FINITE DIFFERENCE DISCRETIZATIONS
OF SOME INITIAL AND BOUNDARY VALUE PROBLEMS
WITH INTERFACE

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ABSTRACT. We analyze the discretization of initial and boundary value problems with a stationary interface in one space dimension for the heat equation, the Schrödinger equation, and the wave equation by finite difference methods. Extending the concept of the elliptic projection, well known from the analysis of Galerkin finite element methods, to our finite difference case, we prove second-order error estimates in space and time in the ℓ^2 norm.

1. INTRODUCTION

Let $\rho > 0$ and $x^* \in (0,1)$. Given $0 < T < \infty$, we consider the following initial and boundary value problem for the heat equation in one space dimension with a stationary (straight) interface. We seek a real-valued function $u = u(x,t)$ for $(x,t) \in [0,1] \times [0,T]$ that satisfies

$$
\begin{aligned}
&u_t = \big(a(x)u_x\big)_x + b(x,t)u && \text{in } \big([0,x^*) \cup (x^*,1]\big) \times [0,T], \\
&a(x^*-)u_x(x^*-,t) = \rho a(x^*+)u_x(x^*+,t), && t \in [0,T], \\
&u(x,0) = v^0(x), && x \in [0,1], \\
&u(0,t) = u(1,t) = 0, && t \in [0,T].
\end{aligned}
$$

(1.1)

Here, $a(x)$ is a positive function on $[0,1]$, smooth on $[0,x^*]$ and on $[x^*,1]$, with a possible discontinuity at x^*, and $b(x,t)$ is nonpositive, smooth on $[0,x^*] \times [0,T]$ and on $[x^*,1] \times [0,T]$, with a possible discontinuity on $\{x^*\} \times [0,T]$. We assume that the coefficients and the initial value v^0 are such that the problem (1.1) possesses a unique solution, *continuous* on $[0,1] \times [0,T]$ and sufficiently smooth for our purposes on $[0,x^*] \times [0,T]$ and on $[x^*,1] \times [0,T]$.

We shall approximate the solution of (1.1) by an implicit finite difference scheme, second-order accurate in space and time. In space, we partition the intervals $[0,x^*]$, resp. $[x^*,1]$, into uniform meshes with meshlengths h_-, resp. h_+. Specifically, we let $J, m \in \mathbb{N}$ and h_-, h_+ be such that $x^* = mh_-$, $mh_- + (J+1-m)h_+ = 1$, and set $x_j := jh_-$, $j = 0, \ldots, m$, —in particular $x_m = x^*$— and $x_j := (j-m)h_+ + x_m$, $j = m+1, \ldots, J+1$. We also let $k := T/N$ and $t^n := nk$, $n = 0, \ldots, N$, for $N \in \mathbb{N}$. If u is the solution of (1.1), denote $u_j^n := u(x_j, t^n)$, $j = 0, \ldots, J+1$, $n = 0, \ldots, N$, and

$u^n := (u_0^n, \ldots, u_{J+1}^n)^T \in \mathbb{R}_0^{J+2}$, where $\mathbb{R}_0^{J+2} := \{v : v = (v_0, \ldots, v_{J+1})^T \in \mathbb{R}^{J+2}, v_0 = v_{J+1} = 0\}$.

Our finite difference method, given by (3.1) in §3 below, produces vectors $U^n = (U_0^n, \ldots, U_{J+1}^n)^T \in \mathbb{R}_0^{J+2}$ for $n = 0, \ldots, N$ that approximate u^n and is based in space on a standard difference approximation of (1.1) (cf. [13]), which to the left and right of the interface $x_m = x^*$ reduces to a simple, locally second-order accurate, centered scheme. Across the interface it becomes a standard simple relation involving the values of U^n at the points x_{m-1}, x_m, x_{m+1}. The time stepping is effected by a Crank–Nicolson procedure. Hence, the scheme requires solving a $J \times J$ tridiagonal linear system of equations at each time step.

In §3 we analyze the convergence of this method in a weighted ℓ^2 norm defined, as is natural to the problem, by

$$\|v\|_h := \left\{ h_- \sum_{j=1}^{m-1} |v_j|^2 + \hat{h}|v_m|^2 + \rho h_+ \sum_{j=m+1}^{J} |v_j|^2 \right\}^{1/2},$$

where we henceforth use the notation $\hat{h} := \frac{1}{2}(h_- + \rho h_+)$. $\|\cdot\|_h$ is given here in general for *complex* vectors in $\mathbb{C}_0^{J+2} := \{v : v = (v_0, \ldots, v_{J+1})^T \in \mathbb{C}^{J+2}, v_0 = v_{J+1} = 0\}$. We shall use complex-valued vectors for approximating the solution of an analogous problem for the Schrödinger equation in §4. In Theorem 3.1 we prove the optimal-order error estimate

$$(1.2) \qquad \max_n \|U^n - u^n\|_h \leq c(k^2 + h^2),$$

where $h := \max(h_-, h_+)$ and c is a constant independent of k and h.

The analysis of numerical methods of finite element and finite difference type for time-dependent interface problems, such as (1.1), has often been considered in the literature. For an error analysis of a finite difference semidiscretization that yields optimal-order error estimates in space, cf. Budak [3]. For full finite difference discretizations see, for instance, Isaacson [5], Samarskii and Fryazinov [10], and Samarskii [8], [9]. Isaacson [5] analyzes the Euler and backward Euler schemes by maximum principle methods and obtains optimal-order error estimates under a restriction implying $k = O(h^2)$. For Crank–Nicolson type schemes, Samarskii and Fryazinov in [10] prove an estimate of $O(k^2 + h^{3/2})$, while in [8], [9] Samarskii obtains the optimal-order error estimate. (Strictly speaking, the problem considered in [8]–[10] is the discretization of equations with discontinuous coefficients, possibly on nonuniform meshes. However the techniques of these papers could be ostensibly extended and applied to the problem at hand.)

