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On the analyticity of certain dissipative-dispersive systems

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Abstract

We study the analyticity properties of solutions of dissipative-dispersive evolutionary equations possessing global attractors. We utilize an analyticity criterion for spatially periodic functions, that involves the rate of growth of the L^2-norm of the n^{th} derivative, as n tends to infinity. This criterion is applied to the dispersively modified Kuramoto–Sivashinsky equation and a general class of semilinear evolutionary pseudo-differential equations, under certain conditions, provided they possess global attractors. The proof is spectral and is fundamentally different from the semigroup approach in Collet *et al* [**3**]; it utilizes an inductive method to show that the analyticity criterion holds.

1. *Introduction*

This study presents analyticity properties of zero mean, spatially $2\pi-$periodic solutions of partial differential equations of the form

$$u_t + uu_x + \mathcal{P}u = 0, \tag{1.1}$$

possessing a global attractor (a compact attracting set). Here $\mathcal{P}$ is a linear pseudo–differential operator defined by its symbol in Fourier space, i.e.,

$$\big(\widehat{\mathcal{P}w}\big)_k = \lambda_k\, \widehat{w}_k, \quad k \in \mathbb{Z},$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \widehat{w}_k\, \mathrm{e}^{\mathrm{i}kx}$, and with λ_k satisfying

$$\mathrm{Re}\,\lambda_k \geq c_1|k|^\gamma, \quad \text{for all } |k| \geq k_0, \tag{1.2}$$

for some positive constants c_1, γ and k_0 a sufficiently large positive integer. Global existence of solutions of (1.1) has been established for $\gamma > 3/2$ (see [**11**]); when $\gamma \geq 2$, it can be deduced from [**4**] that equation (1.1) possesses a global attractor compact in every Sobolev norm. In this work we shall establish analyticity of solutions to (1.1) when $\gamma > 5/2$.

A special case of equation (1.1) is the dispersively modified Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u = 0, \tag{1.3}$$

with $\nu > 0$ and $\mathcal{D}$ a linear antisymmetric pseudo–differential operator; in Fourier space

$$(\widehat{\mathcal{D}w})_k = \mathrm{i}d_k\widehat{w}_k, \quad d_{-k} = -d_k \in \mathbb{R}, \tag{1.4}$$

i.e., $\mathcal{D}$ is dispersive. When $d_k = -k^3$ we obtain the Kawahara equation [**7**, **8**]; another application that emerges from the dynamics of two-phase core-annular flows yields d_k in terms of modified Bessel functions of the first kind [**10**]. Hence, the analysis presented here is applicable to a wide class of models describing different physical applications. Note that

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such spatially extended systems are typically defined on $L-$periodic domains and equations (1.1) and (1.3) have been scaled to have 2π periodicity. This rescaling provides a canonical equation with a "viscosity" parameter $\nu = (2\pi/L)^2$ in front of the highest derivative. It can be deduced from [**4**] that the $2\pi-$periodic solutions of (1.1) possess a global attractor, bounded in every Sobolev norm; in fact such proofs are possible for $\gamma \geq 2$ in (1.2). This Sobolev norm boundedness is used in our analyticity estimates to obtain a lower bound on the band of analyticity.

The present approach is distinct from that in [**3**] which uses semigroup methods on the $L-$periodic KS equation (a special case of (1.3) with $\mathcal{D} \equiv 0$),

$$u_t + uu_x + u_{xx} + u_{xxxx} \;=\; 0.$$

Given the bound (see for example [**2**, **6**, **5**, **9**])

$$\limsup_{t\to\infty} \int_0^L |u(x,t)|^2 dx \;\leq\; R_L^2,$$

the idea is to obtain a lower bound for αt so that the L^2-norm of $v := \mathrm{e}^{\alpha t \boldsymbol{A}} u$ stays bounded. Here $\boldsymbol{A}$ is the pseudo–differential operator, which is defined in the Fourier space as $(\widehat{\boldsymbol{A}u})_k = |k|\widehat{u}_k$, and thus if $u = \sum_{k\in\mathbb{Z}} \widehat{u}_k \mathrm{e}^{\mathrm{i}kqx}$, where $q = 2\pi/L$, then $v = \mathrm{e}^{\alpha t\boldsymbol{A}}u = \sum_{k\in\mathbb{Z}} \widehat{u}_k \mathrm{e}^{\mathrm{i}kqx+\alpha t|k|}$.

