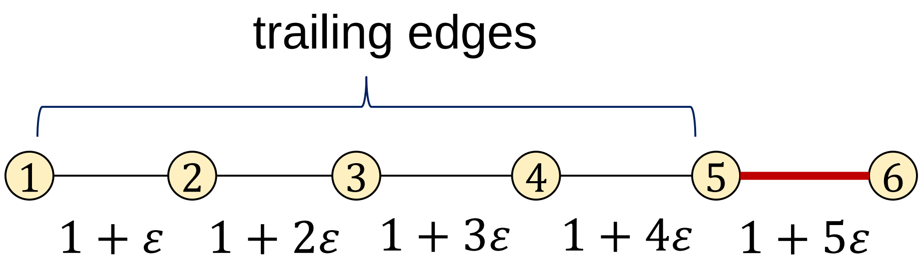


Greedy Weighted Matching Algorithm

1. $M \leftarrow \emptyset$
2. **for** each edge $e \in S$ **do**
3. let C be the set of edges that are in conflict with e
4. **if** $w(e) > (1+\gamma) \cdot w(C)$ **then** add e to M and delete C from M
5. **return** M

Streaming Graph Algorithms

Weighted Matchings



$M = \{(5,6)\}$

$w(M) = 1 + 5\varepsilon$

The problem is that the trailing edges of S that were once inserted into M but removed later may have much larger total weight than the edges added later.

We include e in M If $w(e) > \beta w(C)$ for some constant $\beta = (1 + \gamma) > 1$.

For an edge e define

- $C_0 = \{e\}$
- $C_i =$ edges removed when an edge in C_{i-1} was added to M
- $T_e = C_1 \cup C_2 \cup \cdots$

Then $w(T_e) \le w(e)/\gamma$

Streaming Graph Algorithms

Weighted Matchings

It can be shown that

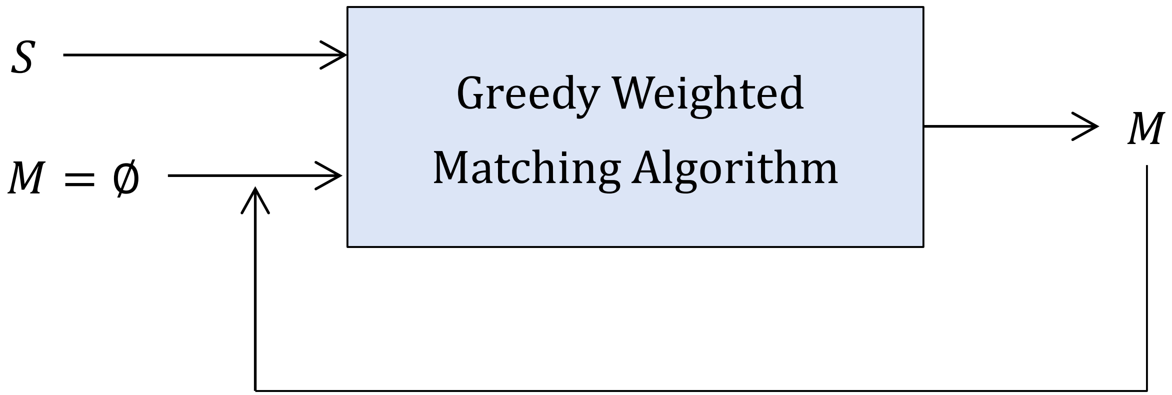
$$w(M^*) \leq (1+\gamma) \cdot \sum_{e \in M} (w(T_e) + 2w(e))$$

By applying a careful charging scheme we get $\frac{w(M^*)}{w(M)} < 5.828$

Streaming Graph Algorithms

Weighted Matchings

Multi-pass Algorithm



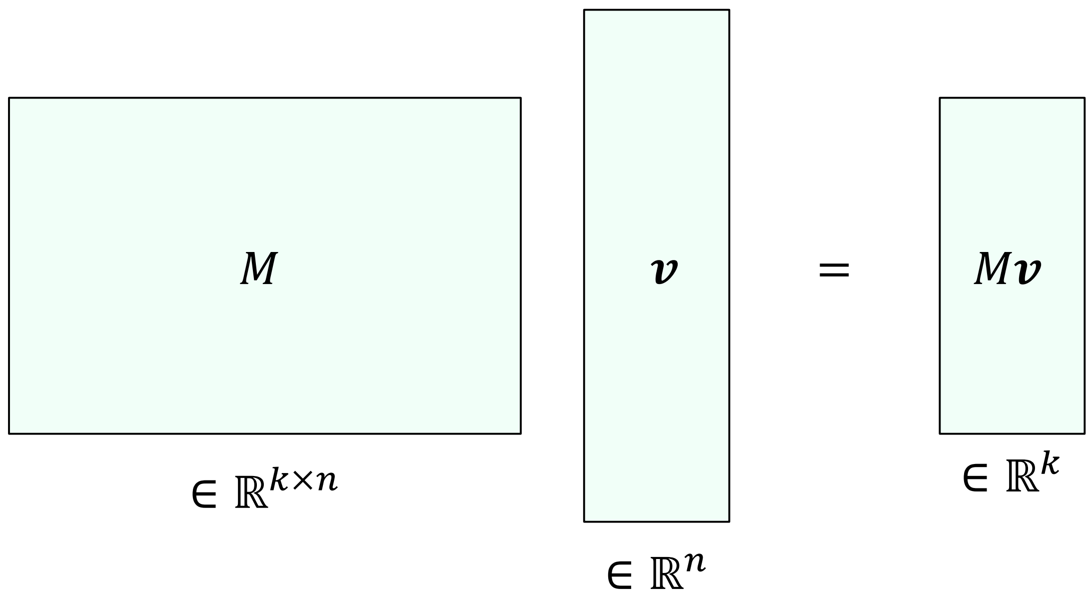
We can get a $(2+\varepsilon)$-approximation with $O(\varepsilon^{-3})$ passes over S, where $\gamma = O(\varepsilon)$