The main contribution of the present paper is an apparently new technique for obtaining the optimal-order error estimate (1.2) and, in particular, the optimal-order term h^2 instead of $h^{3/2}$; the latter bound can be derived in a straightforward manner and is due of course to the interface condition. Our technique depends on considering first an associated two-point boundary value problem whose (complex-valued in general)

solution w, continuous on $[0, 1]$, satisfies

$$
\begin{aligned}
&(a(x)w')' = f(x) \quad \text{in} \ \ [0, x^*) \cup (x^*, 1], \\
&a(x^*-)w'(x^*-) = \rho a(x^*+)w'(x^*+), \\
&w(0) = w(1) = 0.
\end{aligned}
$$

(1.3)

In (1.3), a is as in (1.1), and f is taken to be complex-valued in general (for the purposes of §4), smooth on $[0, x^*]$ and on $[x^*, 1]$, with a possible discontinuity at x^*; w will be supposed to be smooth enough for the purposes of its numerical approximation on $[0, x^*]$ and on $[x^*, 1]$. In §2 we discretize (1.3) by the analog of the scheme used in the space differencing of (1.1) and prove second-order convergence of the resulting discrete approximation to w in the discrete maximum norm and in a discrete H^1 norm. This result is standard, and our proof uses ideas from Samarskii [11, pp. 78–82]. (See also Gartland [4] for a recent treatment of finite difference methods for two-point boundary value problems with interface.) The apparently new step here is to use these estimates in the proof of the error bound (1.2) in Theorem 3.1, where, motivated by the *elliptic projection* concept, well known from the error analysis of Galerkin finite element methods (cf. Wheeler [14]), we define a finite difference analog, an *elliptic approximation* W to the solution u of (1.1), which satisfies the finite difference scheme for a two-point boundary value problem of the form (1.3) with suitable right-hand side depending on u. We then compare U^n to W^n, and the proof of (1.2) is concluded in the standard way by the energy method, with analogous estimates to those of the error analysis of Galerkin methods for parabolic problems (cf. Thomée [12]). It should be pointed out that Samarskii in [8] and [9] had considered comparing the finite-difference approximation of the parabolic problem to the solution of associated discrete stationary problems with suitable right-hand sides, obtained by splitting the truncation error. His approximations, however, differ from the elliptic approximation used here.

In §§4 and 5 the elliptic approximation technique is applied to prove estimates of the form (1.2) for finite difference approximations to the solutions of two other time-dependent problems with stationary interface. In §4 we consider the following initial and boundary-value problem for the *Schrödinger equation*. Using the notation of (1.1), we seek a complex-valued $u = u(x, t)$ for $(x, t) \in [0, 1] \times [0, T]$ that satisfies

$$
\begin{aligned}
&u(t) = i\alpha u_{xx} + i\beta(x, t)u \quad \text{in} \ \ \big([0, x^*) \cup (x^*, 1]\big) \times [0, T], \\
&u_x(x^*-, t) = \rho u_x(x^*+, t), \quad t \in [0, T], \\
&u(x, 0) = v^0(x), \qquad\qquad x \in [0, 1], \\
&u(0, t) = u(1, t) = 0, \qquad t \in [0, T].
\end{aligned}
$$

(1.4)

Here, i is the imaginary unit, α is a nonzero real constant, and $\beta(x, t)$ is real-valued, smooth on $[0, x^*] \times [0, T]$ and on $[x^*, 1] \times [0, T]$, with a possible discontinuity on $\{x^*\} \times [0, T]$. We assume that the coefficients and the (complex-valued) initial data v^0 are such that (1.4) has a unique solution, continuous on $[0, 1] \times [0, T]$ and smooth on $[0, x^*] \times [0, T]$ and on $[x^*, 1] \times [0, T]$. (1.4) occurs in underwater acoustics, as a two-layer horizontal interface problem with a horizontal boundary at $x = 1$ and a free surface at $x = 0$,

for the Schrödinger equation viewed as 'parabolic' approximation to the Helmholtz equation. (For simplicity, we have assumed homogeneous Dirichlet boundary conditions at the endpoints; Neumann or mixed, and also nonhomogeneous, boundary conditions can be analyzed as well, with no additional conceptual complications.) McDaniel and Lee (cf., e.g., [6], [7]) have studied implicit, Crank–Nicolson type finite difference approximations for this problem and used them in computations. In fact, our main motivation for the present work was to prove an error estimate of the type (1.2) for the McDaniel–Lee difference scheme (cf. [7]) for which numerical experiments clearly indicate second-order rate of convergence in x and t. (Actually, our method is slightly different from the one in [7] in that we use a different evaluation of the coefficient $\beta(x, t)$ to render the scheme conservative. However, the proof of optimal-order convergence for the scheme of [7] is very similar to the one for the scheme at hand.) The proof of (1.2) in the present case of the Schrödinger equation follows broadly the lines of the analogous proof for the heat equation and uses energy techniques similar to the ones that we used for Galerkin methods in [1].

Finally, in §5, we consider the analogous problem for the *wave equation*. We seek again a real-valued $u = u(x, t)$ for $(x, t) \in [0, 1] \times [0, T]$, satisfying

$$
\begin{aligned}
& u_{tt} = \big(a(x)u_x\big)_x + b(x, t)u && \text{in } \big([0, x^*) \cup (x^*, 1]\big) \times [0, T], \\
& a(x^*-)u_x(x^*-, t) = \rho a(x^*+)u_x(x^*+, t), && t \in [0, T], \\
(1.5) \quad & u(x, 0) = v^0(x), && x \in [0, 1], \\
& u_t(x, 0) = v^1(x), && x \in [0, 1], \\
& u(0, t) = u(1, t) = 0, && t \in [0, T].
\end{aligned}
$$

Here the notation and assumptions on $a(x)$ and $b(x, t)$ are exactly the same as in the case of the heat equation (1.1). The (real-valued) initial data v^0, v^1 and the coefficients are again assumed to be such that a unique solution u of (1.5) exists and is continuous on $[0, 1] \times [0, T]$ and sufficiently smooth on $[0, x^*] \times [0, T]$ and on $[x^*, 1] \times [0, T]$. We approximate (1.5) by the classical explicit $5-$point Courant–Friedrichs–Lewy difference scheme, modified at the interface x^* as in the previous sections, and prove again an error estimate of the form (1.2) under the stability condition, of course, that $\sqrt{|a|_\infty}\, k/\min(h_-, h_+) \le r_0$ for any fixed $0 < r_0 < 1$. (Here, and in the sequel, $|\cdot|_\infty$ will denote the $L^\infty(0, 1)$ norm.) Now the energy estimate is mildly more complicated, owing to the presence of the time-dependent coefficient $b(x, t)$; again we use ideas from analogous proofs for Galerkin methods, here from Bales, Dougalis and Serbin [2].

It should be pointed out that our technique of comparing the finite difference solution U^n to its elliptic approximation W^n is a general device whose application is not limited to interface problems only. For example, it can be used to prove optimal-order convergence of finite difference approximations to initial and boundary value problems for *nonlinear* PDE's; its application there circumvents many tedious computations involving Taylor expansions.

2. A TWO-POINT BOUNDARY VALUE PROBLEM

In this section we consider the discretization of problem (1.3) by a standard three-point finite difference scheme modified at the interface node x^*. We derive second-order accurate error estimates in the discrete maximum norm and in a discrete H^1 norm. These results will be used later for deriving error estimates for problems (1.1), (1.4), and (1.5).