The analysis obtains a lower bound for αt written as $\alpha_L\, t_L \geq c\, R_L^{-2/5}$, with a positive constant c, and clearly better lower bounds for the size of the band of analyticity emerge when estimates for R_L are improved. The best available analytical estimate is due to Otto [**9**] who finds $R_L = \mathcal{O}(L^{1/2}(\log L)^{5/3})$, and hence the analyticity result can be expressed in terms of the decay of the Fourier modes $\hat{u}_k$,

$$|\hat{u}_k| = \mathcal{O}\big(\exp(-c\, L^{-1/5}(\log L)^{-2/3}\, q|k|)\big), \tag{1.5}$$

i.e., the strip of analyticity is

$$\beta_L \geq c\, L^{-1/5}(\log L)^{-2/3},$$

where c is a positive constant. For comparison purposes with our $2\pi-$periodic solutions, we have repeated the analysis of [**3**] to cast the result in terms of ν. We find that the width of the strip of analyticity, β_ν say, satisfies the bound $\beta_\nu \geq c\,\nu^{41/50}$.

Note that, numerical experiments in [**3**, **12**] suggest that the band of analyticity is independent of L. Interestingly, such numerically obtained optimal bounds persist even in the presence of dispersion, as shown in our computations [**1**]. In what follows we study the analyticity of $2\pi-$periodic solutions of equations (1.1). A special case of this is the dispersively modified KS equation (1.3), and bounds will be expressed in terms of ν. Note that in this case a change of variables affords an interchange between L and ν.

2. *An analyticity criterion*

A real analytic and periodic function $f : \mathbb{R} \to \mathbb{C}$ extends holomorphically in a neighborhood

$$\varOmega_\beta \;=\; \big\{x + \mathrm{i}y : x, y \in \mathbb{R} \;\text{ and }\; |y| < \beta\big\},$$

for some $\beta > 0$. The maximum such $\beta \in (0,\infty]$ is called the *band of analyticity* of f. For completeness, we say that the band of analyticity of f is zero if and only if f is not real analytic. Next we state an analyticity criterion for periodic functions which involves the L^p-norms of their derivatives. The proof is given in the Appendix.

THEOREM 1 **(Analyticity criterion)**. *Let* $u:\mathbb{R}\to\mathbb{C}$ *be an* $L-$*periodic* C^∞ *function and*

$$\mu := \limsup_{n\to\infty}\frac{\left\|u^{(n)}\right\|^{1/n}}{n},$$

with $\|\cdot\|$ *the* L^2-*norm over the interval* $[0,L]$*. Then the band of analyticity* β *of* u *is given by*

$$\beta = \begin{cases} \infty & \text{if} \quad \mu = 0,\\ \dfrac{1}{\mathrm{e}\mu} & \text{if} \quad \mu\in(0,\infty),\\ 0 & \text{if} \quad \mu=\infty. \end{cases}$$

REMARKS 1.

(*i*) According to Theorem 1, the rate of growth of $u^{(n)}$, as n tends to infinity, is in general super–exponential. This rate is exponential if and only if u is a trigonometric polynomial, as it can be readily seen.

(*ii*) Theorem 1 still holds if the L^2-norm of $u^{(n)}$ is replaced by the L^p-norm, for every $p\in[1,\infty]$. (This is explained in the proof of Theorem 1 in the Appendix.)

(*iii*) Another alternative version of our criterion in obtained when μ is defined by

$$\mu = \limsup_{s\to\infty}\frac{\|u\|_{\mathrm{H}^s}^{1/s}}{s},$$

where

$$\|u\|_{\mathrm{H}^s} = \left(\sum_{k\in\mathbb{Z}}(1+k^2)^s\left|\widehat{u}_k\right|^2\right)^{1/2},$$

with $\widehat{u}_k = \frac{1}{L}\int_0^L u(x)\,\mathrm{e}^{-\mathrm{i}kqx}\,dx$ and $q=2\pi/L$. See Remark A.1 in the Appendix.

