On the number of spanning trees of $K_n^m \pm G$ graphs

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Abstract: The K_n-complement of a graph G, denoted by $K_n - G$, is defined as the graph obtained from the complete graph K_n by removing a set of edges that span G; if G has n vertices, then $K_n - G$ coincides with the complement $\overline{G}$ of the graph G. In this paper we extend the previous notion and derive determinant based formulas for the number of spanning trees of graphs of the form $K_n^m \pm G$, where K_n^m is the complete multigraph on n vertices with exactly m edges joining every pair of vertices and G is a multigraph spanned by a set of edges of K_n^m; the graph $K_n^m + G$ (resp. $K_n^m - G$) is obtained from K_n^m by adding (resp. removing) the edges of G. Moreover, we derive determinant based formulas for graphs that result from K_n^m by adding and removing edges of multigraphs spanned by sets of edges of the graph K_n^m. We also prove closed formulas for the number of spanning tree of graphs of the form $K_n^m \pm G$, where G is (i) a complete multipartite graph, and (ii) a multi-star graph. Our results generalize previous results and extend the family of graphs admitting formulas for the number of their spanning trees.

Keywords: Kirchhoff matrix tree theorem, complement spanning tree matrix, spanning trees, K_n-complements, multigraphs.

1 Introduction

The number of spanning trees of a graph G, denoted by $\tau(G)$, is an important, well-studied quantity in graph theory, and appears in a number of applications. Most notable application fields are network reliability [7, 13, 18], enumerating certain chemical isomers [5], and counting the number of Eulerian circuits in a graph [11].

Thus, both for theoretical and for practical purposes, we are interested in deriving formulas for the number of spanning trees of a graph G, and also of the K_n-complement of G; the K_n-*complement* of a graph G, denoted by $K_n - G$, is defined as the graph obtained from the complete graph K_n by removing a set of edges (of the graph K_n) that span G; if G has n vertices, then $K_n - G$ coincides with the complement $\overline{G}$ of the graph G. Many cases have been examined depending on the choice of G. For example, there exist closed formulas for the cases where G is is a pairwise disjoint set of edges [20], a chain of edges [12], a cycle [8], a star [16], a multi-star [15, 21], a multi-complete/star graph [6], a labelled molecular graph [5], and more resent results when G is a circulant graph [9, 22], a quasi-threshold graph [14], and so on (see Berge [2] for an exposition of the main results).

In this paper, we extend the previous notion and consider graphs that result from the complete multigraph K_n^m by removing multiple edges; we denote by K_n^m the complete multigraph on n vertices with exactly m edges joining every pair of vertices. Based on the properties of the Kirchhoff matrix,

which permits the calculation of the number of spanning trees of any given graph, we derive a determinant based formula for the number of spanning trees of the graph $K_n^m - G$, where G is a subgraph of K_n^m, and, thus, it is a multigraph. Note that, if $m = 1$ then $K_n^m - G$ coincides with the graph $K_n - G$.

We also consider graphs that result from the complete multigraph K_n^m by adding multiple edges. More precisely, we consider multigraphs of the form $K_n^m + G$ that results from the complete multigraph K_n^m by adding a set of edges (of the graph K_n^m) that span G. Again, based on the properties of the Kirchhoff matrix, we derive a determinant based formula for the number of spanning trees of the graph $K_n^m + G$. To the best of our knowledge, not as much seems to be known about the number $\tau(K_n^m + G)$. Bedrosian in [1] considered the number $\tau(K_n + G)$ for some simple configurations of G, i.e., when G forms a cycle, a complete graph, or when its vertex set is quite small. More recently, Golin et al. in [9] derive closed formula for the number $\tau(K_n + G)$ using Chebyshev polynomials, introduced in [4], for the case where G forms a circulant graph.

We denote $K_n^m \pm G$ the family of graphs of the forms $K_n^m + G$ and $K_n^m - G$, and derive a determinant based formula for the number $\tau(K_n^m \pm G)$. Moreover, based on these results, we generalize our formulas and extend the family $K_n^m \pm G$ to the more general family of graphs $K_n^{(m)} \pm G$, where $K_n^{(m)}$ is the complete multigraph on n vertices with at least $m \geq 1$ edges joining every pair of vertices.

Based on our results–that is, the determinant based formulas for the number of spanning trees of the family of graphs $\tau(K_n^m \pm G)$, and using standard algebraic techniques, we generalize known closed formulas for the number of spanning trees of simple graphs of the form $K_n - G$. In particular, we derive closed formulas for the number of spanning trees $\tau(K_n^m \pm G)$, in the case where G forms (i) a complete multipartite graph, and (ii) a multi-star graph.