Streaming Graph Algorithms

Graph Sketches

Random linear projection $M : \mathbb{R}^n \to \mathbb{R}^k$, where $k \ll n$

For any vector $\boldsymbol{v} \in \mathbb{R}^n$, the projection $M\boldsymbol{v} \in \mathbb{R}^k$ preserves properties of $\boldsymbol{v}$ with high probability



Many applications: estimating entropy, heavy hitters, estimating norms, fitting polynomials,…

Rich theory: dimensionality reduction, sparse recovery, metric embeddings,…

Streaming Graph Algorithms

Graph Sketches

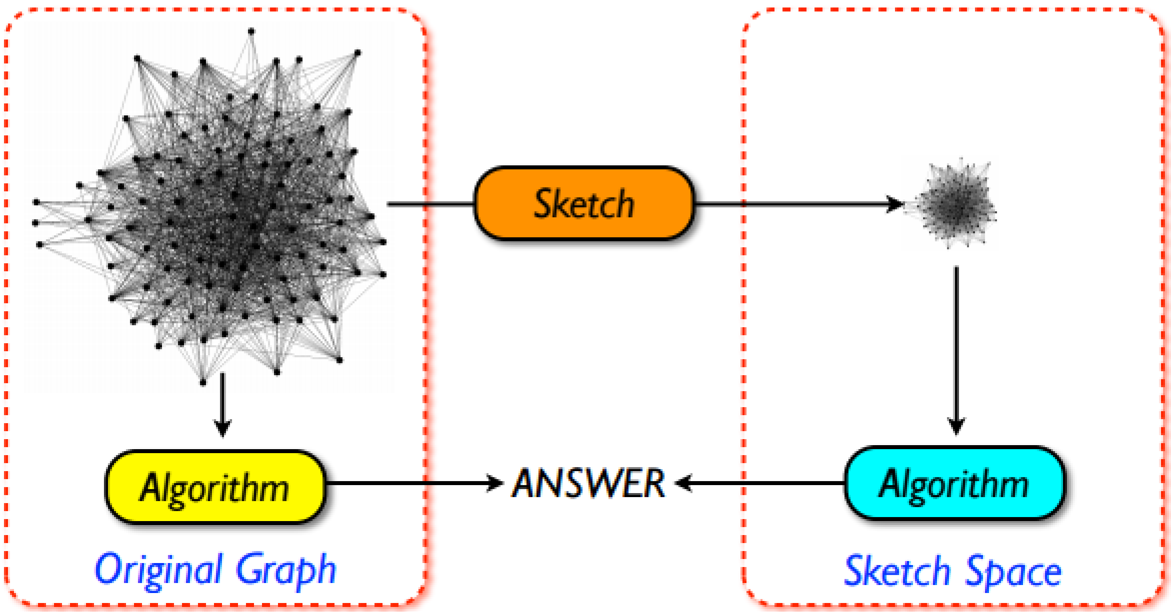
Can we use this approach for graphs?

That is, can we project the adjacency matrix A_G of a graph G to a smaller matrix MA_G, so that we can use MA_G to compute properties of G?

- For a graph G with n vertices, A_G has $\mathrm{O}(n^2)$ dimensions.
- To work in the semi-streaming model we want MA_G to have $\mathrm{O}(n \text{ polylog}(n))$ dimensions.

Streaming Graph Algorithms

Graph Sketches



Picture from https://people.cs.umass.edu/~mcgregor/711S12/lec-2-2.pdf

Streaming Graph Algorithms

Graph Sketches

Dynamic graph stream $S = \langle a_1, a_2, \ldots \rangle$ where $a_i = (e_i, \Delta_\iota)$

e_i = an edge of the graph

$$\Delta_i = \begin{cases} +1, e_i \text{ is inserted} \\ -1, e_i \text{ is deleted} \end{cases}$$

Multiplicity of edge e : $\quad f_e = \sum_{i:\, e_i = e} \Delta_i$

For simplicity we will assume that $f_e \in \{0,1\}$, for all edges e.

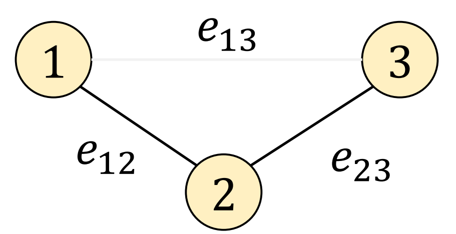
Streaming Graph Algorithms

Graph Sketches

Vector of edge multiplicities $\boldsymbol{f} \in \{0,1\}^{\binom{n}{2}}$

Each entry of $\boldsymbol{f}$ is a multiplicity f_e of a (potential) edge e of G (a simple graph with n vertices has up to $\binom{n}{2}$ edges).

$$\boldsymbol{f} = \begin{pmatrix} f_{e_{12}} \\ f_{e_{13}} \\ f_{e_{23}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$



Streaming Graph Algorithms

Graph Sketches

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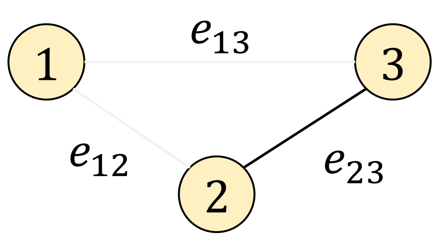
Each entry of $\boldsymbol{f}$ is a multiplicity f_e of a (potential) edge e of G (a simple graph with n vertices has up to $\binom{n}{2}$ edges).