3. *Analyticity of certain dissipative evolutionary systems*

3.1. *The dispersively modified Kuramoto–Sivashinsky equation*

We shall apply our analyticity criterion to $2\pi-$periodic solutions (with zero spatial mean) of (1.3), where $\nu>0$ and $\mathcal{D}$ is a linear antisymmetric pseudo–differential operator with symbol in Fourier space given by (1.4). The operator $\mathcal{D}$ is dispersive and equation (1.3) is known as the dispersively modified Kuramoto–Sivashinsky equation (DKSE). Well–posedness and global existence (in time) of solutions of (1.3) is established in [**11**]. Existence of a global attractor $\mathcal{X}$ (a compact absorbing set) can be derived from the results in [**4**]. In fact, when $t>0$, every solution of (1.3) becomes C^∞ with respect to x. In particular, for every $n\in\mathbb{N}$, there exists an R_n, depending on ν and $\mathcal{D}$ but independent of u_0, such that

$$\limsup_{t\to\infty}\left\|\partial_x^n u(\cdot,t)\right\| \le R_n.$$

Expressing $u(x,t)=\sum_{k\in\mathbb{Z}}\widehat{u}_k(t)\,\mathrm{e}^{\mathrm{i}kx}$, equation (1.3) is transformed into the following infinite dimensional dynamical system

$$\frac{d}{dt}\widehat{u}_k = -\lambda_k\widehat{u}_k - \mathrm{i}k\,\widehat{\varphi}_k, \quad k\in\mathbb{Z}, \tag{3.1}$$

with $\lambda_k = -k^2 + \nu k^4 - \mathrm{i}\, d_k$ and

$$\widehat{\varphi}_k(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}u^2(x,t)\,\mathrm{e}^{-\mathrm{i}kx}\,dx = \frac{1}{2}\sum_{j=1}^{k-1}\widehat{u}_j(t)\widehat{u}_{k-j}(t) + \sum_{j=1}^{\infty}\widehat{u}_{-j}(t)\widehat{u}_{k+j}(t). \tag{3.2}$$

Clearly, (3.1) implies that

$$\widehat{u}_k(t) \;=\; \mathrm{e}^{-\lambda_k t}\widehat{u}_k(0) - ik\int_0^t \mathrm{e}^{-\lambda_k(t-s)}\widehat{\varphi}_k(s)\,ds,$$

and consequently

$$\limsup_{t\to\infty}\big|\widehat{u}_k(t)\big| \;\le\; \frac{|k|}{\operatorname{Re}\lambda_k}\limsup_{t\to\infty}\big|\widehat{\varphi}_k(t)\big|, \tag{3.3}$$

whenever $\operatorname{Re}\lambda_k > 0$. We next define

$$h(s) \;=\; \limsup_{t\to\infty}\left(\sum_{k=1}^{\infty} k^{2s}\big|\widehat{u}_k(t)\big|^2\right)^{1/2}, \quad s\in\mathbb{R}.$$

Note that, if $n\in\mathbb{N}$ and $n\le s$, then

$$2^{1/2}h(s) \;\ge\; \limsup_{t\to\infty}\left(\sum_{k\in\mathbb{Z}} k^{2n}\big|\widehat{u}_k(t)\big|^2\right)^{1/2} \;=\; \limsup_{t\to\infty}\big\|\partial_x^n u(\cdot,t)\big\|.$$

Also,

$$\limsup_{t\to\infty}\big|\widehat{u}_m(t)\big| \;\le\; \frac{h(s)}{|m|^s}, \quad \text{for all } m\in\mathbb{Z}\smallsetminus\{0\}\,. \tag{3.4}$$

Our target is to show the following:

Claim I. *There exist positive constants M and a, such that, for every $s\ge 0$,*

$$h(s) \;\le\; M(as)^s. \tag{3.5}$$

This result in turn implies that

$$\limsup_{n\to\infty}\left(\frac{1}{n}\limsup_{t\to\infty}\big\|\partial_x^n u(\cdot,t)\big\|^{1/n}\right) \;\le\; \limsup_{n\to\infty}\frac{2^{1/(2n)}h^{1/n}(n)}{n} \;\le\; a.$$

By using our analyticity criterion we shall consequently obtain a lower bound for the band of analyticity β of solutions u in the attractor, namely $\beta\ge 1/(\mathrm{e}a)$.