We point out that our proposed formulas express the number of spanning trees $\tau(K_n^m \pm G)$ as a function of the determinant of a matrix that can be easily constructed from the adjacency relation of the graph G. Our results generalize previous results and extend the family of graphs of the form $K_n^m \pm G$ admitting formulas for the number of their spanning trees.

2   Preliminaries

We consider finite undirected simple graphs and multigraphs with no loops; the term *multigraph* is used when multiple edges are allowed in a garph. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively.

The *multiplicity* of a vertex-pair (v, u) of a graph G, denoted by $\ell_G(vu)$, is the number of edges joining the vertices v and u in G. The minimum multiplicity among all the vertex-pairs of G is denoted $\lambda(G)$ while $\Lambda(G)$ is the largest such number. Thus, if $\lambda(G) > 0$, then every pair of vertices in G are connected with at least $\lambda(G)$ edges; if $\Lambda(G) = 1$, then G contains no multiple edges, that is, G is a *simple* graph (note that a simple graph or a multigraph contains no loops). The *degree* of a vertex v of a graph G, denoted by $d_G(v)$, is the number of edges incident with v in G. The minimum degree among the vertices of G is denoted $\delta(G)$ while $\Delta(G)$ is the largest such number.

We denote by $K_n^{(m)}$ the complete multigraph on n vertices with $\lambda(K_n^{(m)}) = m$, where $m \geq 1$; that is, $K_n^{(m)}$ has at least $m \geq 1$ edges joining every pair of its vertices. A complete multigraph on n vertices with exactly m edges joining every pair of its vertices is called *m-complete multigraph* and denoted by K_n^m. Thus, for the m-complete multigraph K_n^m we have that $\lambda(K_n^m) = \Lambda(K_n^m) = m$ and $\delta(K_n^m) = \Delta(K_n^m) = (n-1)m$. Note that, the 1-complete multigraph is the graph K_n. By definition, every complete multigraph $K_n^{(m)}$ contains a subgraph isomorphic to an m-complete multigraph K_n^m.

Let $K_n^{(m)}$ be a complete multigraph and let $\mathcal{C}$ be a set of edges of $K_n^{(m)}$ such that the graph which is obtained from $K_n^{(m)}$ by removing the edges of $\mathcal{C}$ is an m-complete multigraph K_n^m; the graph spanned

by the edges of $\mathcal{C}$ is called a *characteristic* graph of $K_n^{(m)}$ and denoted by $\mathcal{H}(K_n^{(m)})$. By definition, a characteristic graph $\mathcal{H}(K_n^{(m)})$ contains no isolated vertices.

Let G and H be two multigraphs. The graph $G + H$ is defined as follows:

$$V(G + H) \; = \; V(G) \cup V(H)$$

and

$$vu \in E(G + H) \iff vu \in E(G) \;\; \text{or} \;\; vu \in E(H).$$

Both graphs G and H are subgraphs of $G + H$. Moreover, if $v, u \in V(G) \cap V(H)$, then $\ell_{G+H}(vu) = \ell_G(vu) + \ell_H(vu)$.

Let G and H be two multigraphs such that $E(H) \subseteq E(G)$. The graph $G - H$ is defined as the graph obtained from G by removing the edges of H.

Having defined the graphs $G + H$ and $G - H$, it is easy to see that $K_n^{(m)} = K_n^m + \mathcal{H}(K_n^{(m)})$ and $K_n^m = K_n^{(m)} - \mathcal{H}(K_n^{(m)})$. In general, $\mathcal{H}(K_n^{(m)}) \neq K_n^{(m)} - K_n^m$; the equality holds if $\mathcal{H}(K_n^{(m)})$ has n vertices.

The *adjacency matrix* of a multigraph G on n vertices, denoted by $A(G)$, is an $n \times n$ matrix with diagonal elements $A(G)[i,i] = 0$ and off-diagonal elements $A(G)[i,j] = \ell_G(v_i v_j)$. The *degree matrix* of the multigraph G, denoted by $D(G)$, is an $n \times n$ matrix with diagonal elements $D(G)[i,i] = d_G(v_i)$ and off-diagonal elements $D(G)[i,j] = 0$. Throughout the paper empty entries in matrices represent 0s.

For an $n \times n$ matrix M, the $(n-1)$-st order *minor* μ_j^i is the determinant of the $(n-1) \times (n-1)$ matrix obtained from M after having deleted row i and column j; the i-th *cofactor* equals μ_i^i. The *Kirchhoff matrix* $L(G)$ (also known as the *Laplacian matrix*) for a multigraph G on n vertices is an $n \times n$ matrix with elements

$$L(G)[i,j] = \begin{cases} d_G(v_i) & \text{if } i = j, \\ -\ell_G(v_i v_j) & \text{otherwise} \end{cases}$$

where $d_G(v_i)$ is the degree of vertex v_i in the graph G and $\ell_G(v_i v_j)$ is the number of edges joining the vertices v_i and v_j in G. The matrix $L(G)$ is a positive-definite matrix, since it is a symmetric matrix and has nonnegative real eigenvalues. Note that, $L(G) = D(G) - A(G)$.

