LINEARLY IMPLICIT SCHEMES FOR A CLASS OF DISPERSIVE–DISSIPATIVE SYSTEMS[†]

GEORGIOS AKRIVIS AND YIORGOS–SOKRATIS SMYRLIS

Abstract. We consider initial value problems for semilinear parabolic equations, which possess a dispersive term, nonlocal in general. This dispersive term is not necessarily dominated by the dissipative term. In our numerical schemes, the time discretization is done by linearly implicit schemes. More specifically, we discretize the initial value problem by the implicit–explicit Euler scheme and by the two–step implicit–explicit BDF scheme. In this work, we extend the results in [2, 3], where the dispersive term (if present) was dominated by the dissipative one and was not integrated implicitly. We also derive optimal order error estimates. We provide various physically relevant applications of dispersive–dissipative equations and systems fitting in our abstract framework.

1. Introduction

1.1. **Dispersive–dissipative systems.** We consider the time discretization of initial value problems of the form

$$(1.1)\qquad \begin{cases} u'(t)+\mathcal{L}u(t) \;=\; \mathcal{B}\big(t,u(t)\big), \quad 0<t<T,\\ u(0)\;=\;u^0, \end{cases}$$

with $u:[0,T]\to H$, where $\big(H,(\cdot,\cdot)\big)$ is a complex Hilbert space, and $\mathcal{L}$, $\mathcal{B}$ are unbounded (in general) operators on H, with $\mathcal{L}$ linear and $\mathcal{B}$ nonlinear. Problem (1.1) is assumed to possess a smooth solution.

In particular, we assume that $\mathcal{L}$ is normal[1]. We also assume that

$$(1.2)\qquad \operatorname{Re}(\mathcal{L}v,v)\;\ge\;\sigma(v,v), \quad \text{for every } v\in\mathscr{D}(\mathcal{L}),$$

with a positive constant σ. Let

$$\mathcal{A}\;=\;\frac12(\mathcal{L}+\mathcal{L}^\star) \quad\text{and}\quad \mathcal{D}\;=\;\frac12(\mathcal{L}-\mathcal{L}^\star)$$

[†]This work was supported by University of Cyprus grant #8037P-3/311-21028.

2000 *Mathematics Subject Classification.* Primary 65M60, 65M12; Secondary 65L06.

Key words and phrases. dispersive–dissipative systems, semilinear parabolic equations, Kuramoto–Sivashinsky equation, linearly implicit schemes, error estimates.

[1]A densely defined linear operator $\mathcal{L}:\mathscr{D}(\mathcal{L})\to H$ is said to be normal if $\mathcal{L}\mathcal{L}^\star=\mathcal{L}^\star\mathcal{L}$, where $\mathcal{L}^\star$ is the adjoint of $\mathcal{L}$. The last equality implies that $\mathscr{D}(\mathcal{L}\mathcal{L}^\star)=\mathscr{D}(\mathcal{L}^\star\mathcal{L})$. In fact, if $\mathcal{L}$ is normal, then $\mathscr{D}(\mathcal{L})=\mathscr{D}(\mathcal{L}^\star)$, and $\mathscr{D}(\mathcal{L}\mathcal{L}^\star)$ is dense in H. See Kato [12, Chapter V, §3.8].

denote the symmetric and the anti–symmetric part of $\mathcal{L}$, respectively. Clearly, in view also of (1.2), we have

$$(\mathcal{A}v, v) \geq \sigma(v, v) \quad \text{and} \quad \operatorname{Re}(\mathcal{D}v, v) = 0, \quad \text{for all } v \in \mathscr{D}(\mathcal{L}).$$

The symmetric operator $\mathcal{A}$ extends to a self–adjoint (still denoted by $\mathcal{A}$) since it is semi–bounded (cf. Friedrichs extension; see for example Lax [15, Chapter 33.3]). In fact, $\mathcal{A}$ is positive definite.

Let $|\cdot|$ be the norm of H and $\|\cdot\|$ the norm of the space $V = \mathscr{D}(\mathcal{A}^{1/2})$, which is defined by $\|v\| := |\mathcal{A}^{1/2}v|$. We identify H with its dual, and denote by V' the dual of V, again by $(\cdot,\cdot)$ the duality pairing between V' and V, and by $\|\cdot\|_\star$ the dual norm on V', $\|v\|_\star := |\mathcal{A}^{-1/2}v|$.

Let T_u be a tube of radius one around the solution u,

$$T_u := \big\{v \in V : \min_{0\le t\le T} \|v - u(t)\| \le 1\big\}.$$

We assume that the (nonlinear in general) operator $\mathcal{B}(t,\cdot) : \mathscr{D}(\mathcal{L}) \to H$ can be extended to an operator from V into V', for every $t \in [0,T]$. We also require a local Lipschitz condition of the form

$$\|\mathcal{B}(t,v) - \mathcal{B}(t,w)\|_\star \le \lambda\|v - w\| + \mu|v - w| \quad \text{for all } v, w \in T_u, \tag{1.3}$$

to hold, uniformly in $t \in [0,T]$, with two constants λ and μ. Depending on the discretization scheme, it is essential for our analysis that λ be sufficiently small; in any case we assume that λ is less than one, $\lambda < 1$.

The tube T_u is here defined in terms of the norm of V for concreteness. In the fully discrete case, if the discretization in space is based on the finite element method, the proofs can be easily modified to yield error estimates under some mild mesh conditions; the weaker the norm in terms of which T_u is defined, the milder are the required mesh conditions; cf. [3] for details. In particular, if T_u is defined in terms of the norm of H, no mesh–condition at all is needed.

In our applications, H is a Sobolev space $H^s(\mathfrak{X})$, where $\mathfrak{X}$ is a Euclidean space or a torus and $s \in \mathbb{R}$, $\mathcal{A}$ is an elliptic operator of the spatial variable $\boldsymbol{x}$, $\mathcal{D}$ is a dispersive pseudo–differential operator of $\boldsymbol{x}$, i.e., an operator with a symbol of the form $\widehat{\mathcal{D}}(\xi) = \mathrm{i}K(\xi)$, where K is a real–valued function, and $\mathcal{B}(t,\cdot)$ is a nonlinear function of u and partial derivatives of u, with respect to $\boldsymbol{x}$, of orders at most the order of $\mathcal{A}$.

A typical example of such systems is the dispersively modified Kuramoto–Sivashinsky equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u = 0, \tag{1.4}$$

with $2\pi-$periodic initial data, where ν is a positive constant. In this case $H = H^s(T^1)$, where T^1 is the unit circle (or the $1-$dimensional torus), and s a suitable real number. The operator $\mathcal{D}$ is a linear dispersive pseudo–differential operator defined by

$$(\widehat{\mathcal{D}v})_\ell \,=\, \mathrm{i} f(\ell)\, \hat{v}_\ell,$$

where $f : \mathbb{Z} \to \mathbb{R}$. The global well–posedness of the corresponding periodic initial value problem is derived from the work of Tadmor [21]. Equation (1.4) has been derived in the context of interfacial hydrodynamics, when

$$f(\ell) \,=\, \frac{\ell^2 I_1(\ell)}{\ell I_1^2(k) - \ell I_0^2(\ell) + 2I_0(\ell)I_1(\ell)}, \tag{1.5}$$

where $I_\nu = I_\nu(\ell)$ denotes the modified Bessel function of the first kind of order ν (see [11, 18]). A special case of (1.4) is the Kawahara equation

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + \nu u_{xxxx} \,=\, 0, \tag{1.6}$$

which has been derived in the context of falling film flows (see [9, 13, 23]). Another special case of (1.4) is the Benney–Lin equation

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + \nu u_{xxxx} + \varepsilon u_{xxxxx} \,=\, 0, \tag{1.7}$$

which has been derived in the context of one–dimensional evolution of long waves of sufficiently small amplitude in various problems in fluid dynamics (see for example [6, 16]). Global well–posedness of the initial value problem for (1.7) with initial data in $H^s(\mathbb{R})$, $s \geq 0$, has been established by Biagioni & Linares [7].

Further examples will be given in Section 4.

Remark 1.1. It is assumed, for simplicity, that the operator $\mathcal{L}$ is normal. This requirement is also dictated by the applications we have in mind (see Section 4). Nevertheless, our abstract results are valid under a relaxed requirement, namely that $\mathcal{L}$ is *maximal monotone* (*maximal accretive*) and $\mathscr{D}(\mathcal{L}) = \mathscr{D}(\mathcal{L}^\star)$. Recall that an operator $\mathcal{L} : \mathscr{D}(\mathcal{L}) \to H$ is called *monotone*, if $(\mathcal{L}v, v) \geq 0$, for all $v \in \mathscr{D}(\mathcal{L})$, and it is called *maximal monotone* if moreover $\mathcal{I} + \mathcal{L}$ is onto H, where $\mathcal{I}$ is the identity operator in H. In fact, if $\mathcal{L}$ is maximal monotone, then $\mathscr{D}(\mathcal{L})$ is dense in H, $\mathcal{L}$ is *closed* and $\mathcal{I} + \sigma\mathcal{L}$ is onto H, for all positive σ; see [8].

1.2. **Implicit–explicit** (α, β, γ)**–schemes.** Let (α, β) and (α, γ) be an implicit and an explicit, respectively, q–step scheme, characterized by the polynomials

$$\alpha(\zeta) \,=\, \sum_{i=0}^{q} \alpha_i \zeta^i, \quad \beta(\zeta) \,=\, \sum_{i=0}^{q} \beta_i \zeta^i, \quad \gamma(\zeta) \,=\, \sum_{i=0}^{q-1} \gamma_i \zeta^i.$$

We then combine the schemes (α, β) and (α, γ), and construct an implicit–explicit (α, β, γ)–scheme for the discretization of the differential equation in (1.1). The linear part of the equation is discretized by the implicit scheme (α,β) and the nonlinear part by the explicit scheme (α,γ),

$$\sum_{i=0}^{q}\alpha_i\,U^{n+i}+k\sum_{i=0}^{q}\beta_i\,\mathcal{L}U^{n+i}\;=\;k\sum_{i=0}^{q-1}\gamma_i\,\mathcal{B}(t^{n+i},U^{n+i}),$$

for $i=0,\dots,N-q$.

Let us first focus on the case of vanishing dispersive operator $\mathcal{D}=0$. This case has been extensively studied in [2] and [3]; see also [1] for a wider class of schemes. Assume that the implicit scheme (α,β) is *strongly* $A(0)-$*stable*. Then the quantity $K_{(\alpha,\beta,\gamma)}$,

$$K_{(\alpha,\beta,\gamma)}\;:=\;\sup_{x>0}\max_{\zeta\in S^1}\Big|\frac{x\gamma(\zeta)}{\big(\alpha+x\beta\big)(\zeta)}\Big|,$$

with S^1 the unit circle in $\mathbb{C}$, is finite. The implicit–explicit $(\alpha,\beta,\gamma)-$scheme is locally stable in T_u, if

$$\lambda\;<\;\frac{1}{K_{(\alpha,\beta,\gamma)}}. \tag{1.8}$$

Moreover, condition (1.8) is sharp in the following sense: For any $\lambda>1/K_{(\alpha,\beta,\gamma)}$, there exists a (linear) operator $\mathcal{B}$ in (1.1) satisfying (1.3), such that the implicit–explicit $(\alpha,\beta,\gamma)-$scheme is unstable for (1.1); see [3].