The claim will be proved by the following inductive method:

First, we pick $M, a>0$, so that $h(s)\le M(as)^s$, for every $s\in[0,2]$. Suitable values are, for example,

$$M \ge 2^{1/2}R_2 = 2^{1/2}\limsup_{t\to\infty}\|u_{xx}(\cdot,t)\| \quad \text{and} \quad a \;\ge\; 1.$$

Indeed, noting that $(as)^s \ge \mathrm{e}^{-1/(ea)} > 1/2$, for all $a\ge 1$ and $s\ge 0$, we obtain

$$M(as)^s > \frac{M}{2} \ge \frac{1}{\sqrt{2}}\limsup_{t\to\infty}\|u_{xx}(\cdot,t)\| = \limsup_{t\to\infty}\left(\sum_{k=1}^{\infty} k^4\big|\widehat{u}_k(t)\big|^2\right)^{1/2} = h(2)\ge h(s),$$

for all $s\in[0,2]$. Next we shall prove (by selecting a possibly larger a) that (3.5) holds for every $s\in[\sigma,\sigma+1]$, provided that the same inequality holds for every $s\in[0,\sigma]$ and $\sigma\ge 2$. This in turn establishes that (3.5) holds for every $s\ge 0$. It suffices to show the following:

Claim II. *If* (3.5) *holds for every $s\in[0,\sigma]$ and $\sigma\ge 1$, then it also holds for $s=\sigma+1$.*

Proof of Claim II. For every $j=1,\ldots,k-1,$ we have, by virtue of (3.4),

$$\limsup_{t\to\infty}\left|\widehat{u}_j(t)\right| \le \frac{h(\frac{\sigma j}{k})}{j^{\frac{\sigma j}{k}}} \le \frac{M(a\frac{\sigma j}{k})^{\frac{\sigma j}{k}}}{j^{\frac{\sigma j}{k}}},$$

and thus, the first sum in the right-hand side of (3.2) is estimated as follows

$$\begin{aligned}\limsup_{t\to\infty}\sum_{j=1}^{k-1}\left|\widehat{u}_j(t)\right|\left|\widehat{u}_{k-j}(t)\right| &\le \sum_{j=1}^{k-1}\frac{h(\frac{\sigma j}{k})}{j^{\frac{\sigma j}{k}}}\cdot\frac{h\left(\frac{\sigma(k-j)}{k}\right)}{(k-j)^{\frac{\sigma(k-j)}{k}}}\\ &\le \sum_{j=1}^{k-1}\frac{M(a\frac{\sigma j}{k})^{\frac{\sigma j}{k}}}{j^{\frac{\sigma j}{k}}}\cdot\frac{M\left(a\frac{\sigma(k-j)}{k}\right)^{\frac{\sigma(k-j)}{k}}}{(k-j)^{\frac{\sigma(k-j)}{k}}}\\ &= \frac{(k-1)M^2(a\sigma)^\sigma}{k^\sigma} \le \frac{M^2(a\sigma)^\sigma}{k^{\sigma-1}}. \end{aligned}\tag{3.6}$$

For the second sum in the right-hand side of (3.2), using inequality (3.4) and the fact that $\left|\widehat{u}_{-j}(t)\right| = \left|\widehat{u}_j(t)\right|$, we obtain that