Index vector of edge e : $\boldsymbol{i}^e \in \{0,1\}^{\binom{n}{2}}$. The only nonzero entry of $\boldsymbol{i}^e$ is the one that corresponds to edge e.

$$\boldsymbol{i}^{\boldsymbol{e_{23}}} = \begin{pmatrix} i_{e_{12}} \\ i_{e_{13}} \\ i_{e_{23}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Streaming Graph Algorithms

Graph Sketches

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Index vector of edge e : $\boldsymbol{i}^e \in \{0,1\}^{\binom{n}{2}}$. The only nonzero entry of $\boldsymbol{i}^e$ is the one that corresponds to edge e.

Sketch of $\boldsymbol{f}$: $A(\boldsymbol{f}) \in \mathbb{R}^d$, d =dimensionality of the sketch

When we read the next item (e, Δ) from the stream, we can update the sketch as follows:

$$A(\boldsymbol{f}) = A(\boldsymbol{f}) + \Delta \cdot A(\boldsymbol{i}^e)$$

Streaming Graph Algorithms

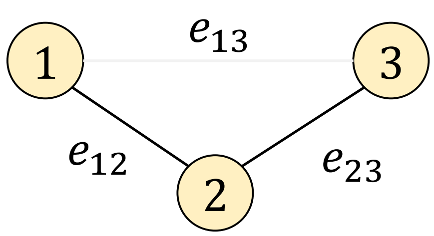
Homomorphic Sketches

Vector of edge multiplicities $\boldsymbol{f} \in \{0,1\}^{\binom{n}{2}}$

Each entry of $\boldsymbol{f}$ is a multiplicity f_e of a (potential) edge e of G (a simple graph with n vertices has up to $\binom{n}{2}$ edges).

For a vertex v let $\boldsymbol{f}^v \in \{0,1\}^{n-1}$ be the restriction of $\boldsymbol{f}$ to the coordinates that involve v (i.e., the $n-1$ edges that can be adjacent to v in G)

$$\boldsymbol{f} = \begin{pmatrix} f_{e_{12}} \\ f_{e_{13}} \\ f_{e_{23}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$



$$\boldsymbol{f}^{\mathbf{1}} = \begin{pmatrix} f_{e_{12}} \\ f_{e_{13}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Streaming Graph Algorithms

Homomorphic Sketches

Vector of edge multiplicities $\boldsymbol{f} \in \{0,1\}^{\binom{n}{2}}$

Each entry of $\boldsymbol{f}$ is a multiplicity f_e of a (potential) edge e of G (a simple graph with n vertices has up to $\binom{n}{2}$ edges).

For a vertex v let $\boldsymbol{f}^v \in \{0,1\}^{n-1}$ be the restriction of $\boldsymbol{f}$ to the coordinates that involve v (i.e., the $n-1$ edges that can be adjacent to v in G)

The sketches of $\boldsymbol{f}$ are formed by concatenation ($\circ$) of the sketches of each $\boldsymbol{f}^v$

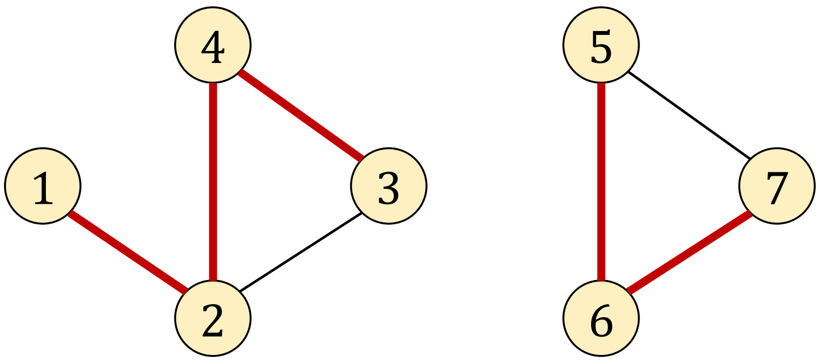
$$A(\boldsymbol{f}) = A_1(\boldsymbol{f}^{v_1}) \circ A_2(\boldsymbol{f}^{v_2}) \circ \cdots \circ A_n(\boldsymbol{f}^{v_n})$$

Homomorphic sketches: For each operation on G there is a corresponding operation on the sketches

Streaming Graph Algorithms

Connectivity via Sketches

We wish to maintain a spanning forest of a graph $G = (V, E)$



Connectivity via Sketches

We wish to maintain a spanning forest of a graph $G = (V, E)$

Let’s begin with a simple (non-sketch) algorithm

Connectivity Algorithm

```
repeat
    for each vertex v of the current graph do
        select an edge incident to v
    contract all selected edges
until the current graph has no edges
```