In the sequel we shall find it convenient to employ difference quotient notation that conforms to the spatial mesh $\{x_j\}, 0 \leq j \leq J+1$, introduced in §1, and takes into account the interface at $x^* = x_m$. For a real or complex $J+2-$vector $(v_0, v_1, \ldots, v_{J+1})^T$ we define first the forward and backward difference quotients ∂v_j, $\bar{\partial} v_j$ as

$$\partial v_j = \begin{cases} (v_{j+1} - v_j)/h_- & \text{if } j = 0, \ldots, m-1, \\ (v_{j+1} - v_j)/h_+ & \text{if } j = m, \ldots, J, \end{cases}$$

$$\bar{\partial} v_j = \begin{cases} (v_j - v_{j-1})/h_- & \text{if } j = 1, \ldots, m, \\ (v_j - v_{j-1})/h_+ & \text{if } j = m+1, \ldots, J+1. \end{cases}$$

Consequently, centered second difference quotients that approximate the function $(a(x) v'(x))'$ at $x_j \neq x_m$ will be denoted by

$$\bar{\partial}\big(a_{j+1/2}\partial v_j\big) = \begin{cases} \big[a_{j+1/2}(v_{j+1} - v_j) - a_{j-1/2}(v_j - v_{j-1})\big]/h_-^2 & \text{if } j = 1, \ldots, m-1, \\ \big[a_{j+1/2}(v_{j+1} - v_j) - a_{j-1/2}(v_j - v_{j-1})\big]/h_+^2 & \text{if } j = m+1, \ldots, J, \end{cases}$$

where we put $a_{j+1/2} = a\big((x_j + x_{j+1})/2\big)$ for $j = 0, \ldots, J$. At the interface we let

$$\delta_\rho\big(a_{m+1/2}\partial v_m\big) := \left\{\rho a_{m+1/2}\Big(\frac{v_{m+1} - v_m}{h_+}\Big) - a_{m-1/2}\Big(\frac{v_m - v_{m-1}}{h_-}\Big)\right\}\Big/\hat{h}.$$

Now let $w_j = w(x_j), j = 0, \ldots, J+1$, where $w(x)$ is the solution of (1.3). We approximate $(w_0, \ldots, w_{J+1})^T$ by $W \in \mathbb{C}_0^{J+2}$, defined by the equations

(2.1)
$$\bar{\partial}\big(a_{j+1/2}\partial W_j\big) = f(x_j) \quad \text{if } 1 \leq j \leq J, \ j \neq m,$$
$$\delta_\rho\big(a_{m+1/2}\partial W_m\big) = \hat{f}(x^*),$$

where we put $\hat{f}(x^*) := \big[h_+\rho f(x^*+) + h_- f(x^*-)\big]/(2\hat{h})$. The $J \times J$ tridiagonal matrix of the linear system (2.1) is obviously invertible, i.e., W is uniquely defined. Introduce in $\mathbb{C}_0^{J+2}$ the sesquilinear form $a_h(\cdot, \cdot)$ by

$$a_h(v, u) := -\Big[h_- \sum_{j=1}^{m-1} \bar{\partial}\big(a_{j+1/2}\partial v_j\big)\bar{u}_j + \hat{h}\delta_\rho\big(a_{m+1/2}\partial v_m\big)\bar{u}_m + \rho h_+ \sum_{j=m+1}^{J} \bar{\partial}\big(a_{j+1/2}\partial v_j\big)\bar{u}_j\Big],$$

where the bar over u_j denotes complex conjugation. It is easily seen that

(2.2) $$\forall v \in \mathbb{C}_0^{J+2} \quad a_h(v, v) = h_- \sum_{j=0}^{m-1} a_{j+1/2}|\partial v_j|^2 + \rho h_+ \sum_{j=m}^{J} a_{j+1/2}|\partial v_j|^2,$$

(2.3) $$\forall v \in \mathbb{C}_0^{J+2} \quad a_h(v, v) \leq \frac{4|a|_\infty}{\big[\min(h_-, h_+)\big]^2}\|v\|_h^2,$$

and that $a_h(\cdot, \cdot)$ is an inner product in $\mathbb{C}_0^{J+2}$.

Further, letting $\rho^* := \max(1, 1/\rho)$, and $\underline{a} := \min_{0 \le x \le 1} a(x)$, we see that the following discrete Sobolev-type inequality holds:

$$(2.4) \qquad \forall v \in \mathbb{C}_0^{J+2} \quad \max_j |v_j|^2 \le (\rho^*/\underline{a}) a_h(v, v).$$

In fact, for $s \le m$ we have

$$2|v_s|^2 = \Big| \sum_{j=0}^{s-1} (v_{j+1} - v_j) \Big|^2 + \Big| \sum_{j=s}^{J} (v_{j+1} - v_j) \Big|^2$$

$$\le s \sum_{j=0}^{s-1} |v_{j+1} - v_j|^2 + 2(m - s) \sum_{j=s}^{m-1} |v_{j+1} - v_j|^2 + 2(J + 1 - m) \sum_{j=m}^{J} |v_{j+1} - v_j|^2$$

$$\le 2(\rho^*/\underline{a}) a_h(v, v).$$

A similar inequality holds for $s > m$, and the result follows. A trivial consequence of (2.4), that will be used in §5, is that

$$(2.5) \qquad \exists c > 0 \ \forall v \in \mathbb{C}_0^{J+2} \quad a_h(v, v) \ge c \|v\|_h^2.$$

Let $e_j := w_j - W_j, j = 0, \ldots, J + 1$. Two approximation results for W are given in the following lemma.

Lemma 2.1. *Let the solution w of (1.3) be continuous in $[0, 1]$ and smooth in $[0, x^*]$ and in $[x^*, 1]$. Then, there exists a constant C, independent of h and w, such that*

$$(2.6) \qquad \max_{0 \le j \le J+1} |e_j| \le ch^2 \sum_{j=1}^{4} |w^{(j)}|_\infty$$

and

$$(2.7) \qquad \big[a_h(e, e) \big]^{1/2} \le ch^2 \sum_{j=1}^{4} |w^{(j)}|_\infty,$$

where $|w^{(j)}|_\infty = \max \big\{ \sup_{0 \le x < x^*} |w^{(j)}(x)|, \sup_{x^* < x \le 1} |w^{(j)}(x)| \big\}$.

Proof. Let us first note that (2.6) follows immediately from (2.7) in view of (2.4). Therefore, it remains to show (2.7). Using (2.1), we have for $1 \le j \le J, j \ne m$,

$$\bar{\partial} \big(a_{j+1/2} \partial e_j \big) = \bar{\partial} \big(a_{j+1/2} \partial w_j \big) - f(x_j).$$

Therefore,

$$(2.8\text{i}) \qquad \bar{\partial} \big(a_{j+1/2} \partial e_j \big) = \begin{cases} h_-^2 g_j & \text{if } 1 \le j \le m - 1, \\ h_+^2 g_j & \text{if } m + 1 \le j \le J, \end{cases}$$

and

$$(2.8\text{ii}) \qquad \hat{h} \delta_\rho \big(a_{m+1/2} \partial e_m \big) = \rho h_+^2 g_m - h_-^2 g_m',$$

where, by Taylor's theorem

$$\max_j |g_j| + |g'_m| \le C \sum_{j=1}^{4} |w^{(j)}|_\infty.$$

Multiplying (2.8i) by $h_- \bar{e}_j$ if $j \le m - 1$ and by $\rho h_+ \bar{e}_j$ if $j \ge m + 1$, and (2.8ii) by $\bar{e}_m$, summing from $j = 1$ to $j = J$ and using the definition of a_h, we obtain

$$(2.9) \qquad a_h(e, e) = h_-^3 \sum_{j=1}^{m-1} g_j \bar{e}_j + \left(\rho h_+^2 g_m - h_-^2 g'_m\right)\bar{e}_m + \rho h_+^3 \sum_{j=m+1}^{J} g_j \bar{e}_j.$$