$$\begin{aligned}\limsup_{t\to\infty}\sum_{j=1}^{\infty}\left|\widehat{u}_j(t)\right|\left|\widehat{u}_{k+j}(t)\right| &\le \limsup_{t\to\infty}\left(\sum_{j=1}^{\infty}\left|\widehat{u}_j(t)\right|^2\right)^{1/2}\limsup_{t\to\infty}\left(\sum_{j=1}^{\infty}\left|\widehat{u}_{k+j}(t)\right|^2\right)^{1/2}\\ &\le h(0)\left(\sum_{j=1}^{\infty}\frac{h^2(\sigma)}{(k+j)^{2\sigma}}\right)^{1/2} \le M\,h(\sigma)\left(\int_0^\infty\frac{dx}{(x+k)^{2\sigma}}\right)^{1/2}\\ &\le M^2(a\sigma)^\sigma\left(\frac{1}{2\sigma-1}\cdot\frac{1}{k^{2\sigma-1}}\right)^{1/2} = \frac{M^2(a\sigma)^\sigma}{(2\sigma-1)^{1/2}k^{\sigma-1/2}}\\ &\le \frac{M^2(a\sigma)^\sigma}{k^{\sigma-1}}. \end{aligned}\tag{3.7}$$

In arriving at the result above, we have used the fact

$$\limsup_{t\to\infty}\sum_{j=1}^{\infty}\left|\widehat{u}_{k+j}(t)\right|^2 \le \sum_{j=1}^{\infty}\limsup_{t\to\infty}\left|\widehat{u}_{k+j}(t)\right|^2 \le \sum_{j=1}^{\infty}\frac{h^2(\sigma)}{(k+j)^{2\sigma}},$$

along with (3.4).

Also, since $\operatorname{Re}\lambda_k = -k^2+\nu k^4$, we have

$$\operatorname{Re}\lambda_k \ge \nu k^4/2 \quad \text{for } k \ge k_0 = \left[(2/\nu)^{1/2}\right]+1. \tag{3.8}$$

Combination of (3.3), (3.6), (3.7) and (3.8) provides that

$$\limsup_{t\to\infty}\left|\widehat{u}_k(t)\right| \le \frac{3M^2(a\sigma)^\sigma}{\nu k^{\sigma+2}} \quad \text{for } k \ge k_0.$$

Thus,

$$\begin{aligned}\limsup_{t\to\infty}\sum_{k=1}^{\infty}k^{2\sigma+2}\left|\widehat{u}_k(t)\right|^2 &\le \sum_{k=k_0}^{\infty}\frac{9M^4(a\sigma)^{2\sigma}}{\nu^2k^2}+\limsup_{t\to\infty}\sum_{k=1}^{k_0-1}k^{2\sigma+2}\left|\widehat{u}_k(t)\right|^2\\ &\le \frac{9M^4(a\sigma)^{2\sigma}}{\nu^2}\sum_{k=k_0}^{\infty}\frac{1}{k^2}+(k_0-1)^{2\sigma-2}\limsup_{t\to\infty}\sum_{k=1}^{k_0-1}k^4\left|\widehat{u}_k(t)\right|^2\\ &\le \frac{9M^4(a\sigma)^{2\sigma}}{\nu^2}\cdot\frac{1}{k_0-1}+(k_0-1)^{2\sigma-2}R_2^2. \end{aligned}$$

Since $h^2(\sigma+1) = \limsup_{t\to\infty} \sum_{k=1}^{\infty} k^{2\sigma+2}\big|\widehat{u}_k(t)\big|^2$, we have

$$h(\sigma+1) \;\le\; \frac{3M^2(a\sigma)^\sigma}{(k_0-1)^{1/2}\nu} + (k_0-1)^{\sigma-1}M \;\le\; \frac{6M^2(a\sigma)^\sigma}{\nu^{3/4}} + \left(\frac{2}{\nu}\right)^{\frac{\sigma-1}{2}} M,$$

since $\sqrt{2}\nu^{-1/2} \ge k_0-1 \ge \frac{1}{2}\nu^{-1/2}$. (The second inequality is valid if and only if $\nu \le 2$.) This inductive step is complete if we can find positive constants M and a satisfying

$$\frac{6M^2(a\sigma)^\sigma}{\nu^{3/4}} + \left(\frac{2}{\nu}\right)^{\frac{\sigma-1}{2}} M \;\le\; M\big(a(\sigma+1)\big)^{\sigma+1} \quad \text{for every } \sigma \ge 1. \tag{3.9}$$