For other classes of implicit–explicit schemes for semilinear parabolic equations we refer to [14, 17, 20]; see also the standard monograph for numerical methods for parabolic equations [22] and references therin.

Our goal here is to extend the results of [3] to problem (1.1). Unfortunately, the schemes analyzed in [3] are in general unstable, when approximating the solutions of (1.1), if the order of the anti–symmetric operator $\mathcal{D}$ is higher than the order of the self–adjoint operator $\mathcal{A}$. Motivated by the results for vanishing $\mathcal{D}$ in [3], we assume in the sequel that (α,β) is strongly $A(0)-$stable and that condition (1.8) is satisfied. Furthermore, due to the presence of the dispersive operator $\mathcal{D}$, we assume that the scheme (α,β) is $A-$stable; we need this condition for stability even in the absence of the nonlinear term ($\mathcal{B}=0$). Consequently, according to the *second Dahlquist barrier*, the highest attainable order of the scheme (α,β) is two. As a result, in contrast with [3], we confine ourselves to low order schemes; more precisely, we shall only analyze the implicit–explicit Euler and the implicit–explicit two-step BDF schemes.

The paper is organized as follows: In Section 2 we consider the implicit–explicit Euler scheme for (1.1). Section 3 is devoted to the second order implicit–explicit BDF scheme. Finally, in Section 4 we apply our abstract results to four examples, a simple system of ODEs, the dispersively modified Kuramoto–Sivashinsky equation, the Topper–Kawahara equation and to systems of Kuramoto–Sivashinsky type equations.

2. Implicit–explicit Euler scheme

In this section we analyze the discretization of (1.1) by the implicit–explicit Euler scheme. Throughout this section, we assume that λ in (1.3) is less than one, $\lambda < 1$.

Let $N \in \mathbb{N}$, $k := T/N$ be the time step, $t^n := nk$, $n = 0, \dots, N$, and u a solution of (1.1). We combine the implicit and explicit Euler schemes for discretizing (1.1) in time, and define approximations U^n to $u^n := u(t^n)$ by

$$\begin{cases} U^{n+1} + k\mathcal{L}U^{n+1} \;=\; U^n + k\mathcal{B}(t^n, U^n), \quad n = 0, \dots, N-1, \\ U^0 = u^0; \end{cases} \tag{2.1}$$

thus we discretize the linear part of the equation by the implicit Euler scheme and the nonlinear part by the explicit Euler scheme.

We assume that

$$\sup_{0<t<T} \Big\| \frac{d}{dt} \mathcal{B}\big(t, u(t)\big) \Big\|_\star \le C, \tag{2.2}$$

for a suitable constant C.

2.1. **Existence and uniqueness of approximations.** We shall show that, given $U^n \in \mathscr{D}(\mathcal{L})$, there exists a unique $U^{n+1} \in \mathscr{D}(\mathcal{L})$ such that

$$U^{n+1} + k\mathcal{L}U^{n+1} \;=\; U^n + k\mathcal{B}(t^n, U^n). \tag{2.3}$$

Clearly, whenever $U^n \in \mathscr{D}(\mathcal{L})$, the right–hand side of (2.3) defines an element $W^n = U^n + k\mathcal{B}(t^n, U^n)$ of the space H. Therefore, existence and uniqueness of approximations reduces to proving that the operator $\mathcal{I} + k\mathcal{L} : \mathscr{D}(\mathcal{L}) \to H$, with $\mathcal{I}$ the identity operator in H, is one–to–one in $\mathscr{D}(\mathcal{L})$ and onto H, for every $k > 0$.

One–to–one: Let $u \in \mathscr{D}(\mathcal{L})$ be such that $(\mathcal{I} + k\mathcal{L})u = 0$. Then

$$0 \;=\; \operatorname{Re}\big((\mathcal{I} + k\mathcal{L})u, u\big) \;=\; \big((\mathcal{I} + k\mathcal{A})u, u\big) \;=\; (u,u) + k(\mathcal{A}u, u) \;\ge\; (u,u),$$

which implies that $u = 0$.

Onto: We shall prove that $\mathscr{R}(\mathcal{I}+k\mathcal{L})$, the range of $\mathcal{I}+k\mathcal{L}$, is a closed subspace of H, which is dense in H, and therefore it coincides with H. If $\mathscr{R}(\mathcal{I}+k\mathcal{L})$ is not dense in H, then the subspace $\big(\mathscr{R}(\mathcal{I}+k\mathcal{L})\big)^\perp$ contains non–zero elements. Let $v \in \big(\mathscr{R}(\mathcal{I}+k\mathcal{L})\big)^\perp$. Then

$$\big((\mathcal{I} + k\mathcal{L})u, v\big) \;=\; 0,$$

for all $u \in \mathscr{D}(\mathcal{L})$, which implies that[2] $v \in \mathscr{D}(\mathcal{L}^\star)$. Thus

$$0 \;=\; \big((\mathcal{I} + k\mathcal{L})u, v\big) \;=\; \big(u, (\mathcal{I} + k\mathcal{L}^\star)v\big),$$

[2]Note that if $\mathcal{T} : \mathscr{D}(\mathcal{T}) \to H$ is an unbounded (densely defined) operator in H, then its adjoint is defined on the set

$$\mathscr{D}(\mathcal{T}^\star) \;=\; \big\{v \in H \;:\; \text{there exists } w \in H \text{ such that } (\mathcal{T}u, v) = (u, w), \text{ for every } u \in \mathscr{D}(\mathcal{T})\big\}.$$

for all $u\in\mathscr{D}(\mathcal{L})$. Since $\mathscr{D}(\mathcal{L})$ is dense in H, then $(\mathcal{I}+k\mathcal{L}^\star)v=0$, which in turn implies that $v=0$. (In order to prove that $\mathcal{I}+k\mathcal{L}^\star$ is one–to–one we use the argument of the proof that $\mathcal{I}+k\mathcal{L}$ is one–to–one and the fact that $\mathcal{L}^{\star\star}=\mathcal{L}$, since $\mathcal{L}$ is closed.) It remains to show that $\mathscr{R}(\mathcal{I}+k\mathcal{L})$ is closed. Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathscr{D}(\mathcal{L})$ be such that $(\mathcal{I}+k\mathcal{L})u_n\to w\in H$. Then

$$\begin{aligned}|(\mathcal{I}+k\mathcal{L})(u_m-u_n)|\cdot|u_m-u_n| &\ge \operatorname{Re}\big((\mathcal{I}+k\mathcal{L})(u_m-u_n),u_m-u_n\big)\\ &= |u_m-u_n|^2+k\operatorname{Re}\big(\mathcal{L}(u_m-u_n),u_m-u_n\big)\\ &\ge (1+k\sigma)|u_m-u_n|^2.\end{aligned}$$

Therefore,

$$|(\mathcal{I}+k\mathcal{L})(u_m-u_n)| \ge (1+k\sigma)|u_m-u_n|,$$

which implies that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. The sequence $\{\mathcal{L}u_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence, since $\{(\mathcal{I}+k\mathcal{L})u_n\}_{n\in\mathbb{N}}$ is convergent. Let $u_n\to u$. Since $\mathcal{L}$ is closed, we have $u\in\mathscr{D}(\mathcal{L})$ and

$$(\mathcal{I}+k\mathcal{L})u = \lim_{n\to\infty}(\mathcal{I}+k\mathcal{L})u_n = w,$$

which concludes the proof that $\mathcal{I}+k\mathcal{L}$ is onto.

Remark 2.1 (Resolvent estimate). The argument we used to prove existence and uniqueness of approximations provides that the resolvent set of the operator $\mathcal{L}$,

$$\varrho(\mathcal{L}) = \big\{z\in\mathbb{C}: z\mathcal{I}-\mathcal{L}:\mathscr{D}(\mathcal{L})\to H \text{ is } 1-1 \text{ and onto}\big\},$$

contains the set $U_\sigma=\{z\in\mathbb{C}:\operatorname{Re} z<\sigma\}$. In fact, we can easily obtain the following bound for the norm of the resolvent of $\mathcal{L}$

$$\big\|(z\mathcal{I}-\mathcal{L})^{-1}\big\| \le \frac{1}{\sigma-\operatorname{Re} z},$$

for every $z\in U_\sigma$, where $\|\cdot\|$ is the operator norm in H.

2.2. **Consistency.** Let E^n denote the consistency error of the implicit–explicit Euler scheme (2.1),

$$kE^n = u^{n+1}+k\mathcal{L}u^{n+1}-u^n-k\mathcal{B}(t^n,u^n), \tag{2.4}$$

$n=0,\dots,N-1$. Using the differential equation in (1.1), we express E^n in the form

$$\begin{aligned}kE^n &= u^{n+1}+k\mathcal{B}(t^{n+1},u^{n+1})-ku'(t^{n+1})-u^n-k\mathcal{B}(t^n,u^n)\\ &= u^{n+1}-u^n-ku'(t^{n+1})+k\big(\mathcal{B}(t^{n+1},u^{n+1})-\mathcal{B}(t^n,u^n)\big)\\ &= kE_1^n+kE_2^n,\end{aligned}$$

with

$$E_1^n := \frac{1}{k}(u^{n+1}-u^n)-u'(t^{n+1}),\quad E_2^n := \mathcal{B}(t^{n+1},u^{n+1})-\mathcal{B}(t^n,u^n).$$

Now,

$$E_1^n \;=\; -\frac{1}{k}\int_{t^n}^{t^{n+1}} (s-t^n)\, u''(s)\, ds,$$

and the regularity assumption

$$\sup_{t\in[0,T]} \|u''(t)\|_\star \;<\; \infty,$$

provides that

$$\max_{0\le n\le N-1} \|E_1^n\|_\star \;\le\; C_1 k,$$

while (2.2) provides that

$$\max_{0\le n\le N-1} \|E_2^n\|_\star \;\le\; C_2 k,$$

for suitable $C_1, C_2 > 0$, and thus we obtain the desired estimate for E^n,

$$\max_{0\le n\le N-1} \|E^n\|_\star \;\le\; Ck, \tag{2.5}$$

for a suitable $C > 0$.