We now essentially derive a discrete H^{-1} estimate for the vector g by introducing an auxiliary vector $\eta \in \mathbb{C}_0^{J+2}$ by the relations

$$(2.10) \qquad \begin{aligned} \eta_{j+1} - \eta_j &= h_-^3 g_j, \qquad j = 1, \dots, m - 1, \\ \rho \eta_{m+1} - \eta_m &= \rho h_+^2 g_m - h_-^2 g'_m, \\ \eta_{j+1} - \eta_j &= h_+^3 g_j, \qquad j = m + 1, \dots, J. \end{aligned}$$

Obviously,

$$(2.11) \qquad \begin{aligned} \eta_s &= -h_+^3 \sum_{j=s}^{J} g_j, \qquad s = m + 1, \dots, J, \\ \eta_s &= -\left(\rho h_+^2 g_m - h_-^2 g'_m\right) - h_-^3 \sum_{j=s}^{m-1} g_j - \rho h_+^3 \sum_{j=m+1}^{J} g_j, \quad s = 1, \dots, m. \end{aligned}$$

Therefore,

$$(2.12) \qquad \max_j |\eta_j| \le Ch^2 \sum_{j=1}^{4} |w^{(j)}|_\infty.$$

Now, using (2.9), (2.10), we have

$$\begin{aligned} a_h(e, e) &= \sum_{j=1}^{m-1} (\eta_{j+1} - \eta_j)\bar{e}_j + \left(\rho \eta_{m+1} - \eta_m\right)\bar{e}_m + \rho \sum_{j=m+1}^{J} (\eta_{j+1} - \eta_j)\bar{e}_j \\ &= -\sum_{j=0}^{m-1} \eta_{j+1}(\bar{e}_{j+1} - \bar{e}_j) - \rho \sum_{j=m}^{J} \eta_{j+1}(\bar{e}_{j+1} - \bar{e}_j). \end{aligned}$$

Therefore, using the Cauchy–Schwarz inequality, (2.12) and (2.2), we obtain

$$\begin{aligned} a_h(e, e) &\le \left\{ h_- \sum_{j=0}^{m-1} |\eta_{j+1}|^2 + \rho h_+ \sum_{j=m}^{J} |\eta_{j+1}|^2 \right\}^{1/2} \left\{ h_- \sum_{j=0}^{m-1} |\partial e_j|^2 + \rho h_+ \sum_{j=m}^{J} |\partial e_j|^2 \right\}^{1/2} \\ &\le Ch^2 \left[a_h(e, e)\right]^{1/2} \sum_{j=1}^{4} |w^{(j)}|_\infty, \end{aligned}$$

and (2.7) follows. □

We close this section with two remarks:

Remark 1. There is an easier, direct way to prove (2.6), which avoids (2.7). In fact, from (2.4), (2.9) we obtain

$$(\underline{a}/\rho^*)\max_j |e_j|^2 \le \max_j |e_j|\Big\{h_-^3\sum_{j=1}^{m-1}|g_j| + \big|\rho h_+^2 g_m - h_-^2 g_m'\big| + \rho h_+^3\sum_{j=m+1}^{J}|g_j|\Big\},$$

and the result follows. We shall use (2.7) only in the case of the wave equation in §5.

Remark 2. Define, for functions u, v in $L^2(0,1)$, the inner product

$$(u,v)_\rho = \int_0^{x^*} u\bar{v}\,dx + \rho\int_{x^*}^1 u\bar{v}\,dx.$$

Then the two-point boundary value problem (1.3) has the following weak formulation: seek w in the Sobolev space $H_0^1(0,1)$ such that for each $v \in H_0^1(0,1)$

$$-(aw', v')_\rho = (f, v)_\rho.$$

Consider now the finite-dimensional subspace S_h of $H_0^1(0,1)$, consisting of piecewise linear functions on the mesh $\{x_j\}, 0 \le j \le J+1$, continuous on $[0,1]$ and vanishing at the endpoints 0 and 1. Endowing S_h with the usual hat function basis $\{\varphi_j\}, 1 \le j \le J$, one obtains a standard finite element approximation $w_h \in S_h$ to w, given by

$$(2.13) \qquad\qquad -(aw_h', \varphi_j')_\rho = (f, \varphi_j)_\rho, \quad 1 \le j \le J.$$

It can be seen now that equations (2.13) reduce exactly to those in (2.1), provided one uses, in each subinterval $[x_i, x_{i+1}]$, the trapezoidal rule to approximate the integrals in the right-hand side of (2.13) and the midpoint rule to approximate those in the left-hand side. The scheme (2.1) is therefore a finite element scheme, modulo quadrature. One could conceivably, therefore, use the techniques of Wheeler [16] to try to derive discrete maximum norm error estimates for (2.1), taking into account, of course, the interface and the use of numerical quadrature.

3. THE HEAT EQUATION

In this section we consider the discretization of problem (1.1) by the Crank–Nicolson method modified at the interface node $x_m = x^*$. For $0 \le n \le N, 0 \le j \le J+1$, we shall approximate u_j^n by U_j^n, where $U^n = (U_0^n, \ldots, U_{j+1}^n)^T \in \mathbb{R}^{J+2}$, starting with $U_j^0 := v_j^0$. We denote $U_j^{n+1/2} = (U_j^{n+1} + U_j^n)/2, t^{n+1/2} = t^n + (k/2), b_j^{n+1/2} = b(x_j, t^{n+1/2})$ and put $\partial_\tau U_j^n = k^{-1}(U_j^{n+1} - U_j^n)$. Then, for $0 \le n \le N-1$, our scheme becomes

$$(3.1) \qquad \begin{aligned} \partial_\tau U_j^n &= \bar{\partial}\big(a_{j+1/2}\partial U_j^{n+1/2}\big) + b_j^{n+1/2}U_j^{n+1/2}, && 1 \le j \le J,\ j \ne m, \\ \partial_\tau U_m^n &= \delta_\rho\big(a_{m+1/2}\partial U_m^{n+1/2}\big) + \hat{b}(x^*, t^{n+1/2})U_m^{n+1/2}, \end{aligned}$$

where, by analogy to $\hat{f}, \hat{b}(x^*, t) := \big[h_-b(x^*-, t) + \rho h_+ b(x^*+, t)\big]/(2\hat{h})$.