Clearly, for every $M>0$, there exists an $a_0>0$, such that (3.9) holds for every $a \ge a_0$. It can be verified that a suitable a_0 is given by

$$a_0 \;=\; c_1 \max\left\{\nu^{-1/2},\, M\nu^{-3/4}\right\},$$

where $M = 2R_2$ and c_1 is a positive constant independent of ν. (This is achieved by making the right-hand side of (3.9) larger than half of each term of the left-hand side, independently.) Therefore, the following has been proved:

THEOREM 2. *Let $\mathcal{X}$ be the global attractor of* (1.3) *for $2\pi-$periodic initial data in* L^2,

$$R_2 \;=\; \sup_{w\in\mathcal{X}} \|\partial_x^2 w\| \quad \text{and} \quad \varrho \;=\; \max\left\{\nu^{-1/2}, R_2\nu^{-3/4}\right\}.$$

Then, every $w\in\mathcal{X}$ extends to a holomorphic function in Ω_β, where $\beta \ge c/\varrho$, for a suitable positive constant c independent of ν. □

3.2. *A class of nonlinear evolutionary pseudo–differential equations*

The technique developed above applies also to a wider class of dissipative evolutionary equations, defined by (1.1), which possess global attractors. It can be derived from [**4**] that periodic solutions of (1.1), with the eigenvalues λ_k of the operator $\mathcal{P}$ satisfying (1.2) with $\gamma \ge 2$, possess a global attractor, bounded in every Sobolev norm. Hence, if $\gamma > 5/2$, a global attractor exists, and it is readily seen that the technique developed to establish the analyticity for DKSE applies, with minor modifications, to the solutions of (1.1). Perhaps the only noteworthy modification of the method is that Claim II will be replaced by:

CLAIM II. *If* (3.5) *holds for every $s \in [0,\sigma]$ and $\sigma \ge 1$, then it also holds for $s = \sigma+\sigma_1$, where $\sigma_1 \in (0, \gamma-5/2)$.*

Appendix

Proof of Theorem 1

Clearly, if $1 \le p \le q \le \infty$, then there exists a positive constant $c_{p,q}$, such that

$$\big\|u^{(n)}\big\|_p \;\le\; L^{\frac{1}{p}-\frac{1}{q}}\big\|u^{(n)}\big\|_q \;\le\; c_{p,q}\big\|u^{(n+1)}\big\|_p, \tag{A.1}$$

for every $n \ge 1$ and $u\in \mathrm{C}^\infty(\mathbb{R})$, which is $L-$periodic, where $\|\cdot\|_p$ is the L^p-norm over the interval $[0,L]$. It is readily seen that (A.1) implies

$$\limsup_{n\to\infty} \frac{\big\|u^{(n)}\big\|_p^{1/n}}{n} \;=\; \limsup_{n\to\infty} \frac{\big\|u^{(n)}\big\|_q^{1/n}}{n}. \tag{A.2}$$

Formula (A.2) implies that it suffices to show the Theorem for the $\infty-$norm, instead of the 2-norm. Combining Stirling's formula, $\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \mathrm{e}$, with (A.2) we obtain

$$\widetilde{\mu} = \limsup_{n\to\infty} \left(\frac{\|u^{(n)}\|_\infty}{n!}\right)^{1/n} = \limsup_{n\to\infty} \frac{n}{(n!)^{1/n}} \cdot \frac{\left(\|u^{(n)}\|_\infty\right)^{1/n}}{n} = \mathrm{e}\mu.$$

Therefore, in order to prove our analyticity criterion (Theorem 1) it suffices to establish the following two Claims:

Claim I. *If* $\widetilde{\mu} < \infty$ *and* $\gamma := \begin{cases} \infty & \text{if} \quad \widetilde{\mu} = 0, \\ \dfrac{1}{\widetilde{\mu}} & \text{if} \quad \widetilde{\mu} > 0, \end{cases}$ *then* u *extends holomorphically in* Ω_γ.

Claim II. *If* $\gamma \in (0,\infty)$ *and* u *extends holomorphically in* Ω_γ, *then* $\widetilde{\mu} \leq 1/\gamma$.