The *Kirchhoff matrix tree theorem* [3] is one of the most famous results in graph theory. It provides a formula for the number of spanning trees of a graph G in terms of the cofactors of G's Kirchhoff matrix; it is stated as follows:

Theorem 2.1. (Kirchhoff Matrix Tree Theorem [3]): *For any multigraph G with $L(G)$ defined as above, the cofactors of $L(G)$ have the same value, and this value equals the number of spanning trees $\tau(G)$ of the multigraph G.*

The Kirchhoff matrix tree theorem provides a powerful tool for computing the number $\tau(G)$ of spanning trees of a graph G. For this computation, we first form the Kirchhoff matrix $L(G)$ of the graph G and obtain the $(n-1) \times (n-1)$ matrix $L_i(G)$ from $L(G)$ by removing its i-th row and column (arbitrary), and then compute the determinant of matrix $L_i(G)$. The operation of removing a row and a column from $L(G)$ may seem somewhat contrived.

The number of spanning trees of a graph G can be computed directly (without removing any row or column) in terms of a matrix $L'(G)$ similar to the Kirchhoff matrix $L(G)$, which is associated with the graph G [19], or, alternatively, it can be computed by defining a characteristic polynomial $\det(L(G) - xI)$ on $L(G)$; the latter approach takes into amount the computation of the eigenvalues of the matrix $L(G)$ (see [4, 5, 9, 18, 22]).

In our work, we express the number of spanning trees of a graph of the form $K_n^m \pm G$, where G is a subgraph of K_n^m on p vertices, in terms of a $p \times p$ matrix $B(G)$ associated with the graph G, and not in terms of an $n \times n$ matrix $L(K_n^m \pm G)$ associated with the whole graph $K_n^m \pm G$.

3 The $K_n^m \pm G$ graphs

In this section, we consider graphs that result from the m-complete multigraph K_n^m by removing or/and adding multiple edges. We are interested in deriving determinant based formulas for the number of spanning trees of the graphs $K_n^m - G$ and $K_n^m + G$, where G is a multigraph spanned by a set of edges $S \subseteq E(K_n^m)$.

3.1 The case $K_n^m - G$

Let G be a multigraph spanned by a set of edges of the graph K_n^m. We derive formulas for the number of spanning trees of the graph $K_n^m - G$; the graph G has $p \le n$ vertices and $\Lambda(G) \le m$.

In order to compute the number $\tau(K_n^m - G)$ we will make use of Theorem 2.1. Thus, we consider the $n \times n$ Kirchhoff matrix $L = L(K_n^m - G)$, which has the form:

$$L = \begin{bmatrix} m(n-1) & -m & \cdots & -m & -m & -m & \cdots & -m \\ -m & m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -m & -m & \cdots & m(n-1) & -m & -m & \cdots & -m \\ -m & -m & \cdots & -m & m(n-1)-d_G(v_1) & & -m+\ell_G(v_j v_i) & \\ -m & -m & \cdots & -m & & m(n-1)-d_G(v_2) & & \\ \vdots & \vdots & \cdots & \vdots & & & \ddots & \\ -m & -m & \cdots & -m & -m+\ell_G(v_i v_j) & & & m(n-1)-d_G(v_p) \end{bmatrix},$$

where $d_G(v_i)$ is the degree of the vertex $v_i \in G$ and $\ell_G(v_i v_j)$ is the multiplicity of the vertices v_i and v_j in G. The entries of the off-diagonal positions $(n-p+i,\ n-p+j)$ of the matrix L are equal to $-m + \ell_G(v_i v_j)$, $1 \le i,\ j \le p$. Note that, the first $n-p$ rows and columns of L correspond to the $n-p$ vertices of the set $V(K_n^m) - V(G)$ and, thus, they have degree $m(n-1)$ in $K_n^m - G$. Since $p \le n$ and $\Lambda(G) \le m$, the matrix L is a positive-definite matrix.