Streaming Graph Algorithms

Connectivity via Sketches

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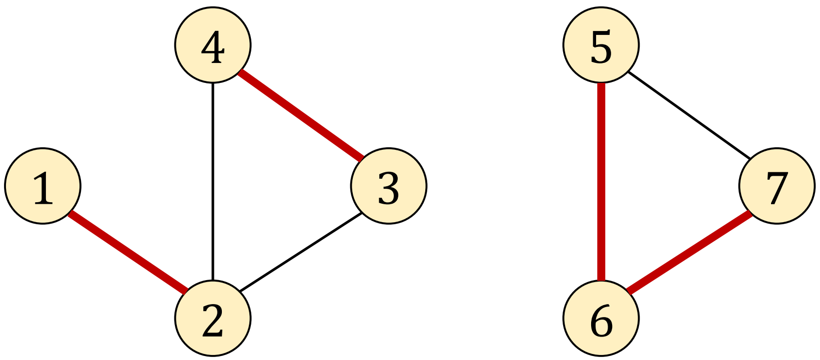

Streaming Graph Algorithms

Connectivity via Sketches

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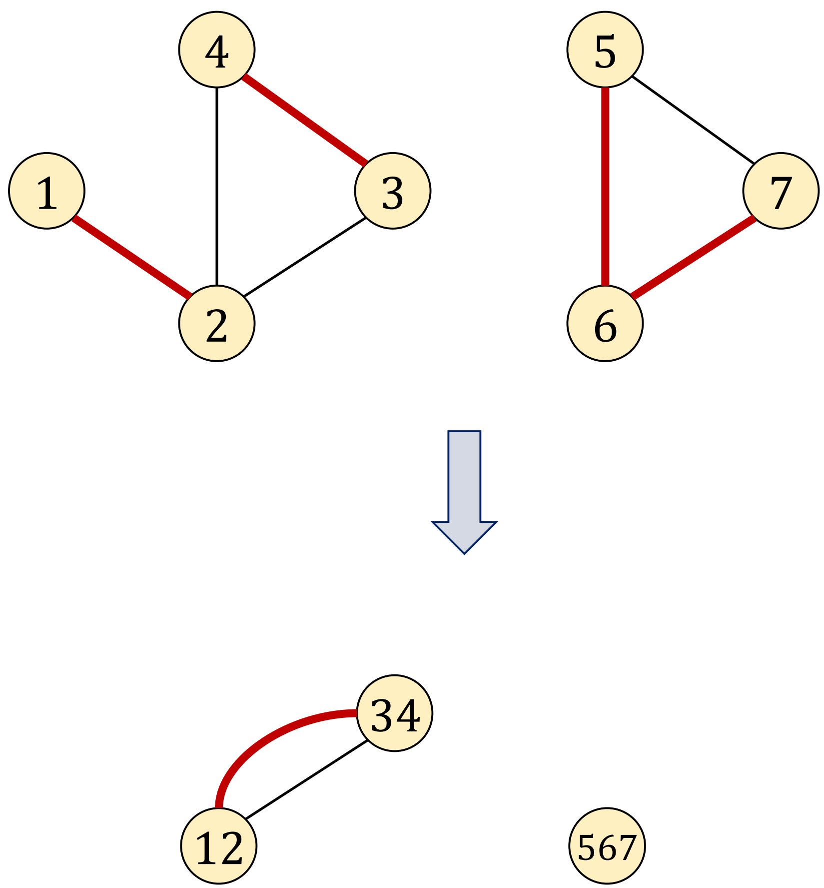

Streaming Graph Algorithms

Connectivity via Sketches

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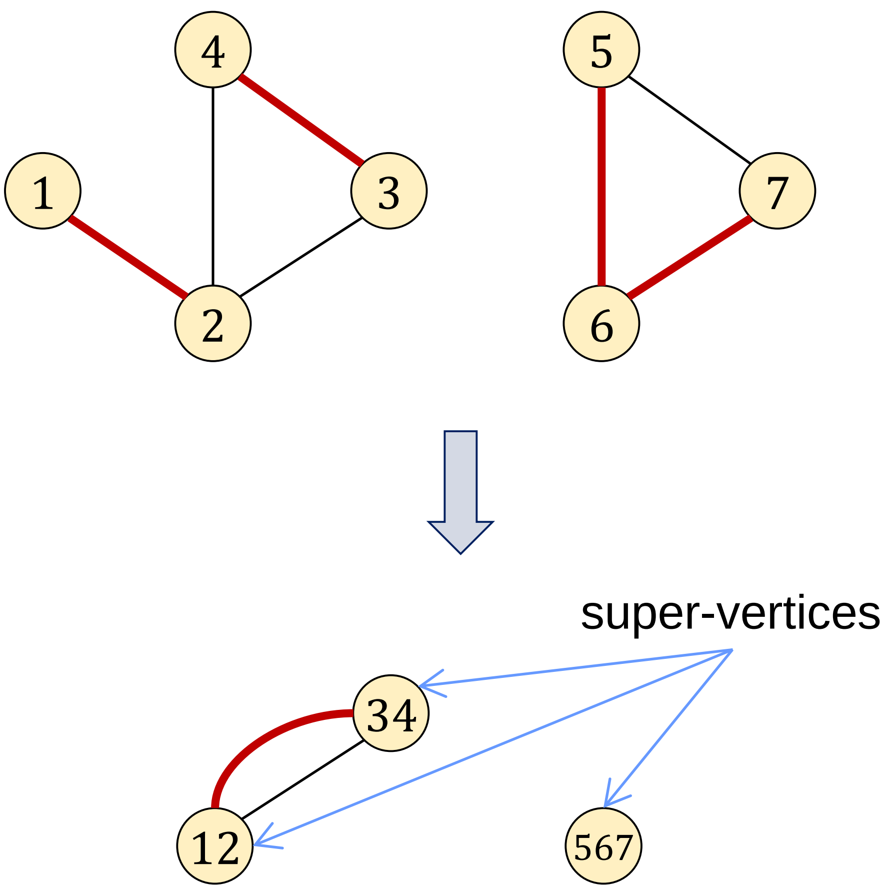