Stability. Let

$$\|g\| := \Big\{\int_0^{x^*}|g(x)|^2\,dx + \rho\int_{x^*}^1|g(x)|^2\,dx\Big\}^{1/2}, \quad g \in L^2(0,1).$$

Multiplying $u_t = (au_x)_x + bu$ first by u, integrating by parts over $[0, x^*]$, and then multiplying the equation by ρu, integrating by parts over $[x^*, 1]$ and using the interface condition, we easily see that $(d/dt)\|u(\cdot, t)\|^2 \leq 0$, i.e.,

$$(3.2) \qquad \|u(\cdot, s)\| \leq \|u(\cdot, t)\|, \quad 0 \leq t \leq s \leq T,$$

and, in particular,

$$(3.3) \qquad \forall t \in [0, T] \quad \|u(\cdot, t)\| \leq \|v^0\|.$$

In the next lemma we prove discrete analogues of (3.2), (3.3) for scheme (3.1).

Lemma 3.1. *The approximate solution U^n of (3.1) satisfies*

$$\|U^{n+1}\|_h \leq \|U^n\|_h, \quad n = 0, \dots, N-1,$$

and, in particular,

$$(3.4) \qquad \|U^n\|_h \leq \|U^0\|_h, \quad n = 0, \dots, N.$$

Proof. Multiplying the first equation in (3.1) by $h_- U_j^{n+1/2}$ if $j \leq m-1$ and by $\rho h_+ U_j^{n+1/2}$ if $j \geq m+1$, and the second by $U_m^{n+1/2}$, summing from $j = 1$ to $j = J$, using (2.2) and the sign conditions on a, b, we obtain $\|U^{n+1}\|_h^2 - \|U^n\|_h^2 \leq 0$, and the results follow. $\qquad\square$

From this lemma it obviously follows that for each $n = 0, \dots, N-1, U^{n+1}$ exists uniquely as solution of the $J \times J$ tridiagonal linear system represented by (3.1).

Convergence. The main result in this section is given in the following theorem.

Theorem 3.1. *Let u satisfy (1.1) and U^n satisfy (3.1). Then there exists a constant $C(u)$, independent of h and k, such that*

$$(3.5) \qquad \max_{0 \leq n \leq N} \|u^n - U^n\|_h \leq C(u)(k^2 + h^2).$$

Proof. For $0 \leq t \leq T$ let $W(t) = (W_0, \dots, W_{J+1})^T \in \mathbb{R}_0^{J+2}$ be the *elliptic approximation* to $u(\cdot, t)$ defined by

$$(3.6) \qquad \begin{aligned} \bar{\partial}\big(a_{j+1/2}\partial W_j\big) &= (Lu)(x_j, t), \qquad 1 \leq j \leq J,\ j \neq m, \\ \delta_\rho\big(a_{m+1/2}\partial W_m\big) &= (\widehat{Lu})(x^*, t), \end{aligned}$$

where $Lu := (a(x)u_x)_x$. With $W^n := W(t^n)$, denote

$$(3.7) \qquad \zeta^n := u^n - W^n, \quad \theta^n := W^n - U^n, \quad n = 0, \dots, N,$$

so that $u^n - U^n = \zeta^n + \theta^n$. According to Lemma 2.1 we have

$$(3.8) \qquad \max_{0 \leq n \leq N} \|\zeta^n\|_h \leq Ch^2.$$

Hence, it remains to estimate $\|\theta^n\|_h$. Using (3.1), (3.6), (1.1), and letting $\theta_j^{n+1/2} = (\theta_j^{n+1} + \theta_j^n)/2$, we obtain for $1 \leq j \leq J, j \neq m$,

$$(3.9) \qquad \partial_\tau \theta_j^n = \bar{\partial}\big(a_{j+1/2}\partial\theta_j^{n+1/2}\big) + b_j^{n+1/2}\theta_j^{n+1/2} + \omega_j^n,$$

where for $j \neq m, \omega_j^n = \omega_{1j}^n + \omega_{2j}^n + \omega_{3j}^n + \omega_{4j}^n + \omega_{5j}^n$, and

$$
\begin{aligned}
\omega_{1j}^n &:= \partial_\tau W_j^n - \partial_\tau u_j^n, \\
\omega_{2j}^n &:= \partial_\tau u_j^n - u_t(x_j, t^{n+1/2}), \\
(3.10) \qquad \omega_{3j}^n &:= (Lu)(x_j, t^{n+1/2}) - \frac{1}{2}\big[(Lu)(x_j, t^n) + (Lu)(x_j, t^{n+1})\big], \\
\omega_{4j}^n &:= -\frac{1}{2}b_j^{n+1/2}\big[(W_j^{n+1} + W_j^n) - (u_j^{n+1} + u_j^n)\big], \\
\omega_{5j}^n &:= -b_j^{n+1/2}\big[\frac{1}{2}(u_j^{n+1} + u_j^n) - u(x_j, t^{n+1/2})\big].
\end{aligned}
$$

Now, we have for $i = 1, 2, 3, 4, 5$,

$$
(3.11\text{i}) \qquad\qquad \max_{j \neq m} |\omega_{ij}^n| \leq C(k^2 + h^2).
$$

In fact, the terms $\omega_{2j}^n, \omega_{3j}^n, \omega_{5j}^n$ are obviously of $O(k^2 + h^2)$. An estimate of the same form for ω_{4j}^n follows immediately from (2.6), while the estimate for ω_{1j}^n results from (2.6) and the commutativity of the elliptic approximation operator and time differentiation.

At the interface, using again (3.1), (3.6) we have

$$
\begin{aligned}
\partial_\tau \theta_m^n &- \delta_\rho\big(a_{m+1/2}\partial\theta_m^{n+1/2}\big) - \hat{b}(x^*, t^{n+1/2})\theta_m^{n+1/2} \\
&= 2\partial_\tau W_m^n - \big[(\widehat{Lu})(x^*, t^n) + (\widehat{Lu})(x^*, t^{n+1})\big] - \hat{b}(x^*, t^{n+1/2})(W_m^{n+1} + W_m^n),
\end{aligned}
$$

and, as before, we see that

$$
(3.12) \qquad \partial_\tau \theta_m^n = \delta_\rho\big(a_{m+1/2}\partial\theta_m^{n+1/2}\big) + \hat{b}(x^*, t^{n+1/2})\theta_m^{n+1/2} + \big(h_-\omega_m^n + \rho h_+\tilde{\omega}_m^n\big)/\hat{h}
$$

with

$$
(3.11\text{ii}) \qquad\qquad |\omega_m^n| + |\tilde{\omega}_m^n| \leq C(k^2 + h^2).
$$