Let L_1 be the $(n-1) \times (n-1)$ matrix obtained from L by removing its first row and column. Then, from Theorem 2.1 we have that
$$\tau(K_n^m - G) = \det(L_1).$$

In order to compute the determinant of the matrix L_1, we add one row and one column to the matrix L_1; the resulting $n \times n$ matrix L_1' has 1 in position $(1,1)$, $-m$ in positions $(1,j)$, $2 \le j \le n$, and 0 in positions $(i,1)$, $2 \le i \le n$. It is easy to see that, $\det(L_1') = \det(L_1)$. Thus, the $n \times n$ matrix L_1' has the following form:

$$
L_1' = \begin{bmatrix}
1 & -m & \cdots & -m & -m & -m & \cdots & -m \\
0 & m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -m & \cdots & m(n-1) & -m & -m & \cdots & -m \\
0 & -m & \cdots & -m & m(n-1)-d_G(v_1) & & & -m+\ell_G(v_jv_i) \\
0 & -m & \cdots & -m & & m(n-1)-d_G(v_2) & & \\
\vdots & \vdots & \cdots & \vdots & & & \ddots & \\
0 & -m & \cdots & -m & -m+\ell_G(v_iv_j) & & & m(n-1)-d_G(v_p)
\end{bmatrix}.
$$

We multiply the first row of resulting matrix L_1' by -1 and add it to the $n-1$ rows. Thus, the determinant of L_1' becomes:

$$
\det(L_1') = \begin{vmatrix}
1 & -m & \cdots & -m & -m & -m & \cdots & -m \\
-1 & mn & & & & & & \\
\vdots & & \ddots & & & & & \\
-1 & & & mn & & & & \\
-1 & & & & mn-d_G(v_1) & & & \ell_G(v_jv_i) \\
-1 & & & & & mn-d_G(v_2) & & \\
\vdots & & & & & & \ddots & \\
-1 & & & & \ell_G(v_iv_j) & & & mn-d_G(v_p)
\end{vmatrix},
$$

where the entries of the off-diagonal positions $(n-p+i,\ n-p+j)$ of the matrix L_1' are equal to $\ell_G(v_iv_j)$, $1 \le i,\ j \le p$. Note that, the first $n-p$ rows of the matrix L_1' have non-zero elements in positions $(1,i)$ and (i,i), $2 \le i \le n-p$. We observe that the sum of all the elements on each row of L_1', except of the first row, is equal to $mn-1$; recall that, $d_G(v_i) = \sum_{1 \le j \le p} \ell_G(v_iv_j)$, for every $v_i \in V(G)$. Thus, we multiply each column of matrix L_1' by $\frac{1}{mn}$ and add it to the first column, and we obtain:

$$
\det(L_1') = \begin{vmatrix}
\frac{1}{n} & -m & \cdots & -m & -m & -m & \cdots & -m \\
0 & mn & & & & & & \\
\vdots & & \ddots & & & & & \\
0 & & & mn & & & & \\
0 & & & & mn-d_G(v_1) & & & \ell_G(v_jv_i) \\
0 & & & & & mn-d_G(v_2) & & \\
\vdots & & & & & & \ddots & \\
0 & & & & \ell_G(v_iv_j) & & & mn-d_G(v_p)
\end{vmatrix} \tag{1}
$$

$$
= m \cdot (mn)^{n-p-2} \cdot \det(B),
$$

where $B = mnI_p - L(G)$ is a $p \times p$ matrix; recall that, $L(G)$ is the Kirchhoff matrix of the multigraph G and thus, $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the degree matrix and the adjacency matrix of G, respectively. Concluding, we obtain the following result.

Lemma 3.1. *Let K_n^m be the m-complete multigraph on n vertices, and let G be a multigraph on p vertices such that $E(G) \subseteq E(K_n^m)$. Then,*

$$\tau(K_n^m - G) = m \cdot (mn)^{n-p-2} \cdot \det(B),$$

where $B = mnI_p - L(G)$ is a $p \times p$ matrix, and $L(G)$ is the Kirchhoff matrix of G.

We note that, for simple graphs K_n and G, Lemma 3.1 has been stated first by Moon in [12] and numerous authors in various guises used it as a constructive tool to obtain formulas for the number of spanning trees of graphs of the type $K_n - G$.

3.2   The case $K_n^m + G$

In this section, we derive a determinant based formula for the number of spanning trees of the graph $K_n^m + G$, where G is a subgraph of K_n^m. Based on Theorem 2.1, we construct the $n \times n$ Kirchhoff matrix $L = L(K_n^m + G)$ associated with the graph $K_n^m + G$; it can be constructed in a fashion similar to that of the previous case $K_n^m - G$. The difference here is the $p \times p$ submatrix which is formed by the last p rows and columns of L. More precisely, the matrix L has the following form:

$$L = \begin{bmatrix} m(n-1) & -m & \cdots & -m & -m & -m & \cdots & -m \\ -m & m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -m & -m & \cdots & m(n-1) & -m & -m & \cdots & -m \\ -m & -m & \cdots & -m & m(n-1)+d_G(v_1) & & & -m-\ell_G(v_jv_i) \\ -m & -m & \cdots & -m & & m(n-1)+d_G(v_2) & & \\ \vdots & \vdots & \cdots & \vdots & & & \ddots & \\ -m & -m & \cdots & -m & -m-\ell_G(v_iv_j) & & & m(n-1)+d_G(v_p) \end{bmatrix}, \quad (2)$$

where the entries of the off-diagonal positions $(n - p + i, \ n - p + j)$ of the matrix L are equal to $-m - \ell_G(v_iv_j)$, $1 \leq i, \ j \leq p$.