2.3. **Local stability.** Assume that $U^0, U^1, \dots, U^N \in T_u$ and $V^0, V^1, \dots, V^N \in T_u$ are implicit–explicit Euler approximations,

$$\begin{cases} U^{n+1} + k\mathcal{L}U^{n+1} \;=\; U^n + k\mathcal{B}(t^n, U^n), \\ V^{n+1} + k\mathcal{L}V^{n+1} \;=\; V^n + k\mathcal{B}(t^n, V^n), \end{cases} \tag{2.6}$$

$n = 0, \dots, N-1$, with starting approximations U^0 and V^0, respectively. Let

$$\vartheta^m \;:=\; U^m - V^m \quad \text{and} \quad b^m \;:=\; \mathcal{B}(t^m, U^m) - \mathcal{B}(t^m, V^m),$$

$m = 0, \dots, N$. Subtracting the second relation of (2.6) from the first, we obtain

$$\vartheta^{n+1} + k\mathcal{L}\vartheta^{n+1} \;=\; \vartheta^n + kb^n. \tag{2.7}$$

Taking here the inner product with ϑ^{n+1}, we get

$$|\vartheta^{n+1}|^2 + k\|\vartheta^{n+1}\|^2 \;=\; \operatorname{Re}\big(\vartheta^n, \vartheta^{n+1}\big) + k\operatorname{Re}\big(b^n, \vartheta^{n+1}\big).$$

Therefore, according to (1.3),

$$\begin{aligned} |\vartheta^{n+1}|^2 + k\|\vartheta^{n+1}\|^2 &\le \frac{1}{2}|\vartheta^n|^2 + \frac{1}{2}|\vartheta^{n+1}|^2 + k\|b^n\|_\star\, \|\vartheta^{n+1}\| \\ &\le \frac{1}{2}|\vartheta^n|^2 + \frac{1}{2}|\vartheta^{n+1}|^2 + k\big(\lambda\|\vartheta^n\| + \mu|\vartheta^n|\big)\|\vartheta^{n+1}\| \\ &\le \frac{1}{2}|\vartheta^n|^2 + \frac{1}{2}|\vartheta^{n+1}|^2 + \frac{\lambda}{2}k\|\vartheta^n\|^2 + \frac{\lambda}{2}k\|\vartheta^{n+1}\|^2 + \frac{\mu}{2\varepsilon}k|\vartheta^n|^2 + \frac{1}{2}\varepsilon\mu k\|\vartheta^{n+1}\|^2, \end{aligned}$$

for any positive ε, i.e.,

$$|\vartheta^{n+1}|^2 + (2-\lambda-\varepsilon\mu)k\|\vartheta^{n+1}\|^2 \;\le\; \Big(1 + \frac{\mu}{\varepsilon}k\Big)|\vartheta^n|^2 + \lambda k\|\vartheta^n\|^2. \tag{2.8}$$

We now fix an ε such that $0<\varepsilon<\frac{2(1-\lambda)}{\mu}$. Then, obviously, $2-\lambda-\varepsilon\mu\geq\lambda$, and (2.8) yields

$$|\vartheta^{n+1}|^2+\lambda k\|\vartheta^{n+1}\|^2 \leq \Big(1+\frac{\mu}{\varepsilon}k\Big)|\vartheta^n|^2+\lambda k\|\vartheta^n\|^2,$$

whence

$$|\vartheta^{n+1}|^2+\lambda k\|\vartheta^{n+1}\|^2 \leq \Big(1+\frac{\mu}{\varepsilon}k\Big)\big(|\vartheta^n|^2+\lambda k\|\vartheta^n\|^2\big). \tag{2.9}$$

Introducing in V the norm $|\!|\!|\cdot|\!|\!|$,

$$|\!|\!|v|\!|\!| := \big(|v|^2+\lambda k\|v\|^2\big)^{1/2},$$

we can rewrite (2.9) in the form

$$|\!|\!|\vartheta^{n+1}|\!|\!|^2 \leq \Big(1+\frac{\mu}{\varepsilon}k\Big)|\!|\!|\vartheta^n|\!|\!|^2.$$

Thus, an obvious induction argument yields

$$|\!|\!|\vartheta^n|\!|\!|^2 \leq \mathrm{e}^{\frac{\mu}{\varepsilon}nk}|\!|\!|\vartheta^0|\!|\!|^2,$$

and we conclude the desired local stability estimate

$$\max_{1\leq n\leq N}|\!|\!|\vartheta^n|\!|\!| \leq \mathrm{e}^{\frac{\mu}{2\varepsilon}T}|\!|\!|\vartheta^0|\!|\!|. \tag{2.10}$$

2.4. **Error estimates.** Let the implicit–explicit Euler approximations $U^0,\dots,U^N$ be given by (2.1). Let $e^n:=u^n-U^n,\ n=0,\dots,N$.

The main result in this section is given in the following proposition:

Proposition 2.1 (Error estimates). *Let the time step k be sufficiently small. Then, we have the local stability estimate*

$$|\!|\!|e^n|\!|\!|^2 \leq \mathrm{e}^{\frac{\mu}{\varepsilon}t^n}\Big(|\!|\!|e^0|\!|\!|^2+\frac{1}{\varepsilon}k\sum_{\ell=0}^{n-1}\|E^\ell\|_\star^2\Big), \tag{2.11}$$

for every $n=0,\dots,N$, and the error estimate

$$\max_{0\leq n\leq N}|u(t^n)-U^n| \leq Ck. \tag{2.12}$$

Proof. In view of the consistency estimate (2.5) and the fact that e^0 vanishes, there exists a constant $C_\star$ such that the right–hand side of (2.11) can be estimated by $C_\star k^2$,

$$\mathrm{e}^{\frac{\mu}{\varepsilon}T}\Big(|\!|\!|e^0|\!|\!|^2+\frac{1}{\varepsilon}k\sum_{\ell=0}^{N-1}\|E^\ell\|_\star^2\Big) \leq C_\star k^2. \tag{2.13}$$

Now, obviously, (2.12) follows immediately from (2.11) and (2.13). Thus, it remains to prove (2.11).

We will use induction, and shall proceed as in the local stability proof, to establish (2.11). Clearly, the estimate (2.11) is valid for $n=0$. Assume that it holds for n,

$0 \le n < N$. Then, according to (2.13) and the induction hypothesis, we have, for k small enough,

$$\|e^n\| \le C_\star k^{1/2} \le 1,$$

and conclude that $U^n \in T_u$.

Let

$$\hat{b}^n := \mathcal{B}(t^n, u^n) - \mathcal{B}(t^n, U^n), \quad n = 0, \dots, N.$$

Subtracting the implicit–explicit Euler scheme in (2.1) from (2.4), we obtain the error equation

$$e^{n+1} + k\mathcal{L}e^{n+1} = e^n + k\hat{b}^n + kE^n, \tag{2.14}$$

$n = 0, \dots, N-1$. Taking in (2.14) the inner product with e^{n+1}, we get

$$|e^{n+1}|^2 + k\|e^{n+1}\|^2 = \operatorname{Re}\big(e^n, e^{n+1}\big) + k \operatorname{Re}\big(\hat{b}^n, e^{n+1}\big) + k \operatorname{Re}\big(E^n, e^{n+1}\big).$$

Therefore,

$$\frac{1}{2}|e^{n+1}|^2 + k\|e^{n+1}\|^2 \le \frac{1}{2}|e^n|^2 + k\|\hat{b}^n\|_\star \|e^{n+1}\| + k\|E^n\|_\star \|e^{n+1}\|; \tag{2.15}$$

see the relation preceding (2.8). Now, according to (1.3), since $U^n \in T_u$,

$$\begin{aligned} k\|\hat{b}^n\|_\star \|e^{n+1}\| &\le k\big(\lambda\|e^n\| + \mu|e^n|\big)\|e^{n+1}\| \\ &\le \frac{\lambda}{2}k\|e^n\|^2 + \frac{\lambda}{2}k\|e^{n+1}\|^2 + \frac{\mu}{2\varepsilon}k|e^n|^2 + \frac{1}{2}\varepsilon\mu k\|e^{n+1}\|^2, \end{aligned}$$

and

$$k\|E^n\|_\star \|e^{n+1}\| \le \frac{1}{2\varepsilon}k\|E^n\|_\star^2 + \frac{1}{2}\varepsilon k\|e^{n+1}\|^2,$$

for any positive ε. Thus, (2.15) yields

$$|e^{n+1}|^2 + (2-\lambda-\varepsilon\mu-\varepsilon)k\|e^{n+1}\|^2 \le \big(1+\frac{\mu}{\varepsilon}k\big)|e^n|^2 + \lambda k\|e^n\|^2 + \frac{1}{\varepsilon}k\|E^n\|_\star^2. \tag{2.16}$$

Now, let ε be sufficiently small, such that $2 - \lambda - (1+\mu)\varepsilon \ge \lambda$; then, from (2.16) we obtain

$$|\!|\!|e^{n+1}|\!|\!|^2 \le \big(1 + \frac{\mu}{\varepsilon}k\big)|\!|\!|e^n|\!|\!|^2 + \frac{1}{\varepsilon}k\|E^n\|_\star^2. \tag{2.17}$$

Using the induction hypothesis, we obtain from (2.17)

$$\begin{aligned} |\!|\!|e^{n+1}|\!|\!|^2 &\le \Big(1 + \frac{\mu}{\varepsilon}k\Big)\mathrm{e}^{\frac{\mu}{\varepsilon}t^n}\Big(|\!|\!|e^0|\!|\!|^2 + \frac{1}{\varepsilon}k\sum_{\ell=0}^{n-1}\|E^\ell\|_\star^2\Big) + \frac{1}{\varepsilon}k\|E^n\|_\star^2 \\ &\le \mathrm{e}^{\frac{\mu}{\varepsilon}t^{n+1}}\Big(|\!|\!|e^0|\!|\!|^2 + \frac{1}{\varepsilon}k\sum_{\ell=0}^{n}\|E^\ell\|_\star^2\Big); \end{aligned}$$

thus, (2.11) is valid for $n+1$ as well, and the proof is complete. ∎

Remark 2.2 (The case $\lambda = 0$)**.** It is easily seen that if $\mathcal{B}$ satisfies (1.3) with $\lambda = 0$, then the local stability estimates (2.10) and (2.11) take the form

$$\max_{1\le n\le N}\big(|\vartheta^n|^2 + k\|\vartheta^n\|^2\big) \le c|\vartheta^0|^2, \tag{2.18}$$

and

$$|e^n|^2 + k\|e^n\|^2 \le c\Big(|e^0|^2 + k\sum_{\ell=0}^{n-1}\|E^\ell\|_\star^2\Big), \tag{2.19}$$

respectively. This is advantageous in the fully discrete case, if the space discretization is based on the finite element method, since one can get by with weaker approximation assumptions for the starting approximation U^0.

3. Implicit–explicit two–step BDF scheme

The two–step BDF scheme is described by the polynomials α and β,

$$\alpha(\zeta) = \frac{3}{2}\zeta^2 - 2\zeta + \frac{1}{2}, \quad \beta(\zeta) = \zeta^2.$$

It is well known that this is a second order scheme, and it is $A-$stable (indeed, $G-$stable) and strongly $A(0)-$stable. Now, for the given α, the only second order explicit two–step scheme (α,γ) is the one with

$$\gamma(\zeta) = 2\zeta - 1;$$

see [2, Remark 3.1]. In this section we will analyze the implicit–explicit two–step BDF scheme (α,β,γ) for (1.1). Again, the linear part of the equation will be discretized by the implicit scheme (α,β), while the nonlinear part by the explicit scheme (α,γ).

Thus, with the notation used in Section 2, we define approximations U^n to $u^n = u(t^n)$ as follows: We let $U^0 := u^0$, perform one step of the implicit–explicit Euler scheme to compute U^1, i.e., we let U^1 be given by

$$U^1 + k\mathcal{L}U^1 = U^0 + k\mathcal{B}\big(t^0, U^0\big), \tag{3.1}$$

and let the approximations $U^2,\dots,U^N$ be given by the implicit–explicit BDF scheme,

$$\frac{3}{2}U^{n+2} - 2U^{n+1} + \frac{1}{2}U^n + k\mathcal{L}U^{n+2} = 2k\mathcal{B}(t^{n+1}, U^{n+1}) - k\mathcal{B}(t^n, U^n), \tag{3.2}$$

or equivalently

$$\Big(\frac{3}{2} + k\mathcal{L}\Big)U^{n+2} = 2U^{n+1} - \frac{1}{2}U^n + 2k\mathcal{B}(t^{n+1}, U^{n+1}) - k\mathcal{B}(t^n, U^n).$$

3.1. **Existence and uniqueness of approximate solutions.** We have seen in Subsection 2.1 that the resolvent set of the operator $\mathcal{L}$ contains all the negative numbers. This implies that, for every $\alpha > 0$ and $v \in H$, the equation

$$(\alpha \mathcal{I} + k\mathcal{L})u = v,$$

possesses a unique solution $u \in \mathscr{D}(\mathcal{L})$, which establishes the existence and uniqueness of approximate solutions for the implicit–explicit two–step BDF scheme.