Streaming Graph Algorithms

Connectivity via Sketches

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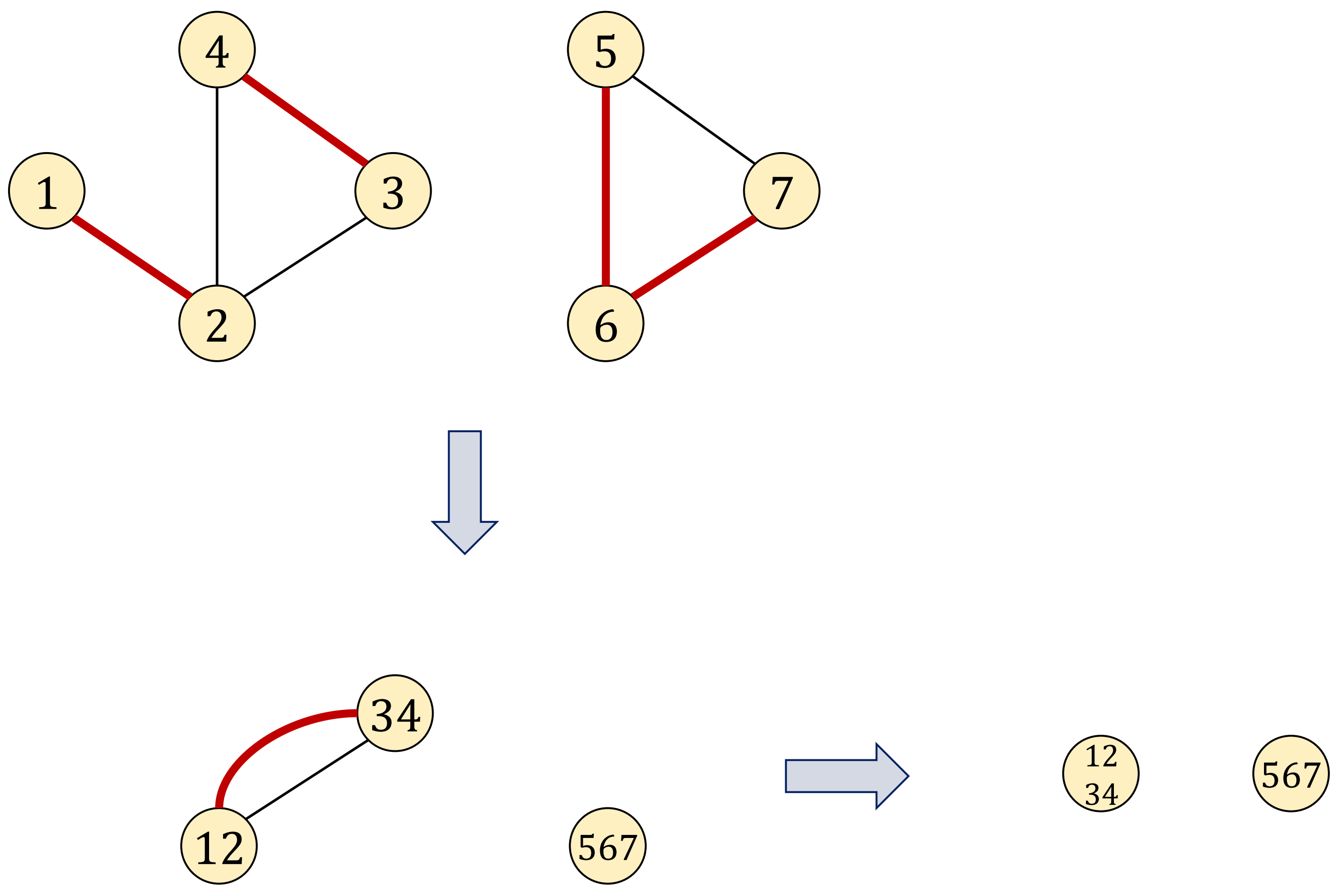