It is straightforward to apply a technique similar to that we have applied for the computation of the determinant of the matrix L_1' in the case of $K_n^m - G$. Thus, in the case of $K_n^m + G$, the determinant of the matrix L_1' of Eq. (1) becomes

$$\det(L_1') = \begin{vmatrix} \frac{1}{n} & -m & \cdots & -m & -m & -m & \cdots & -m \\ 0 & mn & & & & & & \\ \vdots & & \ddots & & & & & \\ 0 & & & mn & & & & \\ 0 & & & & mn+d_G(v_1) & & & -\ell_G(v_jv_i) \\ 0 & & & & & mn+d_G(v_2) & & \\ \vdots & & & & & & \ddots & \\ 0 & & & & -\ell_G(v_iv_j) & & & mn+d_G(v_p) \end{vmatrix}$$

$$= m \cdot (mn)^{n-p-2} \cdot \det(B),$$

where the entries of the off-diagonal positions $(n-p+i, n-p+j)$ of matrix L_1', are equal to $-\ell_G(v_i v_j)$, $1 \leq i, \ j \leq p$. Thus, we obtain the following result.

Lemma 3.2. *Let K_n^m be the m-complete multigraph on n vertices, and let G be a multigraph on p vertices such that $E(G) \subseteq E(K_n^m)$. Then,*

$$\tau(K_n^m + G) = m \cdot (mn)^{n-p-2} \cdot \det(B),$$

where $B = mnI_p + L(G)$ is a $p \times p$ matrix, and $L(G)$ is the Kirchhoff matrix of G.

3.3 The general case $K_n^m \pm G$

We next derive a general formula for both cases $K_n^m + G$ and $K_n^m - G$. To this end, we define a parameter α as follows: (i) $\alpha = 1$, for the case $K_n^m + G$, and (ii) $\alpha = -1$, for the case $K_n^m - G$. In other words, $\alpha = \pm 1$, according to $K_n^m \pm G$. Based on the value of α we conclude with the following result.

Theorem 3.1. *Let K_n^m be the m-complete multigraph on n vertices, and let G be a multigraph on p vertices such that $V(G) \subseteq V(K_n^m)$ and $E(G) \subseteq E(K_n^m)$. Then,*

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-2} \det(B),$$

where $B = mnI_p + \alpha \cdot L(G)$ is a $p \times p$ matrix, $\alpha = \pm 1$ according to $K_n^m \pm G$, and $L(G)$ is the Kirchhoff matrix of G.

In the previous theorem the graph G is a subgraph of K_n^m, and, thus, it has multiplicity $\Lambda(G) \leq m$. It follows that the graph $K_n^m + G$ has multiplicity $\Lambda(K_n^m + G) \leq 2m$. However, we can relax the previous restriction in the case of the graph $K_n^m + G$. It is not difficult to see that the matrix L of Equation (2) remains positive-definite since $\lambda(G) \geq 0$. Thus, we can define the graph G to be a multigraph on p vertices such that $V(G) \subseteq V(K_n^m)$. The following theorem holds.

Theorem 3.2. *Let K_n^m be the m-complete multigraph on n vertices, and let G be a multigraph on p vertices such that $V(G) \subseteq V(K_n^m)$. Then,*

$$\tau(K_n^m + G) = m \cdot (mn)^{n-p-2} \cdot \det(B),$$

where $B = mnI_p + L(G)$ is a $p \times p$ matrix, and $L(G)$ is the Kirchhoff matrix of G.

4 The $K_n^{(m)} \pm G$ graphs

In this section we derive determinant based formulas for the number $\tau(K_n^{(m)} \pm G)$, where $K_n^{(m)}$ is a complete multigraph and G is a subgraph of $K_n^{(m)}$. We first take into consideration the graph $K_n^m + G_1 - G_2$ and derive a determinant based formula for the number $\tau(K_n^m + G_1 - G_2)$, and, then, we derive a formula for the number $\tau(K_n^{(m)} \pm G)$ using the graph $K_n^m + G_1 - G_2$ and a characteristic graph $\mathcal{H}(K_n^{(m)})$.