3.2. **Consistency.** Let E^n denote the consistency error of the implicit–explicit BDF scheme (3.2),

$$kE^n = \frac{3}{2}u^{n+2} - 2u^{n+1} + \frac{1}{2}u^n + k\mathcal{L}u^{n+2} - 2k\mathcal{B}(t^{n+1}, u^{n+1}) + k\mathcal{B}(t^n, u^n), \tag{3.3}$$

$n = 0, \dots, N-2$. Using the differential equation in (1.1), we write this relation in the form

$$\begin{aligned} kE^n &= \frac{3}{2}u^{n+2} - 2u^{n+1} + \frac{1}{2}u^n - ku'(t^{n+2}) \\ &\quad + k\mathcal{B}(t^{n+2}, u^{n+2}) - 2k\mathcal{B}(t^{n+1}, u^{n+1}) + k\mathcal{B}(t^n, u^n) \\ &= kE_1^n + kE_2^n \end{aligned}$$

with

$$\begin{aligned} E_1^n &:= \frac{1}{k}\Big(\frac{3}{2}u^{n+2} - 2u^{n+1} + \frac{1}{2}u^n - ku'(t^{n+2})\Big), \\ E_2^n &:= \mathcal{B}(t^{n+2}, u^{n+2}) - 2\mathcal{B}(t^{n+1}, u^{n+1}) + \mathcal{B}(t^n, u^n). \end{aligned}$$

The quantities E_1^n and E_2^n can be easily rewritten in the form

$$\begin{aligned} E_1^n &= -\int_{t^{n+1}}^{t^{n+2}} (s - t^{n+2})u'''(s)\,ds + \frac{3}{4k}\int_{t^{n+1}}^{t^{n+2}} (s - t^{n+2})^2 u'''(s)\,ds \\ &\quad + \frac{1}{4k}\int_{t^n}^{t^{n+1}} (s - t^n)^2 u'''(s)\,ds, \\ E_2^n &= \int_{t^{n+1}}^{t^{n+2}} (t^{n+2} - s)\frac{d^2}{dt^2}\mathcal{B}\big(s, u(s)\big)\,ds + \int_{t^n}^{t^{n+1}} (s - t^n)\frac{d^2}{dt^2}\mathcal{B}\big(s, u(s)\big)\,ds. \end{aligned}$$

Therefore, under the regularity assumptions

$$\big\|u'''(t)\big\|_\star \le c_1 \quad \text{and} \quad \Big\|\frac{d^2}{dt^2}\mathcal{B}\big(t, u(t)\big)\Big\|_\star \le c_2, \tag{3.4}$$

for all $t \in [0, T]$, we immediately conclude that

$$\max_{0 \le n \le N-2} \|E_1^n\|_\star \le 2c_1k^2 \quad \text{and} \quad \max_{0 \le n \le N-2} \|E_2^n\|_\star \le 2c_2k^2.$$

Thus, we obtain the desired estimate for the consistency error E^n,

$$\max_{0 \le n \le N-2} \|E^n\|_\star \le Ck^2. \tag{3.5}$$

Remark 3.1 (Regularity requirement)**.** Note that the requirements (3.4) can be replaced by slightly weaker $C^{2,1}-$requirements on u and $\mathcal{B}(t,v)$. Similarly also for (2.2).

3.3. **Local stability.** For the scheme under investigation we have $K_{(\alpha,\beta,\gamma)}=3$; see [3, Remark 2.4]. Therefore, in the sequel we assume that (1.3) is satisfied and

$$\lambda<\frac{1}{3}.$$

Let $U^0,\dots,U^N\in T_u$ satisfy (3.1) and (3.2), and $V^0,\dots,V^N\in T_u$ satisfy

$$\frac{3}{2}V^{n+2}-2V^{n+1}+\frac{1}{2}V^n+k\mathcal{L}V^{n+2}=2k\mathcal{B}(t^{n+1},V^{n+1})-k\mathcal{B}(t^n,V^n), \tag{3.6}$$

$n=0,\dots,N-2$. As in Section 2, let

$$\vartheta^m:=U^m-V^m\quad\text{and}\quad b^m:=\mathcal{B}(t^m,U^m)-\mathcal{B}(t^m,V^m),$$

$m=0,\dots,N$. Subtracting (3.6) from (3.2), we obtain

$$\frac{3}{2}\vartheta^{n+2}-2\vartheta^{n+1}+\frac{1}{2}\vartheta^n+k\mathcal{L}\vartheta^{n+2}=2kb^{n+1}-kb^n. \tag{3.7}$$

Now, it is easily seen that

$$\begin{aligned}\operatorname{Re}\Big(\frac{3}{2}\vartheta^{n+2}-2\vartheta^{n+1}+\frac{1}{2}\vartheta^n,\vartheta^{n+2}\Big)&=\frac{5}{4}|\vartheta^{n+2}|^2-|\vartheta^{n+1}|^2-\frac{1}{4}|\vartheta^n|^2\\&-\operatorname{Re}\big((\vartheta^{n+2},\vartheta^{n+1})-(\vartheta^{n+1},\vartheta^n)\big)+\frac{1}{4}|\vartheta^{n+2}-2\vartheta^{n+1}+\vartheta^n|^2;\end{aligned} \tag{3.8}$$

cf. [24]. Let us note that relation (3.8) is due to the $G-$stability of the BDF method (α,β) with the positive definite matrix G,

$$G=\frac{1}{4}\begin{pmatrix}5&-2\\-2&1\end{pmatrix};$$

see [10, Example 6.5].

Taking in (3.7) the inner product with ϑ^{n+2} and using (3.8), we obtain

$$\begin{aligned}&\frac{5}{4}|\vartheta^{n+2}|^2-|\vartheta^{n+1}|^2-\frac{1}{4}|\vartheta^n|^2-\operatorname{Re}\big((\vartheta^{n+2},\vartheta^{n+1})-(\vartheta^{n+1},\vartheta^n)\big)\\&\quad+k\|\vartheta^{n+2}\|^2\le 2k\|b^{n+1}\|_\star\,\|\vartheta^{n+2}\|+k\|b^n\|_\star\,\|\vartheta^{n+2}\|.\end{aligned} \tag{3.9}$$

Now, in view of (1.3), for any positive ε,

$$\begin{aligned}\|b^{n+1}\|_\star\,\|\vartheta^{n+2}\|&\le\lambda\|\vartheta^{n+1}\|\,\|\vartheta^{n+2}\|+\mu|\vartheta^{n+1}|\,\|\vartheta^{n+2}\|\\&\le\frac{\lambda}{2}\|\vartheta^{n+1}\|^2+\frac{\lambda}{2}\|\vartheta^{n+2}\|^2+\frac{1}{2}\frac{\mu^2}{\varepsilon}|\vartheta^{n+1}|^2+\frac{1}{2}\varepsilon\|\vartheta^{n+2}\|^2;\end{aligned}$$

similarly,

$$\|b^n\|_\star\,\|\vartheta^{n+2}\|\le\frac{\lambda}{2}\|\vartheta^n\|^2+\frac{\lambda}{2}\|\vartheta^{n+2}\|^2+\frac{1}{4}\frac{\mu^2}{\varepsilon}|\vartheta^n|^2+\varepsilon\|\vartheta^{n+2}\|^2.$$

Therefore, (3.9) yields

$$\begin{aligned}&\frac{5}{4}\big(|\vartheta^{n+2}|^2-|\vartheta^{n+1}|^2\big)+\frac{1}{4}\big(|\vartheta^{n+1}|^2-|\vartheta^{n}|^2\big)\\ &\qquad-\operatorname{Re}\big((\vartheta^{n+2},\vartheta^{n+1})-(\vartheta^{n+1},\vartheta^{n})\big)+\Big(1-\frac{3}{2}\lambda-2\varepsilon\Big)k\|\vartheta^{n+2}\|^2\\ &\qquad\le\lambda k\|\vartheta^{n+1}\|^2+\frac{\lambda}{2}k\|\vartheta^{n}\|^2+C_\varepsilon k\big(|\vartheta^{n+1}|^2+|\vartheta^{n}|^2\big),\end{aligned}\tag{3.10}$$

with $C_\varepsilon:=\frac{1}{2}\frac{\mu^2}{\varepsilon}$.

Now, let ε be sufficiently small such that $1-2\varepsilon-\frac{3}{2}\lambda\ge\frac{3}{2}\lambda$. Then, from (3.10) we get

$$\begin{aligned}&\frac{5}{4}\big(|\vartheta^{n+2}|^2-|\vartheta^{n+1}|^2\big)+\frac{1}{4}\big(|\vartheta^{n+1}|^2-|\vartheta^{n}|^2\big)-\operatorname{Re}\big((\vartheta^{n+2},\vartheta^{n+1})-(\vartheta^{n+1},\vartheta^{n})\big)\\ &\qquad+k\frac{\lambda}{2}\Big(3\big(\|\vartheta^{n+2}\|^2-\|\vartheta^{n+1}\|^2\big)+\big(\|\vartheta^{n+1}\|^2-\|\vartheta^{n}\|^2\big)\Big)\\ &\qquad\le C_\varepsilon k\big(|\vartheta^{n+1}|^2+|\vartheta^{n}|^2\big).\end{aligned}\tag{3.11}$$

Summing in (3.11) from $n=0$ to $n=\ell$, we obtain

$$\begin{aligned}&\frac{5}{4}\big(|\vartheta^{\ell+2}|^2-|\vartheta^{1}|^2\big)+\frac{1}{4}\big(|\vartheta^{\ell+1}|^2-|\vartheta^{0}|^2\big)-\operatorname{Re}\big(\vartheta^{\ell+2},\vartheta^{\ell+1}\big)+k\frac{3\lambda}{2}\|\vartheta^{\ell+2}\|^2\\ &\qquad\le 2C_\varepsilon k\sum_{n=0}^{\ell+1}|\vartheta^n|^2+k\frac{\lambda}{2}\big(3\|\vartheta^1\|^2+\|\vartheta^0\|^2\big)-\operatorname{Re}\big(\vartheta^1,\vartheta^0\big),\end{aligned}$$

whence, easily,

$$\begin{aligned}\frac{1}{4}|\vartheta^{\ell+2}|^2+k\frac{3\lambda}{2}\|\vartheta^{\ell+2}\|^2\le\ &\frac{3+2\sqrt{2}}{4}\big(|\vartheta^1|^2+|\vartheta^0|^2\big)\\ &+2C_\varepsilon k\sum_{n=0}^{\ell+1}|\vartheta^n|^2+k\frac{\lambda}{2}\big(3\|\vartheta^1\|^2+\|\vartheta^0\|^2\big).\end{aligned}$$

Therefore, we have

$$|\!|\!|\vartheta^{\ell+1}|\!|\!|^2\le Ck\sum_{n=0}^{\ell}|\vartheta^n|^2+c\big(|\!|\!|\vartheta^0|\!|\!|^2+|\!|\!|\vartheta^1|\!|\!|^2\big),$$

$\ell=1,\dots,N-1$. Now, a straightforward application of the discrete Gronwall inequality leads to the desired local stability estimate

$$|\!|\!|\vartheta^{n}|\!|\!|^2\le C\big(|\!|\!|\vartheta^1|\!|\!|^2+|\!|\!|\vartheta^0|\!|\!|^2\big),\tag{3.12}$$

for every $n=1,\dots,N$.