To derive (3.11ii), we write $\omega_m^n, \tilde{\omega}_m^n$ as sums of terms, as was done before. These terms are estimated easily, and (3.11ii) follows because only differences in t are involved. From (3.9) and (3.12) we obtain now, as in the stability proof,

$$
\begin{aligned}
(2k)^{-1}\big(\|\theta^{n+1}\|_h^2 - \|\theta^n\|_h^2\big) &\leq h_- \sum_{j=1}^{m-1} \omega_j^n \theta_j^{n+1/2} + \big(h_-\omega_m^n + \rho h_+\tilde{\omega}_m^n\big)\theta_m^{n+1/2} \\
&\quad + \rho h_+ \sum_{j=m+1}^{J} \omega_j^n \theta_j^{n+1/2}.
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality to the right-hand side, and using (3.11i) and (3.11ii), we obtain

$$
\|\theta^{n+1}\|_h - \|\theta^n\|_h \leq Ck(k^2 + h^2)
$$

and conclude

$$
(3.13) \qquad\qquad \max_n \|\theta^n\|_h \leq C(k^2 + h^2).
$$

The result now follows from (3.8), (3.13), and the proof is complete. $\qquad\qquad \square$

Remark 3. In analogy to, and using the notation of Remark 2 in the previous section, consider the Galerkin-finite element formulation of (1.1), with respect to the inner product $(\cdot, \cdot)_\rho$, coupled with a Crank–Nicolson type time-stepping procedure, given by

$$(3.14) \qquad (\partial_\tau U^n, \varphi_j)_\rho + (aU_x^{n+1/2}, \varphi_j')_\rho - (b(\cdot, t^{n+1/2})U^{n+1/2}, \varphi_j)_\rho = 0,$$

for $0 \le n \le N-1, 1 \le j \le J$, where $U^n = U^n(x)$ is the fully discrete approximation of $u(x, t^n)$ in $S_h, \partial_\tau U^n = k^{-1}(U^{n+1} - U^n)$, and $U^{n+1/2} = (U^{n+1} + U^n)/2$. Use the trapezoidal rule on each subinterval $[x_j, x_{j+1}]$ to approximate the integrals in the first and third term in (3.14) (which amounts to lumping into the diagonal the associated matrices), and the midpoint rule to approximate the integrals in the second term. It is seen that (3.14) reduces to the scheme (3.1) which, therefore, may be considered in this sense as a fully discrete finite-element type method, provided one defines U^0 as the interpolant onto S_h of $v^0(x)$. Modulo interfaces and quadrature errors, therefore, one could conceivably derive discrete maximum norm error estimates for (3.1), using the techniques of Wheeler [15]. Similar remarks can be made in the case of the wave equation of §5.

4. THE SCHRÖDINGER EQUATION

This section is devoted to the discretization of problem (1.4) by the Crank–Nicolson method modified at the interface node x^*. For $0 \le n \le N, 0 \le j \le J + 1$, we shall approximate u_j^n by U_j^n, where $U^n = (U_0^n, \ldots, U_{J+1}^n)^T \in \mathbb{C}_0^{J+2}$, starting with $U_j^0 := v_j^0$. As in the previous section, we let

$$U_j^{n+1/2} = (U_j^{n+1} + U_j^n)/2, \quad t^{n+1/2} = t^n + (k/2),$$
$$\beta_j^{n+1/2} = \beta(x_j, t^{n+1/2}), \quad \partial_\tau U_j^n = k^{-1}(U_j^{n+1} - U_j^n).$$

Our scheme, for $0 \le n \le N - 1$, is:

$$(4.1) \qquad \begin{aligned} \partial_\tau U_j^n &= i\alpha\bar{\partial}\big(\partial U_j^{n+1/2}\big) + i\beta_j^{n+1/2}U_j^{n+1/2}, \qquad 1 \le j \le J, \ j \ne m, \\ \partial_\tau U_m^n &= i\alpha\delta_\rho\big(\partial U_m^{n+1/2}\big) + i\hat{\beta}(x^*, t^{n+1/2})U_m^{n+1/2}, \end{aligned}$$

where we recall that $\bar{\partial}(\partial v_j) = (v_{j+1} - 2v_j + v_{j-1})/h^2$ (where $h = h_-$ if $j \le m - 1$ and $h = h_+$ if $j \ge m + 1$) and

$$\delta_\rho(\partial v_m) = \Big[\rho\frac{v_{m+1} - v_m}{h_+} - \frac{v_m - v_{m-1}}{h_-}\Big]/\hat{h}.$$

Stability. Multiply first $u_t = i\alpha u_{xx} + i\beta u$ by $\bar{u}$, integrate over $[0, x^*]$ by parts and take real parts. Then perform the analogous operations on $[x^*, 1]$ after multiplying the equation by $\rho\bar{u}$. Add the resulting equations, use the interface condition and obtain $(d/dt)\|u(\cdot, t)\|^2 = 0$, i.e.,

$$(4.2) \qquad \|u(\cdot, t)\| = \|v^0\|, \quad t \in [0, T].$$

A discrete analog of (4.2) is given in the following lemma.

Lemma 4.1. *If U^n satisfies (4.1), then*

(4.3)
$$\|U^n\|_h = \|U^0\|_h, \quad n = 0, \dots, N.$$

Proof. Multiplying the first equation in (4.1) by $h_-\bar{U}_j^{n+1/2}$ if $j \leq m-1$ and by $\rho h_+ \bar{U}_j^{n+1/2}$ if $j \geq m+1$, and the second by $\bar{U}_m^{n+1/2}$, summing from $j = 1$ to $j = J$, using (2.2) and taking real parts, we obtain $\|U^{n+1}\|_h = \|U^n\|_h$, and the result follows.   □

Using Lemma 4.1, it is easily seen that $U^n, n = 0, \dots, N$, are uniquely defined as solutions, for each n, of the $J \times J$ tridiagonal linear system represented by (4.1).

Convergence. The proof of the following result is similar to the proof of Theorem 3.1 and will be omitted.

Theorem 4.1. *Let u be the solution of (1.4), and U^n of (4.1). Then, there exists a constant $C(u)$, independent of k and h, such that*

(4.4)
$$\max_{0 \leq n \leq N} \|u^n - U^n\|_h \leq C(u)(k^2 + h^2).$$

5. The wave equation

In this section we consider the discretization of problem (1.5) by the standard, explicit Courant–Friedrichs–Lewy five-point scheme, modified at the interface node $x_m = x^*$. Letting $U_j^0 := v_j^0$, we construct approximations $U^n \in \mathbb{R}_0^{J+2}$ to u^n as follows: letting $b_j^n := b(x_j, t^n)$, we first compute U_j^1 by the formulas

(5.1)
$$U_j^1 := U_j^0 + kv_j^1 + \frac{k^2}{2}\big[\bar{\partial}\big(a_{j+1/2}\partial U_j^0\big) + b_j^0 U_j^0\big], \quad 1 \leq j \leq J,\ j \neq m,$$
$$U_m^1 := U_m^0 + kv_m^1 + \frac{k^2}{2}\big[\delta_\rho\big(a_{m+1/2}\partial U_m^0\big) + \hat{b}(x^*, 0)U_m^0\big].$$

Then, for $1 \leq n \leq N-1$, denoting $\delta_\tau^2 U_j^n = k^{-2}\big(U_j^{n+1} - 2U_j^n + U_j^{n-1}\big)$, we let

(5.2)
$$\delta_\tau^2 U_j^n = \bar{\partial}\big(a_{j+1/2}\partial U_j^n\big) + b_j^n U_j^n, \quad 1 \leq j \leq J,\ j \neq m,$$
$$\delta_\tau^2 U_m^n = \delta_\rho\big(a_{m+1/2}\partial U_m^n\big) + \hat{b}(x^*, t^n)U_m^n.$$

Convergence.