Streaming Graph Algorithms

Connectivity via Sketches

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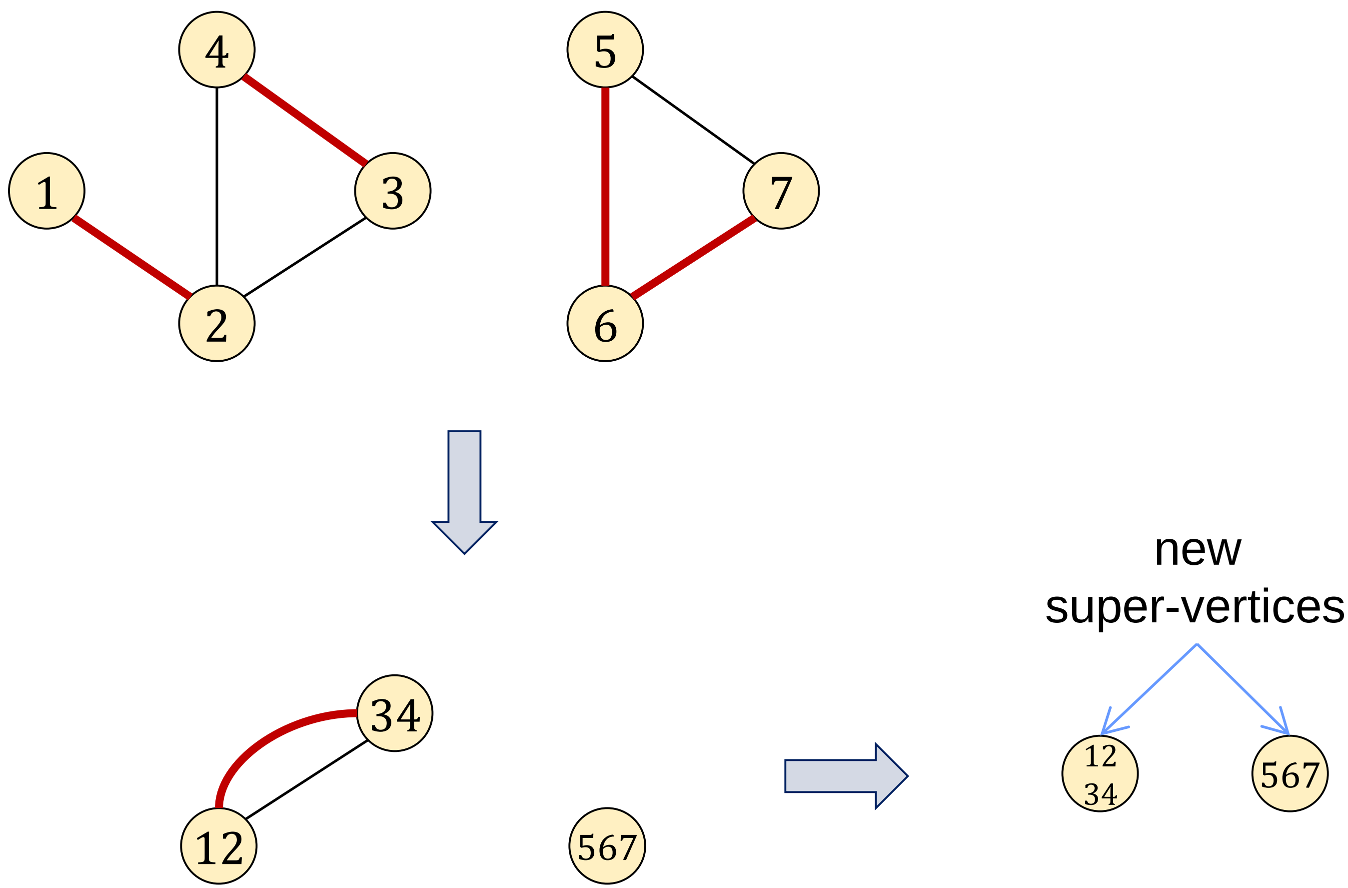

Streaming Graph Algorithms

Connectivity via Sketches

We wish to maintain a spanning forest of a graph $G = (V, E)$

Let's begin with a simple (non-sketch) algorithm

Connectivity Algorithm

1. **repeat**
2.   **for** each vertex v of the current graph **do**
3.   select an edge incident to v
4.   contract all selected edges
5. **until** the current graph has no edges

Finds the connected components of G, and a spanning forest, in $O(\log n)$ rounds

Streaming Graph Algorithms

Connectivity via Sketches

To design an algorithm that uses sketches we have to:

1. Define an appropriate graph representation
2. Apply ℓ_0-sampling via linear sketches

Streaming Graph Algorithms

Connectivity via Sketches

To design an algorithm that uses sketches we have to:

1. Define an appropriate graph representation
2. **Apply ℓ_0-sampling via linear sketches**

ℓ_0-sampling

Let $K = \mathrm{polylog}(N)$. There is a distribution over matrices $M \in \mathbb{R}^{K \times N}$ such that for any $\boldsymbol{x} \in \mathbb{R}^N$, a random non-zero element of $\boldsymbol{x}$ can be reconstructed from $M\boldsymbol{x}$ with high probability.

Streaming Graph Algorithms

Connectivity via Sketches

To design an algorithm that uses sketches we have to:

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$$\boldsymbol{a}^i_{(j,k)} = \begin{cases} +1, & \text{if } i = j < k \text{ and } (v_j, v_k) \in E \\ -1, & \text{if } j < k = i \text{ and } (v_j, v_k) \in E \\ 0, & \text{otherwise} \end{cases}$$

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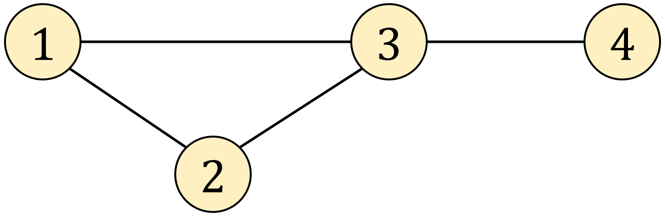