Now, let V^1 and V^0 be related by

$$V^1+k\mathcal{L}V^1=V^0+k\mathcal{B}(t^0,V^0),\tag{3.13}$$

i.e., starting with initial value V^0 we obtain V^1 by performing one step with the implicit–explicit Euler scheme to the differential equation in (1.1); see (3.1) and (2.3). Subtracting (3.13) from (3.1), we obtain

$$\vartheta^1 + k\mathcal{L}\vartheta^1 \;=\; \vartheta^0 + kb^0;$$

see (2.7). Thus, we conclude

$$\|\!|\vartheta^1|\!\|^2 \;\le\; \big(1+\frac{\mu}{\varepsilon}k\big)\|\!|\vartheta^0|\!\|^2; \tag{3.14}$$

cf. (2.9). Now, (3.12) and (3.14) yield

$$\|\!|\vartheta^n|\!\| \;\le\; C\|\!|\vartheta^0|\!\|, \tag{3.15}$$

for every $n = 1, \dots, N$.

3.4. **Error estimates.** Let the implicit–explicit BDF2 approximations $U^0, \dots, U^N$ be given by (3.1) and (3.2). Let

$$e^n \;:=\; u^n - U^n, \qquad \hat{b}^n \;:=\; \mathcal{B}(t^n, u^n) - \mathcal{B}(t^n, U^n),$$

for $n = 0, \dots, N$.

The main result in this section is given in the following proposition:

Proposition 3.1 (Error estimates). *Let the time step k be sufficiently small. Then, we have the local stability estimate*

$$\|\!|e^n|\!\|^2 \;\le\; C\Big(\|\!|e^0|\!\|^2 + \|\!|e^1|\!\|^2 + k\sum_{\ell=0}^{n-2}\|E^\ell\|_\star^2\Big), \tag{3.16}$$

for every $n = 0, \dots, N$, and the error estimate

$$\max_{0\le n\le N} |u(t^n) - U^n| \;\le\; Ck^2. \tag{3.17}$$

Proof. First, we will estimate $\|\!|e^1|\!\|$. Denoting by $\tilde{E}^1$ the consistency error of the first step of the backward Euler method,

$$k\tilde{E}^1 \;=\; u^1 + k\mathcal{L}u^1 - u^0 - \mathcal{B}(t^0, u^0),$$

we have, in view of (3.1),

$$e^1 + k\mathcal{A}e^1 + k\mathcal{D}e^1 \;=\; k\tilde{E}^1,$$

and thus

$$|e^1|^2 + k\|e^1\|^2 \;=\; k(\tilde{E}^1, e^1) \;\le\; \frac{1}{2}k^2|\tilde{E}^1|^2 + \frac{1}{2}|e^1|^2,$$

whence

$$\|\!|e^1|\!\|^2 \;\le\; k^2|\tilde{E}^1|^2. \tag{3.18}$$

Under suitable regularity assumptions, we obtain, as in the derivation of (2.5), that

$$|\tilde{E}^1| \;\le\; Ck,$$

which combined with (3.18) provides that

$$\|\!|e^1\|\!|^2 \le Ck^4. \tag{3.19}$$

Now, in view of the consistency estimate (3.5), the estimate (3.19), and the fact that e^0 vanishes, there exists a constant $C_\star$ such that the right–hand side of (3.16) can be estimated by $C_\star k^4$,

$$C\Big(\|\!|e^0\|\!|^2 + \|\!|e^1\|\!|^2 + k\sum_{\ell=0}^{N-2}\|E^\ell\|_\star^2\Big) \le C_\star k^4. \tag{3.20}$$

Inequality (3.17) follows immediately from (3.16) and (3.20). Thus, it remains to prove (3.16).

We will use induction, and shall proceed as in the local stability proof, to establish (3.16). Clearly, the estimate (3.16) is valid for $n=0$ and $n=1$. Assume that it holds for $n=0,\dots,\ell+1,\ 0\le\ell\le N-2$. Then, according to (3.19), (3.20) and the induction hypothesis, we have, for k small enough,

$$\|e^n\| \le C_\star k^{3/2} \le 1, \quad n=0,\dots,\ell+1,$$

and conclude that $U^n\in T_u$, $n=0,\dots,\ell+1$. Let now $n=0,\dots,\ell+1$. Subtracting the implicit–explicit BDF2 scheme in (3.2) from (3.3), we obtain the error equation

$$\frac{3}{2}e^{n+2} - 2e^{n+1} + \frac{1}{2}e^n + k\mathcal{L}e^{n+2} = 2k\hat{b}^{n+1} - k\hat{b}^n + kE^n, \tag{3.21}$$

$n=0,\dots,\ell+1$. Taking in (3.21) the inner product with e^{n+2}, we get

$$\begin{aligned}&\frac{5}{4}|e^{n+2}|^2 - |e^{n+1}|^2 - \frac{1}{4}|e^n|^2 - \operatorname{Re}\big((e^{n+2},e^{n+1}) - (e^{n+1},e^n)\big)\\ &+ k\|e^{n+2}\|^2 \le 2k\|\hat{b}^{n+1}\|_\star\,\|e^{n+2}\| + k\|\hat{b}^n\|_\star\,\|e^{n+2}\| + k\|E^n\|_\star\,\|e^{n+2}\|;\end{aligned} \tag{3.22}$$

cf. (3.8) and (3.9). Now, in view of (1.3), for any positive ε,

$$\begin{aligned}\|\hat{b}^{n+1}\|_\star\,\|e^{n+2}\| &\le \frac{1}{2}\Big(\lambda\|e^{n+1}\|^2 + \lambda\|e^{n+2}\|^2 + \frac{\mu^2}{\varepsilon}|e^{n+1}|^2 + \varepsilon\|e^{n+2}\|^2\Big),\\ \|\hat{b}^n\|_\star\,\|e^{n+2}\| &\le \frac{1}{2}\Big(\lambda\|e^n\|^2 + \lambda\|e^{n+2}\|^2 + \frac{1}{2}\frac{\mu^2}{\varepsilon}|e^n|^2 + 2\varepsilon\|e^{n+2}\|^2\Big),\end{aligned}$$

and

$$\|E^n\|_\star\,\|e^{n+2}\| \le \frac{1}{4\varepsilon}\|E^n\|_\star^2 + \varepsilon\|e^{n+2}\|^2;$$

cf. the estimates preceding (3.10). Therefore, (3.22) yields

$$\begin{aligned}&\frac{5}{4}\big(|e^{n+2}|^2 - |e^{n+1}|^2\big) + \frac{1}{4}\big(|e^{n+1}|^2 - |e^n|^2\big)\\ &\quad - \operatorname{Re}\big((e^{n+2},e^{n+1}) - (e^{n+1},e^n)\big) + \Big(1 - \frac{3}{2}\lambda - 3\varepsilon\Big)k\|e^{n+2}\|^2\\ &\quad \le \lambda k\|e^{n+1}\|^2 + \frac{\lambda}{2}k\|e^n\|^2 + C_\varepsilon k\big(|e^{n+1}|^2 + |e^n|^2\big) + \tilde{C}_\varepsilon k\|E^n\|_\star^2,\end{aligned} \tag{3.23}$$

with $C_\varepsilon := \frac{1}{2}\frac{\mu^2}{\varepsilon}$ and $\tilde C_\varepsilon := \frac{1}{2\varepsilon}$.

Now, let ε be sufficiently small such that $1-3\varepsilon-\dfrac{3}{2}\lambda \geq \dfrac{3}{2}\lambda$. Then, from (3.23) we get

$$\begin{aligned}
&\frac{5}{4}\big(|e^{n+2}|^2-|e^{n+1}|^2\big)+\frac{1}{4}\big(|e^{n+1}|^2-|e^{n}|^2\big)-\operatorname{Re}\big((e^{n+2},e^{n+1})-(e^{n+1},e^{n})\big)\\
&\qquad+k\frac{\lambda}{2}\Big(3\big(\|e^{n+2}\|^2-\|e^{n+1}\|^2\big)+\big(\|e^{n+1}\|^2-\|e^{n}\|^2\big)\Big)\\
&\qquad\leq C_\varepsilon k\big(|e^{n+1}|^2+|e^{n}|^2\big)+\tilde C_\varepsilon k\|E^n\|_\star^2.
\end{aligned}\tag{3.24}$$

Summing in (3.24) from $n=0$ to $n=m,\ m\leq \ell$, we obtain

$$\begin{aligned}
&\frac{5}{4}\big(|e^{m+2}|^2-|e^{1}|^2\big)+\frac{1}{4}\big(|e^{m+1}|^2-|e^{0}|^2\big)-\operatorname{Re}\big(e^{m+2},e^{m+1}\big)+k\frac{3\lambda}{2}\|e^{m+2}\|^2\\
&\leq 2C_\varepsilon k\sum_{n=0}^{m+1}|e^n|^2+k\frac{\lambda}{2}\big(3\|e^1\|^2+\|e^0\|^2\big)-\operatorname{Re}\big(e^1,e^0\big)+2\tilde C_\varepsilon k\sum_{n=0}^{m}\|E^n\|_\star^2,
\end{aligned}$$

which provides that

$$\begin{aligned}
\frac{1}{4}|e^{m+2}|^2+k\frac{3\lambda}{2}\|e^{m+2}\|^2 \;&\leq\; \frac{3+2\sqrt{2}}{4}\big(|e^1|^2+|e^0|^2\big)\\
+2C_\varepsilon k\sum_{n=0}^{m+1}|e^n|^2&+k\frac{\lambda}{2}\big(3\|e^1\|^2+\|e^0\|^2\big)+2\tilde C_\varepsilon k\sum_{n=0}^{m}\|E^n\|_\star^2.
\end{aligned}$$

With obvious notation, we conclude that

$$|\!|\!|e^{m+2}|\!|\!|^2 \;\leq\; c\Big(|\!|\!|e^{0}|\!|\!|^2+|\!|\!|e^{1}|\!|\!|^2+k\sum_{n=0}^{m+1}|\!|\!|e^{n}|\!|\!|^2+k\sum_{n=0}^{\ell}\|E^n\|_\star^2\Big),\tag{3.25}$$

for every $m=0,\dots,\ell$. The desired estimate (3.16) for $n=\ell+2$ follows now from (3.25) by a straightforward application of the discrete Gronwall inequality: With

$$\alpha_m \;:=\; k\sum_{n=0}^{m+1}|\!|\!|e^{n}|\!|\!|^2,$$

we have $\alpha_{m+1}-\alpha_m = k|\!|\!|e^{m+2}|\!|\!|^2$, whence (3.25) yields

$$\alpha_{m+1} \;\leq\; (1+ck)\alpha_m+ck\Big(|\!|\!|e^{0}|\!|\!|^2+|\!|\!|e^{1}|\!|\!|^2+k\sum_{n=0}^{\ell}\|E^n\|_\star^2\Big),\tag{3.26}$$

for every $m=0,\dots,\ell$, and, finally,

$$|\!|\!|e^{\ell+2}|\!|\!|^2 \;\leq\; c\Big(|\!|\!|e^{0}|\!|\!|^2+|\!|\!|e^{0}|\!|\!|^2+\alpha_{\ell+1}+k\sum_{n=0}^{\ell}\|E^n\|_\star^2\Big).\tag{3.27}$$