Theorem 5.1. *Let u be the solution of (1.5) and U^n satisfy (5.1), (5.2). For fixed $0 < r_0 < 1$ let $k|a|_\infty^{1/2}/\min(h-, h+) \leq r_0$. Then there exists a constant $C(u)$, independent of k and h, such that*

(5.3)
$$\max_{0 \leq n \leq N} \|u^n - U^n\|_h \leq C(u)(k^2 + h^2).$$

Proof. Let $W^n \in \mathbb{R}_0^{J+2}$ be the elliptic approximation to $u(\cdot, t^n)$ given by (3.6) and set

(5.4)
$$\zeta^n := u^n - W^n, \quad \theta^n := W^n - U^n, \quad n = 0, \dots, N,$$

so that $u^n - U^n = \zeta^n + \theta^n$. According to Lemma 2.1 we have

(5.5)
$$\max_{0 \leq n \leq N} \|\zeta^n\|_h \leq Ch^2.$$

Hence, it remains to estimate $\|\theta^n\|_h$. Using (5.2), (3.6), we obtain for $1 \leq j \leq J, j \neq m, n = 1, \ldots, N-1, Lu = (au_x)_x$,

$$\delta_\tau^2 \theta_j^n - \bar{\partial}\big(a_{j+1/2}\partial\theta_j^n\big) - b_j^n \theta_j^n = \delta_\tau^2 W_j^n - (Lu)(x_j, t^n) - b_j^n W_j^n,$$

i.e.,

(5.6) $$\delta_\tau^2 \theta_j^n = \bar{\partial}\big(a_{j+1/2}\partial\theta_j^n\big) + b_j^n \theta_j^n + \omega_j^n, \quad j \neq m,$$

where for $j \neq m, \omega_j^n = \omega_{1j}^n + \omega_{2j}^n + \omega_{3j}^n$ with

(5.7) $$\begin{aligned} \omega_{1j}^n &:= \delta_\tau^2 W_j^n - \delta_\tau^2 u_j^n, \\ \omega_{2j}^n &:= \delta_\tau^2 u_j^n - u_{tt}(x_j, t^n), \\ \omega_{3j}^n &:= b_j^n(u_j^n - W_j^n). \end{aligned}$$

Now, we have for $i = 1, 2, 3,$

(5.8i) $$\max_{j \neq m} |\omega_{ij}^n| \leq C(k^2 + h^2).$$

In fact, ω_{2j}^n is obviously estimated by Ck^2, and a bound of the form $C(k^2 + h^2)$ for ω_{3j}^n follows immediately from Lemma 2.1. Finally, to estimate ω_{1j}^n, we use Lemma 2.1 and the fact that

$$\delta_\tau^2 u_j^n = k^{-2} \int_{-k}^{k} (k - |\tau|) u_{tt}(x_j, t^n + \tau)\, d\tau.$$

Further, for $j = m$, using again (5.2), (3.6), we have

$$\begin{aligned} &\delta_\tau^2 \theta_m^n - \delta_\rho\big(a_{m+1/2}\partial\theta_m^n\big) - \hat{b}(x^*, t^n)\theta_m^n \\ &\quad = \delta_\tau^2 W_m^n - u_{tt}(x^*, t^n) + \hat{b}(x^*, t^n)u_m^n - \hat{b}(x^*, t^n)W_m^n. \end{aligned}$$

As before, we see that

(5.9) $$\delta_\tau^2 \theta_m^n = \delta_\rho\big(a_{m+1/2}\partial\theta_m^n\big) + \hat{b}(x^*, t^n)\theta_m^n + \big(h_-\omega_m^n + \rho h_+ \tilde{\omega}_m^n\big)/\hat{h}$$

with

(5.8ii) $$|\omega_m^n| + |\tilde{\omega}_m^n| \leq C(k^2 + h^2).$$

Introducing the bilinear, symmetric, positive semidefinite form $b_h^n(\cdot, \cdot)$ in $\mathbb{R}_0^{J+2}$ by

$$b_h^n(v, w) = -h_- \sum_{j=1}^{m-1} b_j^n v_j w_j - \hat{h}\hat{b}(x^*, t^n)v_m w_m - \rho h_+ \sum_{j=m+1}^{J} b_j^n v_j w_j,$$

multiplying (5.6) by $h_-(\theta_j^{n+1} - \theta_j^{n-1}) = h_-\big[(\theta_j^{n+1} - \theta_j^n) + (\theta_j^n - \theta_j^{n-1})\big]$ if $j \leq m-1$ and by $\rho h_+(\theta_j^{n+1} - \theta_j^{n-1})$ if $j \geq m+1$, multiplying (5.9) by $(\theta_m^{n+1} - \theta_m^{n-1})$, and summing

from $j = 1$ to $j = J$, we obtain

$$k^{-2}\big(\|\theta^{n+1} - \theta^n\|_h^2 - \|\theta^n - \theta^{n-1}\|_h^2\big) = -\big[a_h(\theta^n, \theta^{n+1} - \theta^{n-1}) + b_h^n(\theta^n, \theta^{n+1} - \theta^{n-1})\big]$$

$$+ h_- \sum_{j=1}^{m-1} \omega_j^n(\theta_j^{n+1} - \theta_j^{n-1}) + \big(h_- \omega_m^n + \rho h_+ \tilde{\omega}_m^n\big)(\theta_m^{n+1} - \theta_m^{n-1})$$

$$+ \rho h_+ \sum_{j=m+1}^{J} \omega_j^n(\theta_j^{n+1} - \theta_j^{n-1}).$$

Therefore, applying the Cauchy–Schwarz inequality and using (5.8), we have

$$
\begin{aligned}
(5.10) \quad \|\theta^{n+1} - \theta^n\|_h^2 - \|\theta^n - \theta^{n-1}\|_h^2 &\leq -k^2\big[a_h(\theta^{n+1}, \theta^n) - a_h(\theta^n, \theta^{n-1})\big] \\
&\quad - k^2\big[b_h^n(\theta^{n+1}, \theta^n) - b_h^n(\theta^n, \theta^{n-1})\big] + Ck^2(k^2 + h^2)\|\theta^{n+1} - \theta^{n-1}\|_h.
\end{aligned}
$$