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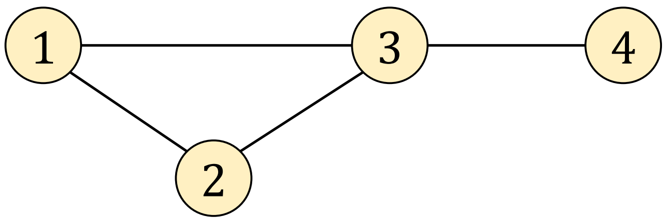

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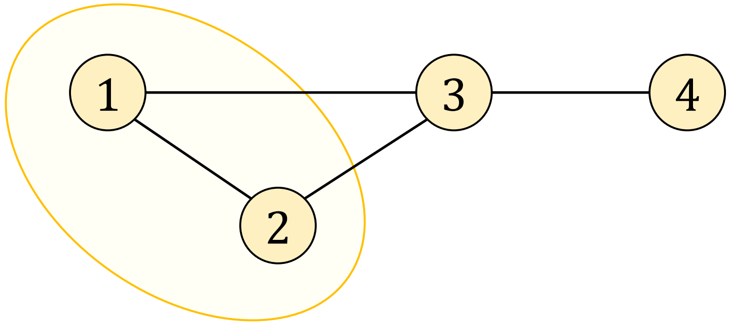

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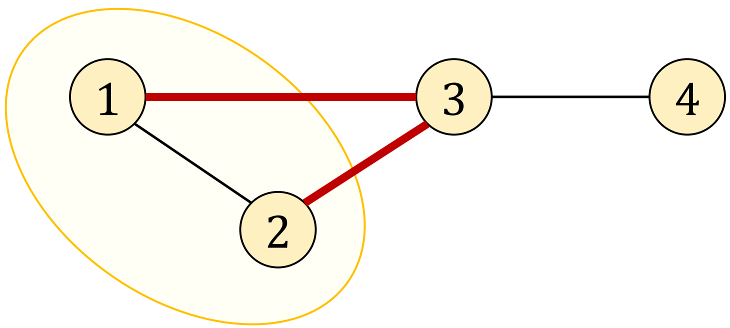

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Thus $\sum_{v_i \in U} M\boldsymbol{a}^{\boldsymbol{i}} = M\left(\sum_{v_i \in U} \boldsymbol{a}^{\boldsymbol{i}}\right)$ gives a random edge in $\delta_G(U)$

Streaming Graph Algorithms

Connectivity via Sketches

Connectivity via Sketches Algorithm I: Compute the Sketches in a Single Pass

1. Choose $t = O(\log n)$
2. **for** $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,t$ **do**
3. Construct the random projection $M_j \boldsymbol{a}^i$
4. **for** $i = 1,2,\ldots,n$ **do**
5. Compute $A_i(\boldsymbol{f}^{v_i}) = (M_1 \boldsymbol{a}^i) \circ (M_2 \boldsymbol{a}^i) \circ \cdots \circ (M_t \boldsymbol{a}^i)$

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- Each sketch A_i has dimension $O(\text{polylog} n)$
- Since there are n sketches, the required space is $O(n \text{ polylog} n)$

Streaming Graph Algorithms

Connectivity via Sketches

Connectivity via Sketches Algorithm II: Emulate Connectivity Algorithm

1. Let $\hat{V} = V$ be the initial set of super-vertices
2. **for** $i = 1,2, \ldots, t$ **do**
3. **for** each super-vertex $U \in \hat{V}$ **do**
4. use $\sum_{v_i \in U} M\boldsymbol{a}^{\boldsymbol{i}}$ to sample an edge between U and another super-vertex W
5. collapse U and W to form a new super-vertex

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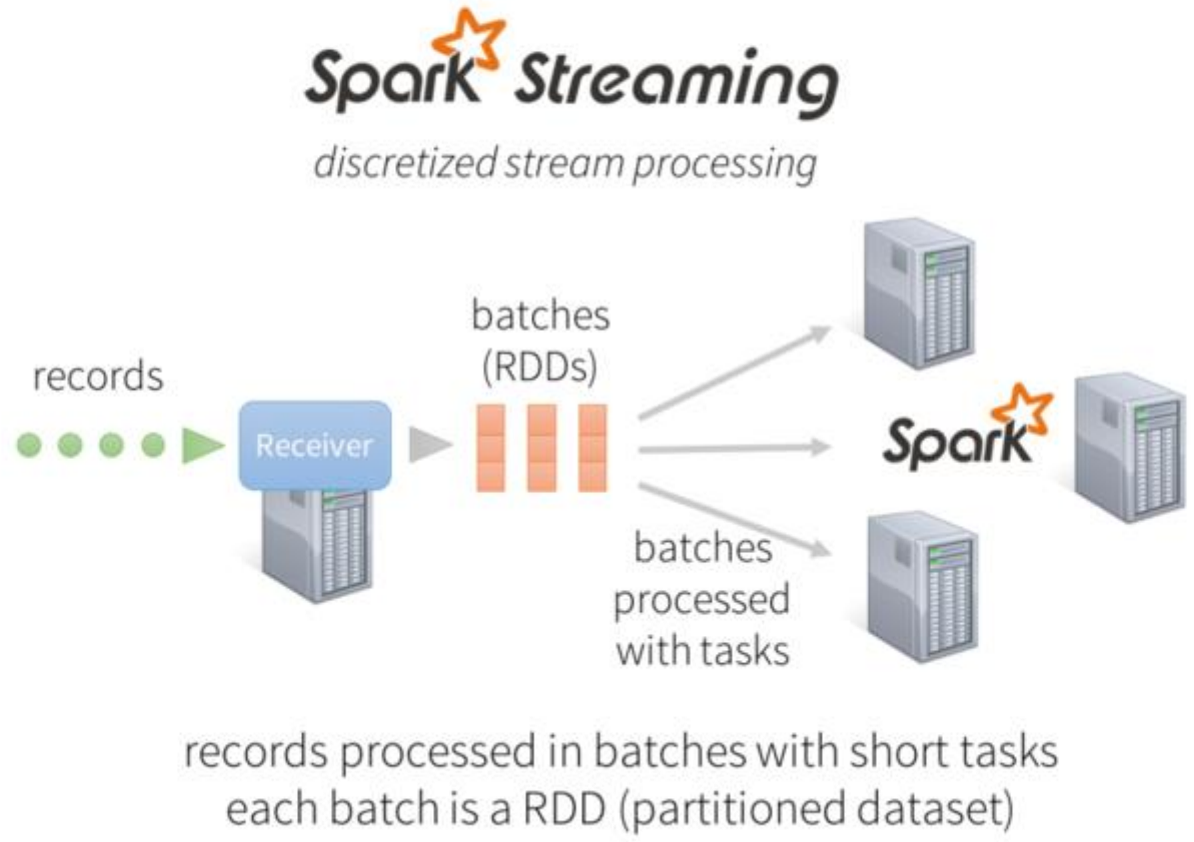
The update time (to process the next edge in S) is $O(\mathrm{polylog}\, n)$