Using (3.26) we can estimate $\alpha_{\ell+1}$; this estimate combined with (3.27) leads to the desired estimate (3.16) for $n=\ell+2$ and the proof is complete. □

Remark 3.2 (The case $\lambda = 0$)**.** It is easily seen that if $\mathcal{B}$ satisfies (1.3) with $\lambda = 0$, then the local stability estimates (3.15) and (3.16) take the form

$$\max_{1\le n\le N} \big(|\vartheta^n|^2 + k\|\vartheta^n\|^2\big) \le c|\vartheta^0|^2,$$

and

$$|e^n|^2 + k\|e^n\|^2 \le c\Big(|e^0|^2 + |e^1|^2 + k\sum_{\ell=0}^{n-2} \|E^\ell\|_\star^2\Big),$$

respectively. This is advantageous in the fully discrete case, if the space discretization is based on the finite element method, since one can get by with weaker approximation assumptions for the starting approximations U^0 and U^1; cf. Remark 2.2. □

4. Examples

In this section we present four examples of equations satisfying our assumptions. The first example is a simple system of ODEs; we include it to justify the stability requirements on the implicit schemes. The other three examples, namely the dispersively modified Kuramoto–Sivashinsky equation, the Topper–Kawahara equation and a systems of Kuramoto–Sivashinsky type equations, are physically relevant dispersive-dissipative equations and systems.

To avoid repetitions, let us introduce the following notation: For $2\pi-$periodic functions v of one or two variables, we denote by $\hat v_\ell$ and $\hat v_{j\ell}$ their Fourier coefficients,

$$v(x) = \sum_{\ell\in\mathbb{Z}} \hat v_\ell\, \mathrm{e}^{\mathrm{i}\ell x}, \qquad v(x,y) = \sum_{j,\ell\in\mathbb{Z}} \hat v_{j\ell}\, \mathrm{e}^{\mathrm{i}(jx+\ell y)}.$$

4.1. **A simple finite–dimensional example.** Let

$$\mathcal{A} = \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 0 & b\\ -b & 0\end{pmatrix},$$

with $a, b \in \mathbb{R}$ and $a > 0$. Then, the system of ODEs

$$u' + \mathcal{L}u = 0, \tag{4.1}$$

with $\mathcal{L} = \mathcal{A} + \mathcal{D}$, satisfies our hypotheses, since the matrix $\mathcal{A}$ is symmetric and positive definite, and the matrix $\mathcal{D}$ is anti–symmetric.

The eigenvalues of the matrix $\mathcal{L}$ are $\lambda_1 = a + \mathrm{i}b$ and $\lambda_2 = a - \mathrm{i}b$. As a and b vary, the eigenvalues cover the right complex half–plane. Consequently, the $(\alpha,\beta)-$scheme is unconditionally stable for equations of the form (4.1), if and only if it is $A-$stable; our assumption on the $A-$stability of the $(\alpha,\beta)-$scheme is motivated by this fact.

More generally, $\mathcal{A}$ in (4.1) could be any positive definite symmetric $d \times d$ matrix and $\mathcal{D}$ any anti–symmetric matrix.

4.2. **The dispersively modified Kuramoto–Sivashinsky equation.** A typical infinite dimensional example of a problem of the form (1.1) is the initial value problem for the dispersively modified Kuramoto–Sivashinsky equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u \; = \; 0, \tag{4.2}$$

where $u = u(x,t)$ is $2\pi-$periodic in x and ν a positive constant. In (4.2), $\mathcal{D}$ is a linear dispersive pseudo–differential operator defined by

$$(\widehat{\mathcal{D}v})_\ell \; = \; \mathrm{i} f(\ell)\, \hat{v}_\ell,$$

with f a given real–valued function.

We rewrite (4.2) in the form

$$u_t + \frac{1}{\nu}u + u_{xx} + \nu u_{xxxx} + \mathcal{D}u \; = \; \frac{1}{\nu}u - uu_x, \tag{4.3}$$

and will show that (4.3) fits into our abstract framework. First, we let the operators $\mathcal{L}$ and $\mathcal{B}$ be given by

$$\mathcal{L}v \; = \; \Big(\frac{1}{\nu}v + v_{xx} + \nu v_{xxxx}\Big) + \mathcal{D}v, \quad \mathcal{B}(v) \; = \; \frac{1}{\nu}v - vv_x,$$

and write (4.3) in the form

$$u_t + \mathcal{L}u \; = \; \mathcal{B}(u).$$

Let $L^2(T^1)$ be the space of $2\pi-$periodic square integrable functions, and denote by $(\cdot,\cdot)$ its inner product defined by

$$(u,v) \; = \; \frac{1}{2\pi}\int_0^{2\pi} u(x)\,\overline{v}(x)\,dx \; = \; \sum_{\ell\in\mathbb{Z}} \hat{u}_\ell\,\overline{\hat{v}}_\ell. \tag{4.4}$$

Clearly,

$$\mathcal{L}^\star v \; = \; \Big(\frac{1}{\nu}v + v_{xx} + \nu v_{xxxx}\Big) - \mathcal{D}v.$$

In fact, $(\widehat{\mathcal{L}v})_\ell = m(\ell)\hat{v}_\ell,\ \ell\in\mathbb{Z}$, where

$$m(\ell) \; = \; \frac{1}{\nu} + \ell^2 + \nu\ell^4 + \mathrm{i} f(\ell),$$

and $(\widehat{\mathcal{L}^\star v})_\ell = \overline{m}(\ell)\hat{v}_\ell$, with $\overline{m}$ the complex conjugate of m. In particular,

$$\begin{aligned}\mathscr{D}(\mathcal{L}) \; &= \; \big\{v\in L^2(0,2\pi) : \{m(\ell)\hat{v}_\ell\}_{\ell\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\big\} \\ &= \; \big\{v\in L^2(0,2\pi) : \{\overline{m}(\ell)\hat{v}_\ell\}_{\ell\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\big\} \; = \; \mathscr{D}(\mathcal{L}^\star).\end{aligned}$$

Note that $\mathcal{L}$ is a normal operator since

$$\mathscr{D}(\mathcal{L}^\star\mathcal{L}) \; = \; \big\{v\in L^2(0,2\pi) : \{\overline{m}(\ell)m(\ell)\hat{v}_\ell\}_{\ell\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\big\} \; = \; \mathscr{D}(\mathcal{L}\mathcal{L}^\star),$$

and

$$(\widehat{\mathcal{L}^\star\mathcal{L}v})_\ell \; = \; |m(\ell)|^2\hat{v}_\ell \; = \; (\widehat{\mathcal{L}\mathcal{L}^\star v})_\ell,$$

for every $v\in\mathscr{D}(\mathcal{L}^\star\mathcal{L})$. Therefore, we let the operator $\mathcal{A}$ be given by

$$\mathcal{A}v \;:=\; \frac{1}{\nu}v + v_{xx} + \nu v_{xxxx}. \tag{4.5}$$

For $s\in\mathbb{R}$, let $H^s(T^1)$ denote the periodic Sobolev space of order s with norm $\|\cdot\|_{H^s}$ defined by

$$\|v\|_{H^s} \;:=\; \Big(\sum_{\ell\in\mathbb{Z}}(1+\ell^2)^s|\hat{v}_\ell|^2\Big)^{1/2}.$$

Clearly, $H^0(T^1) = L^2(T^1)$, and the norm induced by the inner product in (4.4), which we shall be denoting by $|\cdot|$, coincides with $\|\cdot\|_{H^0}$. It is readily seen that $V := \mathscr{D}(\mathcal{A}^{1/2}) = H^2(T^1)$. Let $\|\cdot\|$ be the norm defined by

$$\|v\| \;=\; \Big(\nu|v_{xx}|^2 - |v_x|^2 + \frac{1}{\nu}|v|^2\Big)^{1/2}.$$

This norm is in fact equivalent to $\|\cdot\|_{H^2}$. Also,

$$(\mathcal{A}v,v) \;\geq\; \frac{1}{2}\Big(\nu|v_{xx}|^2 + \frac{1}{\nu}|v|^2\Big) \qquad \text{for all } \; v\in V;$$

thus, in particular, $\mathcal{A}$ is positive definite; see [5].

Furthermore, obviously $\mathcal{B} : V \to H$. Also, with T_u the tube around the solution u defined in terms of the norm of H, i.e.,

$$T_u \;:=\; \big\{v\in V : \min_t |v-u(t)| \leq 1\big\}, \tag{4.6}$$

and $\|\cdot\|_\star$ the dual norm on V', we have

$$\|\mathcal{B}(v)-\mathcal{B}(w)\|_\star \;\leq\; \mu|v-w| \quad \text{for all } \; v,w\in T_u, \tag{4.7}$$

with

$$\mu \;=\; \frac{1}{\sqrt{\nu}}\Big(2\sqrt{\pi}\,\big(1+\max_{0\leq t\leq T}|u(t)|\big) + \sqrt{2}\Big);$$

see [5]. Hence, $\mathcal{B}$ satisfies the local Lipschitz condition (1.3) with $\lambda = 0$.

Remark 4.1. [The Kawahara equation] Of particular interest is the periodic initial value problem for the Kawahara equation (1.6), where the dispersive term is dominated by the dissipative term. This equation may be discretized by higher order implicit–explicit multistep schemes, as we will see here. First, we write the equation in the form

$$u_t + \frac{1}{\nu}u + u_{xx} + \nu u_{xxxx} \;=\; \frac{1}{\nu}u - \delta u_{xxx} - uu_x.$$

Thus, with the self–adjoint and positive definite operator $\mathcal{A}$ given in (4.5), $\mathcal{D}=0$, and the nonlinear operator $\mathcal{B}$,

$$\mathcal{B}v \;:=\; \frac{1}{\nu}v - \delta v_{xxx} - vv_x,$$

the Kawahara equation takes the form

$$u_t + \mathcal{A}u \;=\; \mathcal{B}(u). \tag{4.8}$$

By periodicity, for $v, \tilde{v}, w \in H^2(T^1)$,

$$\big(\mathcal{B}(v) - \mathcal{B}(\tilde{v}), w\big) \;=\; \frac{1}{\nu}(v - \tilde{v}, w) - \delta(v_x - \tilde{v}_x, w_{xx}) + \frac{1}{2}(v^2 - \tilde{v}^2, w_x),$$

whence,

$$\begin{aligned}\big(\mathcal{B}(v) - \mathcal{B}(\tilde{v}), w\big) \;&\le\; \frac{1}{\nu}|v - \tilde{v}|\,|w| + |\delta|\,|v_x - \tilde{v}_x|\,|w_{xx}| \\ &\quad + \frac{1}{2}|v + \tilde{v}|\,|v - \tilde{v}|\,\|w_x\|_{L^\infty}.\end{aligned}$$

Now, for $v \in H^2(T^1)$,

$$\begin{aligned}|v_x|^2 \;&=\; \frac{1}{2\pi}\int_0^{2\pi} v_x(x)\,\overline{v}_x(x)\,dx \;=\; -\frac{1}{2\pi}\int_0^{2\pi} v(x)\,\overline{v}_{xx}(x)\,dx \;=\; -(v, v_{xx}) \\ &\le\; |v|\,|v_{xx}|,\end{aligned}$$

whence, for any positive $\tilde{\varepsilon}$,

$$|v_x| \;\le\; \frac{1}{4\tilde{\varepsilon}}|v| + \tilde{\varepsilon}|v_{xx}|;$$

proceeding now as in the derivation of (2.15) in [5], cf. also (4.7), we easily see that

$$\|\mathcal{B}(v) - \mathcal{B}(\tilde{v})\|_\star \;\le\; \varepsilon\|v - \tilde{v}\| + C_\varepsilon|v - \tilde{v}| \quad \text{ for all } v, \tilde{v} \in T_u,$$

with the tube T_u as in (4.6), for any positive ε and a constant C_ε depending on δ, ε and the value of $\max_{0\le t\le T}|u(t)|$.