4.1 The case $K_n^m + G_1 - G_2$

Here, we consider graphs that result from the m-complete multigraph K_n^m by adding multiple edges of a graph G_1 and removing multiple edges from a graph G_2. Let G_1 be a multigraph on p_1 vertices, such that $V(G_1) \subseteq V(K_n^m)$, and let G_2 be a multigraph on p_2 vertices, such that $V(G_2) \subseteq V(K_n^m)$ and $E(G_2) \subseteq E(K_n^m + G_1)$. Here, we focus on the graph $K_n^m + G_1 - G_2$, which is obtained from the m-complete multigraph K_n^m by adding the edges of the graph G_1, and, then, removing from the resulting graph $K_n^m + G_1$ the edges of G_2; that is, $K_n^m + G_1 - G_2 = (K_n^m + G_1) - G_2$.

It is easy to see that, $(K_n^m + G_1) - G_2 \neq K_n^m + (G_1 - G_2)$ since, in general, the graph G_2 is not a subgraph of G_1. Moreover, $V(G_1) \neq V(G_2)$.

For the pair of multigraphs (G_1, G_2) we define the *union-stable* graphs G_1^* and G_2^* of (G_1, G_2) as follows: G_1^* is the multigraph that results from G_1 by adding in $V(G_1)$ the vertices of the set $V(G_2) - V(G_1)$ and G_2^* is the multigraph that results from G_2 by adding in $V(G_2)$ the vertices of the set $V(G_1) - V(G_2)$. Thus, $V(G_1^*) = V(G_2^*)$.

It is easy to see that, both G_1^* and G_2^* are multigraphs on $p = |V(G_1) \cup V(G_2)|$ vertices, with at least $p - p_1$ and $p - p_2$ isolated vertices, respectively. Since $K_n^m + G_1^* - G_2^* = K_n^m + G_1 - G_2$, we focus on the graph $K_n^m + G_1^* - G_2^*$.

Based on Theorem 2.1, we construct the $n \times n$ Kirchhoff matrix $L = L(K_n^m + G_1^* - G_2^*)$ associated with the graph $K_n^m + G_1^* - G_2^*$; it is similar to that of the case of $K_n^m \pm G$. The difference here is the $p \times p$ submatrix which is formed by the last p rows and columns of L, where $p = |V(G_1) \cup V(G_2)|$. More precisely, the matrix L has the following form:

$$
L = \begin{bmatrix}
m(n-1) & -m & \cdots & -m & -m & -m & \cdots & -m \\
-m & m(n-1) & \cdots & -m & -m & -m & \cdots & -m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-m & -m & \cdots & m(n-1) & -m & -m & \cdots & -m \\
-m & -m & \cdots & -m & B'(G_1^*, G_2^*)[1,1] & & B'(G_1^*, G_2^*)[j,i] & \\
-m & -m & \cdots & -m & & B'(G_1^*, G_2^*)[2,2] & & \\
\vdots & \vdots & \cdots & \vdots & & & \ddots & \\
-m & -m & \cdots & -m & B'(G_1^*, G_2^*)[i,j] & & B'(G_1^*, G_2^*)[p,p]
\end{bmatrix},
$$

where the $p \times p$ submatrix $B'(G_1^*, G_2^*)$ has elements

$$
B'(G_1^*, G_2^*)[i,j] = \begin{cases} m(n-1) + d_{G_1^*}(v_i) - d_{G_2^*}(v_i) & \text{if } i = j, \\ -m - \ell_{G_1^*}(v_i v_j) + \ell_{G_2^*}(v_i v_j) & \text{otherwise.} \end{cases}
$$

We note that $\ell_{G_1^*}(v_i v_j)$ and $\ell_{G_2^*}(v_i v_j)$ are the number of edges of the vertices v_i and v_j in G_1^* and G_2^*, respectively. The entries $d_{G_1^*}(v_i)$ and $d_{G_2^*}(v_i)$, $1 \leq i \leq p$, are the degrees of vertex v_i of G_1^* and G_2^*, respectively. Note that, $V(G_1^*) = V(G_2^*)$.

It is straightforward to apply a technique similar to that we have applied for the computation of the determinant of the matrix L_1' in the case of $K_n^m - G$. Thus, in the case of $K_n^m + G_1^* - G_2^*$, the determinant of the matrix L_1' of Eq. (1) becomes

$$
\det (L'_1) \;=\; \begin{vmatrix}
\frac{1}{n} & -m & \cdots & -m & -m & & -m & \cdots & -m \\
0 & mn & & & & & & & \\
\vdots & & \ddots & & & & & & \\
0 & & & mn & & & & & \\
0 & & & & B(G_1^*,G_2^*)[1,1] & & & & B(G_1^*,G_2^*)[j,i] \\
0 & & & & & B(G_1^*,G_2^*)[2,2] & & & \\
\vdots & & & & & & & \ddots & \\
0 & & & & B(G_1^*,G_2^*)[i,j] & & & & B(G_1^*,G_2^*)[p,p]
\end{vmatrix}
$$

$$
=\; m \cdot (mn)^{n-p-2} \cdot \det\left(B(G_1^*,G_2^*)\right),
$$

where the $p \times p$ submatrix $B(G_1^*,G_2^*)$ has elements

$$
B(G_1^*,G_2^*)[i,j] = \begin{cases} mn + d_{G_1^*}(v_i) - d_{G_2^*}(v_i) & \text{if } i = j, \\ -\ell_{G_1^*}(v_i v_j) + \ell_{G_2^*}(v_i v_j) & \text{otherwise.} \end{cases}
$$

Since $B(G_1^*,G_2^*) = mnI_p + L(G_1^*) - L(G_2^*)$, we set $B = B(G_1^*,G_2^*)$ and obtain the following result.