Proof of Claim I. It can be readily seen that the function

$$U(x+\mathrm{i}y) = \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!}(\mathrm{i}y)^n$$

is well defined (cf. $n^{\text{th}}-$root test for series) and differentiable, with respect to x and y, for every $(x,y)\in\mathbb{R}\times(-\gamma,\gamma)$, and satisfies the Cauchy–Riemann equations ($U_y = \mathrm{i}U_x$). Therefore, U is holomorphic in Ω_γ, and since $U(x) = u(x)$, for $x\in\mathbb{R}$, then u extends holomorphically in Ω_γ.

Proof of Claim II. Let U be holomorphic in Ω_γ and agree with u in $\mathbb{R}$, and $\varepsilon\in(0,\gamma)$. Set $M_\varepsilon = \max\{|U(x+\mathrm{i}y)| : x \in [0,L] \text{ and } |y| \leq \gamma-\varepsilon\}$. We have $M_\varepsilon = \sup_{z\in\overline{\Omega}_{\gamma-\varepsilon}} |U(z)|$, since U is also $L-$periodic. Also, for every $x\in\mathbb{R}$ and $n\in\mathbb{N}$, we have

$$u^{(n)}(x) = U^{(n)}(x) = \frac{n!}{2\pi\mathrm{i}} \int_{|z-x|=\gamma-\varepsilon} \frac{U(z)}{(z-x)^{n+1}}\,dz, \quad \text{whence} \quad \left|u^{(n)}(x)\right| \leq \frac{n! M_\varepsilon}{(\gamma-\varepsilon)^n},$$

and thus

$$\widetilde{\mu} = \limsup_{n\to\infty} \left(\frac{\left\|u^{(n)}\right\|_\infty}{n!}\right)^{1/n} \leq \frac{1}{\gamma-\varepsilon},$$

for every $\varepsilon\in(0,\gamma)$. Consequently $\widetilde{\mu} \leq 1/\gamma$. □

Remark A.1. Another alternative version of this analyticity criterion is obtained when the L^2-norm is replaced by suitable Sobolev norms, due to the fact that

$$\limsup_{s\to\infty} \frac{\|u\|_{\mathrm{H}^s}^{1/s}}{s} = \limsup_{n\to\infty} \frac{\left\|u^{(n)}\right\|^{1/n}}{n} = \mu. \tag{A.3}$$

In order to prove this, we first observe that

$$\limsup_{s\to\infty} \frac{\|u\|_{\mathrm{H}^s}^{1/s}}{s} \geq \limsup_{n\to\infty} \frac{\|u^{(n)}\|^{1/n}}{n},$$

since $\|u\|_{\mathrm{H}^n} \geq \|u^{(n)}\|$. On the other hand, the definition of μ implies that, for every $\varepsilon > 0$, there exists an $M_\varepsilon > 0$, such that

$$\left\|u^{(n)}\right\| \leq M_\varepsilon\big(n(\mu+\varepsilon)\big)^n, \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}\|u\|_{\mathrm{H}^n}^2 &= \sum_{k\in\mathbb{Z}}(1+k^2)^n\big|\widehat{u}_k\big|^2 = \sum_{k\in\mathbb{Z}}\sum_{j=0}^n\binom{n}{j}k^{2j}\big|\widehat{u}_k\big|^2 = \sum_{j=0}^n\binom{n}{j}\big\|u^{(j)}\big\|^2\\ &\le \sum_{j=0}^n\binom{n}{j}M_\varepsilon^2\big(j(\mu+\varepsilon)\big)^{2j} \le M_\varepsilon^2\sum_{j=0}^n\binom{n}{j}\big(n(\mu+\varepsilon)\big)^{2j} = M_\varepsilon^2\Big(1+\big(n(\mu+\varepsilon)\big)^2\Big)^n,\end{aligned}$$

and consequently

$$\frac{\|u\|_{\mathrm{H}^n}^{1/n}}{n} \le \frac{M_\varepsilon^{1/n}\Big(1+\big(n(\mu+\varepsilon)\big)^2\Big)^{1/2}}{n} \longrightarrow \mu+\varepsilon, \quad \text{for every } \varepsilon>0.$$

Thus $\dfrac{\|u\|_{\mathrm{H}^n}^{1/n}}{n} \le \mu$. The treatment of the case of general $s>0$ is straight–forward.

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