Thus, all linearly implicit schemes of [1], and, in particular, the implicit–explicit $p-$step BDF schemes, with $p = 1, \dots, 6$, see [5], are locally stable for (4.8) and consequently suitable for the discretization of the Kawahara equation.

4.3. **Topper–Kawahara equation.** In the case of falling film flows, in two space dimensions, Topper & Kawahara [23] derived a rather general evolution equation for the liquid interface which takes the form

$$u_t + uu_x + \alpha u_{xx} + \beta\varDelta u + \gamma\varDelta^2 u + \delta\varDelta u_x \;=\; 0, \tag{4.9}$$

where $\gamma > 0$, $\varDelta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ the Laplacian, x is in the direction of the flow, while y is the transverse coordinate.

We consider an initial value problem for the Topper–Kawahara equation (4.9) with periodic boundary conditions; $u(x, y, t)$ is $2\pi-$periodic in both variables x and y, i.e.,

$$u(x + 2\pi, y, t) \;=\; u(x, y, t) \quad \text{and} \quad u(x, y + 2\pi, t) \;=\; u(x, y, t), \quad \text{for all } x, y \in \mathbb{R}.$$

The $2\pi-$periodic functions in x and y variables can be thought of as functions with domain the two–dimensional torus T^2. For $s \in \mathbb{R}$, we denote by $H^s(T^2)$ the periodic

Sobolev space of order s in two dimensions with norm[3]

$$\|v\|_{H^s} := \Big(\sum_{j,\ell\in\mathbb{Z}}(1+j^2+\ell^2)^s\,|\hat{v}_{j\ell}|^2\Big)^{1/2}. \tag{4.10}$$

Clearly, $H^s(T^2)$ is a Hilbert space, for every $s\in\mathbb{R}$. Let $H := H^0(T^2) = L^2(T^2)$. Then the norm of H, which we shall be denoting by $|\cdot|$, is induced by the inner product

$$(u,v) = \frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} u(x,y)\,\overline{v}(x,y)\,dx\,dy = \sum_{j,\ell\in\mathbb{Z}}\hat{u}_{j\ell}\,\overline{\hat{v}}_{j\ell}.$$

As in Remark 4.1, we shall see here that the Topper–Kawahara equation may be discretized by higher order implicit–explicit multistep schemes. We first write (4.9) in the form

$$u_t + cu + \alpha u_{xx} + \beta\varDelta u + \gamma\varDelta^2 u = -\delta\varDelta u_x - uu_x + cu, \tag{4.11}$$

where c is a positive constant, which will be determined later. With the self-adjoint operator $\mathcal{A}$,

$$\mathcal{A}v := \gamma\varDelta^2 v + \beta\varDelta v + \alpha v_{xx} + cv,$$

$\mathcal{D}=0$, and the nonlinear operator $\mathcal{B}$,

$$\mathcal{B}(v) := -\delta\varDelta v_x - vv_x + cv,$$

equation (4.11) can be written in the form

$$u_t + \mathcal{A}u = \mathcal{B}(u). \tag{4.12}$$

Next, let us first show that the operator $\mathcal{A}$ is positive definite, when c is sufficiently large. For $v\in V = H^2(T^2)$, we have

$$(\mathcal{A}v,v) = \gamma(\varDelta v,\varDelta v) - \beta(\nabla v,\nabla v) - \alpha(v_x,v_x) + c(v,v),$$

i.e.,

$$(\mathcal{A}v,v) = \gamma|\varDelta v|^2 - \beta|\nabla v|^2 - \alpha|v_x|^2 + c|v|^2. \tag{4.13}$$

Obviously,

$$|v|^2 = \sum_{j,\ell\in\mathbb{Z}}|\hat{v}_{j\ell}|^2,\quad |\nabla v|^2 = \sum_{j,\ell\in\mathbb{Z}}(j^2+\ell^2)\,|\hat{v}_{j\ell}|^2 \text{ and } |\varDelta v|^2 = \sum_{j,\ell\in\mathbb{Z}}(j^2+\ell^2)^2\,|\hat{v}_{j\ell}|^2,$$

and using the fact that

$$j^2+\ell^2 \le \frac{1}{4\varepsilon} + \varepsilon(j^2+\ell^2)^2,$$

[3]Note that, if s is a non–negative integer, then $\|\cdot\|_{H^s}$ is equivalent to the norm defined by

$$\|u\|_s = \Big(\sum_{|\alpha|\le s}\int_0^{2\pi}\!\!\int_0^{2\pi}|D^\alpha u(x,y)|^2\,dx\,dy\Big)^{1/2}.$$

we derive the inequality

$$|\nabla v|^2 \le \frac{1}{4\varepsilon}|v|^2 + \varepsilon|\Delta v|^2, \tag{4.14}$$

which holds for every $\varepsilon > 0$. Hence, from (4.13) we obtain

$$(\mathcal{A}v, v) \ge \big(\gamma - (|\beta| + |\alpha|)\varepsilon\big)\,|\Delta v|^2 + \Big(c - \frac{1}{4\varepsilon}\Big)|v|^2.$$

Choosing here, for instance,

$$\varepsilon := \frac{\gamma}{2(|\beta| + |\alpha|)} \quad \text{and} \quad c := \frac{\gamma}{2} + \frac{1}{4\varepsilon},$$

we easily see that

$$(\mathcal{A}v, v) \ge \frac{\gamma}{2}\big(|v|^2 + |\Delta v|^2\big),$$

and conclude the coercivity of $\mathcal{A}$,

$$(\mathcal{A}v, v) \ge \tilde{\gamma}\|v\|_{H^2}^2 \quad \text{for all } v \in V,$$

for a suitable positive constant $\tilde{\gamma}$.

Finally, concerning (1.3), we let the tube T_u around the solution u be defined in terms of the L^∞−norm, i.e.,

$$T_u := \big\{\, v \in V \,:\, \min_t \|v - u(t)\|_{L^\infty} \le 1 \,\big\}, \tag{4.15}$$

and denote by $\|\cdot\|_\star$ the dual norm on V'. For convenience, we split $\mathcal{B}$ into two parts, $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, with $\mathcal{B}_1(v) := -vv_x + cv$ and $\mathcal{B}_2(v) := -\delta\Delta v_x$. Now, for $v, \tilde{v}, w \in V$, we have

$$\begin{aligned}\big(\mathcal{B}_1(v) - \mathcal{B}_1(\tilde{v}), w\big) &= \frac{1}{2}(v^2 - \tilde{v}^2, w_x) + c\,(v - \tilde{v}, w)\\ &\le \frac{1}{2}\|v + \tilde{v}\|_{L^\infty}\,|v - \tilde{v}|\,|w_x| + c\,|v - \tilde{v}|\,|w|\\ &\le \Big(\frac{1}{4}\|v + \tilde{v}\|_{L^\infty}^2 + c^2\Big)^{1/2}\,|v - \tilde{v}|\,\|w\|_{H^1},\end{aligned}$$

whence, easily

$$\|\mathcal{B}_1(v) - \mathcal{B}_1(\tilde{v})\|_\star \le \mu_1|v - \tilde{v}| \quad \text{for al } v, w \in T_u, \tag{4.16}$$

with

$$\mu_1 = \sqrt{\tilde{\gamma}}\,\Big(\frac{1}{4}\big(2 + \max_{0\le t\le T}\|u(t)\|_{L^\infty}^2\big)^2 + c^2\Big)^{1/2}.$$

Furthermore, by elementary calculations we obtain from (4.14)

$$|v_x| \le \frac{1}{4\varepsilon}|v| + \varepsilon\|v\|_{H^2}, \tag{4.17}$$

for any positive ε, which in turn yields that

$$\|\mathcal{B}_2(v) - \mathcal{B}_2(\tilde{v})\|_\star \le |\delta|\Big(\varepsilon|v - \tilde{v}|_{H^2} + \frac{1}{4\varepsilon}|v - \tilde{v}|\Big) \quad \text{for al } v, w \in V. \tag{4.18}$$

We infer from (4.16) and (4.18) that $\mathcal{B}$ satisfies the local Lipschitz condition (1.3) for any positive λ. Hence, all implicit–explicit multistep methods considered in [3], and, indeed, the wider class of methods considered in [1] are suitable for the discretization of the periodic initial value problem for the Topper–Kawahara equation.

Remark 4.2. It follows immediately from (4.16) that it is possible for the Topper–Kawahara equation as well to have a local Lipschitz condition (1.3) with $\lambda = 0$. This can be done by considering $\mathcal{B}_2$ as a dispersive operator, $\mathcal{D} := \mathcal{B}_2$, and, consequently, letting $\mathcal{B} := \mathcal{B}_1$. As already mentioned, this allows one to get by by less stringent conditions on the starting approximations; see Remarks 2.2 and 3.1 as well as [1]. The price we have to pay if we choose this splitting, however, is that we have to confine ourselves to implicit–explicit multistep schemes of first– or second–order.

4.4. **Systems of Kuramoto–Sivashinsky type equations.** When surfactants are present in axisymmetric core annular flows, the spatiotemporal evolution of the interface and the local surfactant concentration on it are given by the system

$$\begin{cases} u_t + \nu u_{xxxx} + u_{xx} + uu_x + \mathit{\Gamma}_{xx} + \mathcal{D}u \;=\; 0, \\ \mathit{\Gamma}_t - \eta \mathit{\Gamma}_{xx} + (u\mathit{\Gamma})_x \;=\; 0, \end{cases} \tag{4.19}$$

where $u = u(x,t)$ denotes the scaled interfacial amplitude as before, $\mathit{\Gamma} = \mathit{\Gamma}(x,t)$ is the surfactant concentration at any point on the interface, $\mathcal{D}$ is the pseudo–differential operator defined by

$$(\widehat{\mathcal{D}v})_\ell \;=\; \mathrm{i} f(\ell)\, \hat{v}_\ell,$$

with f given by (1.5), and ν and η are positive diffusion constants. This equation was derived by Kas–Danouche, Papageorgiou & Siegel [11].