Streaming Graph Algorithms

Concluding remarks

- Many graph algorithms in the data stream model are known for basic problems. E.g., estimating connectivity, approximating distances, finding approximate matchings, counting subgraphs,…
- But limited work on directed graphs!
- Space constraints: semi-stream model not suited for sparse graphs ($m = O(n \operatorname{polylog} n)$)

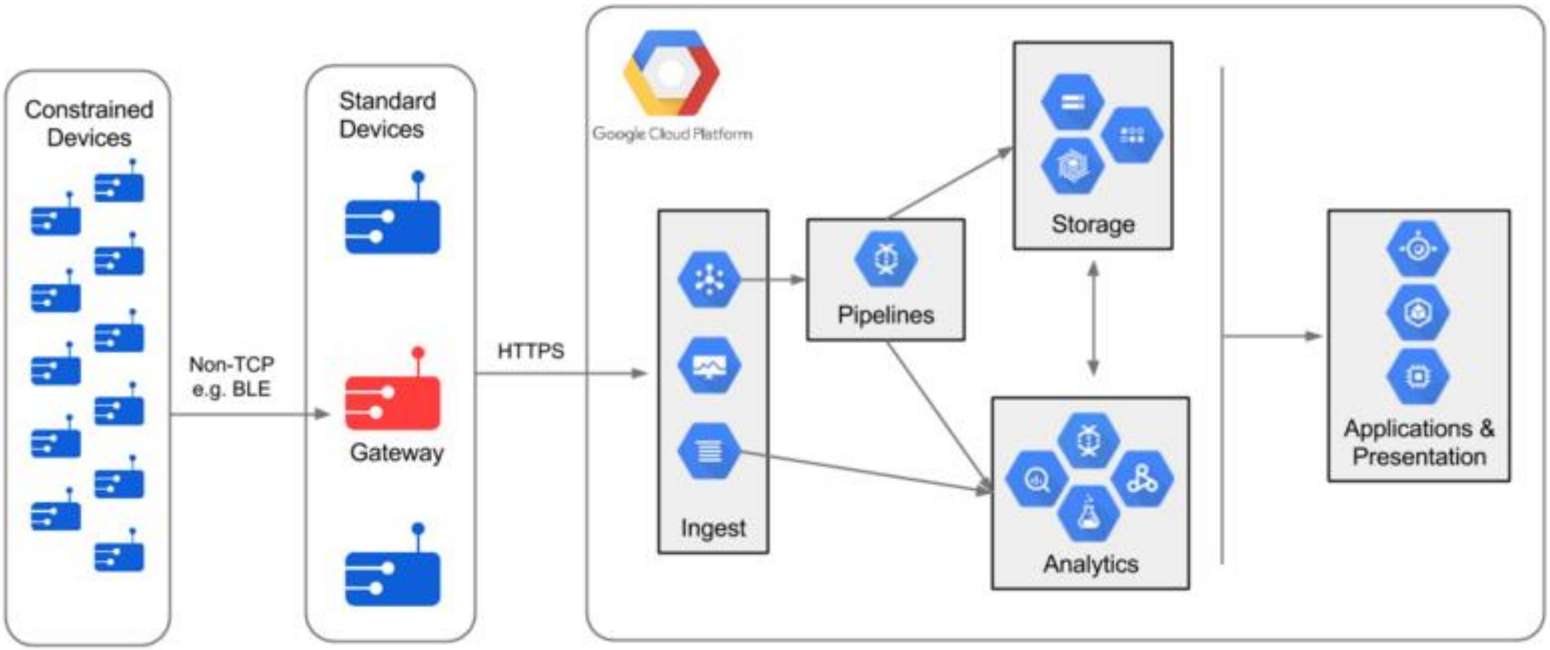
Streaming Architectures



Picture from https://databricks.com/blog/2015/07/30/diving-into-spark-streamings-execution-model.html

Streaming Architectures

Google Cloud Platform



https://cloud.google.com/solutions/architecture/streamprocessing