Theorem 4.1. *Let K_n^m be the m-complete multigraph on n vertices, and let G_1, G_2 be two multigraphs such that $V(G_1) \subseteq V(K_n^m)$ and $E(G_2) \subseteq E(K_n^m + G_1)$. Then,*

$$
\tau(K_n^m + G_1 - G_2) = m \cdot (mn)^{n-p-2} \det(B),
$$

where $p = |V(G_1) \cup V(G_2)|$, $B = mnI_p + L(G_1^*) - L(G_2^*)$ is a $p \times p$ matrix, $L(G_1^*)$ and $L(G_2^*)$ are the Kirchhoff matrices of the union-stable graphs G_1^* and G_2^* of (G_1,G_2), respectively.

4.2 The general case $K_n^{(m)} \pm G$

Let $K_n^{(m)}$ be a complete multigraph on n vertices and let G be a subgraph of $K_n^{(m)}$. We will show that the previous theorem provides the key idea for computing the number $\tau(K_n^{(m)} \pm G)$, where $K_n^{(m)} \pm G$ is the graph that results from $K_n^{(m)}$ by adding or removing the edges of G. Since $\lambda(K_n^{(m)}) > 0$, we have that $K_n^{(m)} = K_n^m + \mathcal{H}(K_n^{(m)})$, where $\mathcal{H}(K_n^{(m)})$ is a characteristic graph of $K_n^{(m)}$. Then we have that,

$$
K_n^{(m)} \pm G = K_n^m + \mathcal{H}(K_n^{(m)}) \pm G.
$$

The addition of the edges of G in the graph $K_n^{(m)}$, implies that $K_n^{(m)} + G = K_n^m + G'$, where the graph $G' = \mathcal{H}(K_n^{(m)}) + G$. Thus, for the computation of the number $\tau(K_n^{(m)} + G)$ we can apply Theorem 3.2. On the other hand, in the case of removal the edges of G from the graph $K_n^{(m)}$, for the computation of the number $\tau(K_n^{(m)} - G)$ we can apply Theorem 4.1, since $K_n^{(m)} - G = K_n^m + \mathcal{H}(K_n^{(m)}) - G$. Concluding we have the following result.

Lemma 4.1. *Let $K_n^{(m)}$ be a complete multigraph on n vertices and $\mathcal{H}(K_n^{(m)})$ be a characteristic graph of $K_n^{(m)}$, and let G be a subgraph of $K_n^{(m)}$. Then,*

$$
\tau(K_n^{(m)} \pm G) = m \cdot (mn)^{n-p-2} \det(B),
$$

where $p = |V(\mathcal{H}(K_n^{(m)})) \cup V(G)|$, $B = mnI_p + L(\mathcal{H}(K_n^{(m)})^*) \pm L(G^*)$ is a $p \times p$ matrix and $L(\mathcal{H}(K_n^{(m)})^*)$ and $L(G^*)$ are the Kirchhoff matrices of the union-stable graphs $\mathcal{H}(K_n^{(m)})^*$ and G^* of $(\mathcal{H}(K_n^{(m)}), G)$, respectively.

Note that, we consider the graph $K_n^{(m)} \pm G$ and therefore G must be a subgraph of $K_n^{(m)}$. However in the case of the $K_n^{(m)} + G$ graph, similar to Theorem 3.2, it is obvious that G can be a graph spanned by any set of edges joining the vertices of $K_n^{(m)}$.

5 Classes of graphs

In this section, we generalize known closed formulas for the number of spanning trees of families of graphs of the form $K_n - G$. As already mentioned in the introduction there exist many cases for the $\tau(K_n - G)$, depending on the choice of G. The purpose of this section is to prove closed formulas for $\tau(K_n^m \pm G)$, by applying similar techniques to that of the case of $K_n - G$ and, most important, by using results of Section 3. Thus we derive closed formulas for the number of spanning trees $\tau(K_n^m \pm G)$, in the cases where G forms (i) a complete multipartite graph, and (ii) a multi-star graph.