For reasons that will become apparent in the sequel, we write (4.19) as

$$\begin{cases} u_t + \nu u_{xxxx} + u_{xx} + \dfrac{1}{\nu}u + \mathcal{D}u \;=\; -uu_x + \dfrac{1}{\nu}u - \mathit{\Gamma}_{xx}, \\ \mathit{\Gamma}_t - \eta\big(\mathit{\Gamma}_{xx} - \mathit{\Gamma}\big) \;=\; -(u\mathit{\Gamma})_x + \eta\mathit{\Gamma}, \end{cases} \tag{4.20}$$

where $(x,t) \in \mathbb{R} \times [0,\infty)$. We write (4.20) in the form

$$\boldsymbol{u}_t + \boldsymbol{\mathcal{L}}\boldsymbol{u} \;=\; \mathcal{B}(\boldsymbol{u}), \tag{4.21}$$

where $t \in [0,\infty)$. As in Subsection 4.2, let $H^s(T^1)$ be the periodic Sobolev space of order s in one variable and $\|\cdot\|_{H^s}$ its norm. Let also $(\cdot,\cdot)$ be the inner product of $L^2(T^1) = H^0(T^1)$, as defined by (4.4), and $|\cdot|$ its induced norm.

Clearly, $(\mathcal{D}v, v) = 0$, for all $v \in \mathscr{D}(\mathcal{D}) \subset L^2(T^1)$. We also introduce the linear operators $\mathcal{A}_1 : H^4(T^1) \to L^2(T^1)$ and $\mathcal{A}_2 : H^2(T^1) \to L^2(T^1)$ by

$$\mathcal{A}_1 v \;:=\; \nu v_{xxxx} + v_{xx} + \frac{1}{\nu}v, \quad \mathcal{A}_2 v \;:=\; -\eta(v_{xx} - v).$$

Obviously, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are self–adjoint. Furthermore, it is easily seen that

$$(\mathcal{A}_1 v, v) \ \geq\ \frac{1}{2}\Big(\nu|v_{xx}|^2 + \frac{1}{\nu}|v|^2\Big), \qquad \text{for all } \ v \in H^2(T^1)$$

and

$$(\mathcal{A}_2 v, v) \ =\ \eta\big(|v_x|^2 + |v|^2\big), \qquad \text{for all } \ v \in H^1(T^1).$$

Hence, $\mathcal{A}_1$ and $\mathcal{A}_2$ are positive definite. Here and in the sequel we also denote by $(\cdot,\cdot)$ the duality pairing between $H^{-1}(T^1)$ and $H^1(T^1)$, as well as between $H^{-2}(T^1)$ and $H^2(T^1)$.

Let $\boldsymbol{H} := L^2(T^1)\times L^2(T^1)$ and denote by $\langle\cdot,\cdot\rangle$ the product inner product,

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle \ :=\ (u_1, v_1) + (u_2, v_2),$$

with $u_1, u_2, v_1, v_2 \in L^2(T^1)$ the components of $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. To formulate our problem in the form (4.21) in the framework of the Hilbert space $\big(\boldsymbol{H}, \langle\cdot,\cdot\rangle\big)$, we introduce the operator $\mathcal{A} : \mathscr{D}(\mathcal{A}) = H^4(T^1)\times H^2(T^1) \to \boldsymbol{H}$,

$$\mathcal{A}\boldsymbol{u} \ :=\ \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ =\ \begin{pmatrix} \mathcal{A}_1 u_1 \\ \mathcal{A}_2 u_2 \end{pmatrix}.$$

It is readily seen that $\mathcal{A}$ is self–adjoint and positive definite. Let

$$\boldsymbol{V} \ :=\ H^2(T^1)\times H^1(T^1) \ =\ \mathscr{D}(\mathcal{A}^{1/2}),$$

and $\boldsymbol{V}'$ be the dual of $\boldsymbol{V}$, $\boldsymbol{V}' = H^{-2}(T^1)\times H^{-1}(T^1)$. We denote the norms in $\boldsymbol{V}$ and $\boldsymbol{V}'$ by $\|\cdot\|$ and $\|\cdot\|_\star$, respectively,

$$\|\boldsymbol{u}\| \ =\ \big(|\mathcal{A}_1^{1/2}u_1|^2 + |\mathcal{A}_2^{1/2}u_2|^2\big)^{1/2}, \quad \|\boldsymbol{u}\|_\star \ =\ \big(|\mathcal{A}_1^{-1/2}u_1|^2 + |\mathcal{A}_2^{-1/2}u_2|^2\big)^{1/2},$$

where $\boldsymbol{u} = (u_1, u_2)$. Furthermore, let $\mathcal{B} : \mathscr{D}(\mathcal{A}) \to \boldsymbol{H}$,

$$\mathcal{B}(\boldsymbol{u}) \ =\ -\begin{pmatrix} u_1(u_1)_x - \dfrac{1}{\nu}u_1 + (u_2)_{xx} \\[2ex] (u_1u_2)_x - \eta u_2 \end{pmatrix}.$$

Obviously, $\mathcal{B}$ can be extended to a map from $\boldsymbol{V}$ to $\boldsymbol{V}'$. With this notation, and $u_1 = u, u_2 = \Gamma$, the system (4.20) can be written in the form (4.21) with

$$\mathcal{L}(\boldsymbol{u}) \ =\ \mathcal{A}(\boldsymbol{u}) + \begin{pmatrix} \mathcal{D}u_1 \\ 0 \end{pmatrix}.$$

Let T_u be a tube around the solution $\boldsymbol{u}$, defined in terms of the L^∞−norm,

$$T_{\boldsymbol{u}} \ :=\ \big\{\boldsymbol{v} \in \boldsymbol{V} : \inf_{t\geq 0}\|v_i - u_i(t)\|_{L^\infty} \leq 1, \ i = 1, 2\,\big\}. \tag{4.22}$$

It is then easily seen that

$$\|\mathcal{B}(\boldsymbol{v}) - \mathcal{B}(\tilde{\boldsymbol{v}})\|_\star \ \leq\ C|\boldsymbol{v} - \tilde{\boldsymbol{v}}| \quad \text{for all } \boldsymbol{v}, \tilde{\boldsymbol{v}} \in T_{\boldsymbol{u}};$$

cf. [4]. Here $|\cdot|$ is the norm defined by

$$|\boldsymbol{u}| := \big(|u_1|^2 + |u_2|^2\big)^{1/2},$$

where $\boldsymbol{u} = (u_1, u_2) \in L^2(T^1) \times L^2(T^1)$, and the constant C depends on ν, η and the upper bound of the $L^\infty-$norm of the components of the exact solution $\boldsymbol{u}$; hence, $\mathcal{B}$ satisfies the local Lipschitz condition (1.3) with $\lambda = 0$.

Remark 4.3 (The tubes T_u)**.** We emphasize that in all applications in Subsections 4.2–4.4 the tubes T_u are defined in terms of norms weaker than the corresponding norm of V; see (4.6), (4.15) and (4.22).

Acknowledgment. Parts of this work were undertaken while the first–named author was Visiting Professor in the Department of Mathematics and Statistics, University of Cyprus. The authors are grateful to Professors Michel Crouzeix and Demetrios T. Papageorgiou for useful discussions concerning the abstract formulation and the dispersive–dissipative systems, respectively.

References

1. Akrivis, G., Crouzeix, M.: Linearly implicit methods for nonlinear parabolic equations. Math. Comp.**73**, 613–635 (2004)
2. Akrivis, G., Crouzeix, M., Makridakis, Ch.: Implicit–explicit multistep finite element methods for nonlinear parabolic problems. Math. Comp. **67**, 457–477 (1998)
3. Akrivis, G., Crouzeix, M., Makridakis, Ch.: Implicit–explicit multistep methods for quasilinear parabolic equations. Numer. Math. **82**, 521–541 (1999)
4. Akrivis, G., Papageorgiou, D.T., Smyrlis, Y.-S.: Linearly implicit methods for a semilinear parabolic system arising in two–phase flows. To appear in IMA J. Numer. Anal. (doi: 10.1093/imanum/drp034)
5. Akrivis, G., Smyrlis, Y.-S.: Implicit–explicit BDF spectral methods for the Kuramoto–Sivashinsky equation. Appl. Numer. Math. **51**, 151–169 (2004)
6. Benney, D.J.: Long waves on liquid films. J. Math. and Phys. **45**, 150–155 (1966)
7. Biagioni, H.A., Linares, F.: On the Benney–Lin and Kawahara equations. J. Math. Anal. Appl. **211**, 131–152, (1997)
8. Brezis, H.: Analyse Fonctionnelle, Théorie et Applications. Masson, Paris (1983)
9. Frenkel, A.L., Indireshkumar, K.: Wavy film flows down an inclined plane: perturbation theory and general evolution equation for the film thickness. Phys. Rev. E (3) **60**, 4143–4157 (1999)
10. Hairer E., Wanner, G.: Solving Ordinary Differential Equations II: Stiff and Differential–Algebraic Problems, 2nd rev. edn. Springer–Verlag, Berlin (1991)
11. Kas–Danouche, S., Papageorgiou, D.T., Siegel, M.: A mathematical model for core–annular flows with surfactants. Divulg. Mat. **12**, 117–138 (2004)
12. Kato, T.: Perturbation theory for linear operators, 2nd edn. Springer–Verlag, Berlin (1976)
13. Kawahara T., Toh, S.: On some properties of solutions to a nonlinear evolution equation including long–wavelength instability. Nonlinear wave motion, Pitman Monogr. Surveys Pure Appl. Math., vol. 43, pp. 95–117, Longman Sci. Tech., Harlow (1989)

14. Lang, J.: Adaptive Multilevel Solution of Nonlinear Parabolic PDE Systems. Theory, Algorithm, and Applications. Lecture Notes in Computational Science and Engineering, vol. 16, Springer–Verlag, Berlin (2001)
15. Lax, P.D.: Functional Analysis. Wiley-Interscience (John Wiley & Sons), New York (2002)
16. Lin, S.P.: Finite amplitude side–band stability of a viscous film. J. Fluid Mech. **63**, 417–429 (1974)
17. Lubich, C., Ostermann, A.: Linearly implicit time discretization of non-linear parabolic equations. IMA J. Numer. Anal. **15**, 555–583 (1995)
18. Papageorgiou, D.T., Maldarelli, C., Rumschitzki, D.S.: Nonlinear interfacial stability of cone–annular film flow. Phys. Fluids **A2**, 340–352 (1990)
19. Reed, M., Simon, B.: Methods of modern mathematical physics. I Functional analysis, 2nd edn. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York (1980)
20. Steihaug, T., Wolfbrandt, A.: An attempt to avoid exact Jacobian and nonlinear equations in the numerical solution of stiff differential equations. Math. Comp. **33**, 521–534 (1979)
21. Tadmor, E.: The well–posedness of the Kuramoto–Sivashinsky equation. SIAM J. Math. Anal. **17**, 884–893 (1986)
22. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems, 2nd edn. Springer–Verlag, Berlin (2006)
23. Topper, J., Kawahara, T.: Approximate equations for long nonlinear waves on a viscous fluid. J. Phys. Soc. Japan **44**, 663–666 (1978)
24. Zlámal, M.: Finite element methods for nonlinear parabolic equations. RAIRO **11**, 93–107 (1977)

Computer Science Department, University of Ioannina, 451 10 Ioannina, Greece
E-mail address: `akrivis@cs.uoi.gr`

Department of Mathematics and Statistics, University of Cyprus, P.O.Box 20537, 1678 Nicosia, Cyprus
E-mail address: `smyrlis@ucy.ac.cy`