Summing from $n = 1$ to $n = \ell \leq N - 1$ then yields

$$
\begin{aligned}
(5.11) \quad &\|\theta^{\ell+1} - \theta^\ell\|_h^2 + k^2\big[a_h(\theta^{\ell+1}, \theta^\ell) + b_h^{\ell+1}(\theta^{\ell+1}, \theta^\ell)\big] \\
&\leq \|\theta^1 - \theta^0\|_h^2 + k^2\big[a_h(\theta^0, \theta^1) + b_h^0(\theta^0, \theta^1)\big] \\
&\quad + Ck(k^2 + h^2) \max_{0 \leq n \leq N-1} \|\theta^{n+1} - \theta^n\|_h + k^2 \sum_{n=1}^{\ell} \big[b_h^{n+1}(\theta^{n+1}, \theta^n) - b_h^n(\theta^{n+1}, \theta^n)\big].
\end{aligned}
$$

Now, using (2.5),

$$
\begin{aligned}
\sum_{n=1}^{\ell} \big[b_h^{n+1}(\theta^{n+1}, \theta^n) - b_h^n(\theta^{n+1}, \theta^n)\big] &\leq Ck \sum_{n=1}^{\ell} \|\theta^{n+1}\|_h \|\theta^n\|_h \\
&\leq Ck \sum_{n=1}^{\ell} \big(\|\theta^{n+1} + \theta^n\|_h^2 + \|\theta^{n+1} - \theta^n\|_h^2\big) \\
&\leq Ck \sum_{n=1}^{\ell} \big[a_h(\theta^{n+1} + \theta^n, \theta^{n+1} + \theta^n) + \|\theta^{n+1} - \theta^n\|_h^2\big].
\end{aligned}
$$

Further, using (2.3),

$$
\begin{aligned}
&\|\theta^{\ell+1} - \theta^\ell\|_h^2 + k^2\big[a_h(\theta^{\ell+1}, \theta^\ell) + b_h^{\ell+1}(\theta^{\ell+1}, \theta^\ell)\big] \\
&= \|\theta^{\ell+1} - \theta^\ell\|_h^2 + \frac{k^2}{4}\big[a_h(\theta^{\ell+1} + \theta^\ell, \theta^{\ell+1} + \theta^\ell) - a_h(\theta^{\ell+1} - \theta^\ell, \theta^{\ell+1} - \theta^\ell)\big] \\
&\quad + \frac{k^2}{4}\big[b_h^{\ell+1}(\theta^{\ell+1} + \theta^\ell, \theta^{\ell+1} + \theta^\ell) - b_h^{\ell+1}(\theta^{\ell+1} - \theta^\ell, \theta^{\ell+1} - \theta^\ell)\big] \\
&\geq \Big[1 - r_0^2 + \frac{k^2}{4} \min_{x,t} b(x,t)\Big]\|\theta^{\ell+1} - \theta^\ell\|_h^2 + \frac{k^2}{4}a_h(\theta^{\ell+1} + \theta^\ell, \theta^{\ell+1} + \theta^\ell).
\end{aligned}
$$

Therefore, (5.11) yields

$$\left[1 - r_0^2 + \frac{k^2}{4} \min_{x,t} b(x,t)\right] \|\theta^{\ell+1} - \theta^{\ell}\|_h^2 + \frac{k^2}{4} a_h(\theta^{\ell+1} + \theta^{\ell}, \theta^{\ell+1} + \theta^{\ell})$$

$$\leq \|\theta^1 - \theta^0\|_h^2 + k^2\left[a_h(\theta^0, \theta^1) + b_h^0(\theta^0, \theta^1)\right] + Ck(k^2 + h^2) \max_{0 \leq n \leq N-1} \|\theta^{n+1} - \theta^n\|_h$$

$$+ Ck \sum_{n=1}^{\ell} \left[\|\theta^{n+1} - \theta^n\|_h^2 + \frac{k^2}{4} a_h(\theta^{n+1} + \theta^n, \theta^{n+1} + \theta^n)\right].$$

For k sufficiently small, we note that $1 - r_0^2 + \frac{k^2}{4} \min_{x,t} b(x,t) > 0$. Then, applying the discrete Gronwall lemma yields, for $n = 1, \ldots, N-1$,

$$\|\theta^{n+1} - \theta^n\|_h^2 \leq C\Big\{ \|\theta^1 - \theta^0\|_h^2 + k^2\big|a_h(\theta^0, \theta^1) + b_h^0(\theta^0, \theta^1)\big|$$

$$+ k(k^2 + h^2) \max_{0 \leq \ell \leq N-1} \|\theta^{\ell+1} - \theta^{\ell}\|_h\Big\},$$

i.e.,

$$(5.12) \quad \max_{0 \leq n \leq N-1} \|\theta^{n+1} - \theta^n\|_h^2 \leq C\Big\{ \|\theta^1 - \theta^0\|_h^2 + k^2\big|a_h(\theta^0, \theta^1) + b_h^0(\theta^0, \theta^1)\big| + k^2(k^2 + h^2)^2\Big\}.$$

Using the Cauchy–Schwarz inequality and (2.3), we have

$$k^2|a_h(\theta^0, \theta^1 - \theta^0)| \leq k^2 a_h(\theta^0, \theta^0) + r_0^2 \|\theta^1 - \theta^0\|_h^2.$$

Hence, from (5.12),

$$(5.13) \quad \begin{aligned} \max_{0 \leq n \leq N-1} &\|\theta^{n+1} - \theta^n\|_h^2 \\ &\leq C\Big\{ \|\theta^1 - \theta^0\|_h^2 + k^2\big[a_h(\theta^0, \theta^0) + |b_h^0(\theta^0, \theta^1)|\big] + k^2(k^2 + h^2)^2\Big\}. \end{aligned}$$

According to Lemma 2.1 we have

$$(5.14) \qquad\qquad a_h(\theta^0, \theta^0) + \|\theta^0\|_h^2 \leq Ch^4,$$

and using the fact that

$$\theta_j^1 = (W_j^1 - u_j^1) + (u_j^1 - U_j^1) = (W_j^1 - u_j^1) + O\big(k(k^2 + h^2)\big),$$

we get

$$(5.15) \qquad\qquad \|\theta^1\|_h \leq Ch^2.$$

Further, since

$$\theta_j^1 - \theta_j^0 = (W_j^1 - W_j^0) + (u_j^1 - u_j^0) + O\big(k(k^2 + h^2)\big),$$

we obtain

$$(5.16) \qquad\qquad \|\theta^1 - \theta^0\|_h \leq Ck(k^2 + h^2).$$

Finally, using (5.14)–(5.16) in (5.13) yields

$$\max_{0 \leq n \leq N-1} \|\theta^{n+1} - \theta^n\|_h \leq Ck(k^2 + h^2),$$

i.e.,

$$\|\theta^{n+1}\|_h \leq \|\theta^n\|_h + Ck(k^2 + h^2), \quad n = 0, \ldots, N-1.$$

Hence

$$\max_{0\leq n\leq N} \|\theta^n\|_h \leq C(k^2 + h^2),$$

and the proof is complete. $\square$

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