5.1 Complete multipartite graphs

A graph is defined to be a *complete multipartite* (or complete k-partite) if there is a partition of its vertex set into k disjoint sets such that no two vertices of the same set are adjacent and every pair of vertices of different sets are adjacent. We denote a complete multipartite graph on p vertices by $K_{m_1, m_2, \ldots, m_k}$, where $p = m_1 + m_2 + \cdots + m_k$.

Let $G = K_{m_1, m_2, \ldots, m_k}$ be a complete multipartite graph on p vertices. In [17] it has been proved that the number of spanning trees of $K_n - G$ is given by the following formula:

$$\tau(K_n - G) = n^{n-p-1}(n-p)^{k-1} \prod_{i=1}^{k} (n - (p - m_i))^{m_i - 1},$$

where p is the number of vertices of G.

In this section, we extend the previous result by deriving a closed formula for the number of spanning trees of the graphs $K_n^m \pm G$, where G a complete multipartite graph on $p \leq n$ vertices. From Theorem 3.1, we construct the $p \times p$ matrix $B(G)$ and add one row and column to the matrix $B(G)$; the resulting $(p+1) \times (p+1)$ matrix $B'(G)$ has 1 in position $(1,1)$, α in positions $(1,j)$, $2 \leq j \leq p+1$, and 0 in positions $(i,1)$, $2 \leq i \leq p+1$; recall that, $\alpha = \pm 1$. Thus, the resulting matrix $B'(G)$ has the following form:

$$B'(G) = \begin{bmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\ & M_1 & -\alpha & \cdots & -\alpha \\ & -\alpha & M_2 & \cdots & -\alpha \\ & \vdots & \vdots & \ddots & \vdots \\ & -\alpha & -\alpha & \cdots & M_k \end{bmatrix},$$

where the diagonal $m_i \times m_i$ submatrices M_i have diagonal elements $mn + \alpha \cdot (p - m_i)$, $1 \leq i \leq k$. Note that, $\det(B(G)) = \det(B'(G))$.

In order to compute the determinant of the matrix $B'(G)$ we add the first row to the next p rows. We multiply the $2, 3, \ldots, p+1$ columns of the matrix $B'(G)$ by $-1/(mn + \alpha \cdot p)$ and add them to the first column; note that, the sum of each row of the matrix $B'(G)$ is equal to $mn + \alpha \cdot p$. Thus, the determinant of matrix $B'(G)$ becomes:

$$\det\left(B'(G)\right) = \begin{vmatrix} 1 - \frac{\alpha \cdot p}{mn + \alpha \cdot p} & \alpha & \alpha & \cdots & \alpha \\ & M_1' & & & \\ & & M_2' & & \\ & & & \ddots & \\ & & & & M_k' \end{vmatrix}$$

$$= \frac{mn}{mn + \alpha \cdot p} \cdot \det\left(M_1'\right) \cdot \det\left(M_2'\right) \cdot \cdots \cdot \det\left(M_k'\right), \tag{3}$$

where the $m_i \times m_i$ submatrices M_i', $1 \le i \le k$, have the following form:

$$M_i' = \begin{vmatrix} mn + \alpha \cdot (p - m_i) + \alpha & \alpha & \cdots & \alpha \\ \alpha & mn + \alpha \cdot (p - m_i) + \alpha & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & mn + \alpha \cdot (p - m_i) + \alpha \end{vmatrix}.$$

For the determinant of matrix M_i' we multiply the first row by -1 and add it to the next $m_i - 1$ rows. Then, we add the columns of matrix M_i' to the first column. Observing that $mn + \alpha \cdot (p - m_i) + \alpha \cdot m_i = mn + \alpha \cdot p$, we obtain

$$\det\left(M_i'\right) = (mn + \alpha \cdot p) \cdot (mn + \alpha \cdot (p - m_i))^{m_i - 1}.$$

Thus, from Equation (3) we have the following result.

Theorem 5.1. *Let $G = K_{m_1, m_2, \ldots, m_k}$ be a complete multipartite graph on $p = m_1 + m_2 + \cdots + m_k$ vertices. Then,*

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-1}(mn + \alpha \cdot p)^{k-1} \prod_{i=1}^{k} (mn + \alpha \cdot (p - m_i))^{m_i - 1},$$

where $p \le n$ and $\alpha = \pm 1$ according to $K_n^m \pm G$.

Remark 5.1. The class of complete multipartite graphs contains the class of c-split graphs (complete split graphs); a graph is defined to be a c-split graph if there is a partition of its vertex set into a stable set S and a complete set K and every vertex in S is adjacent to all the vertices in K [10].

Thus, a c-split graph G on p vertices and $V(G) = K + S$ is a complete multipartite graph $K_{m_1, m_2, \ldots, m_k}$ with $m_1 = |S|$, $m_2 = m_3 = \cdots = m_k = 1$ and $k = |K| + 1$. A closed formula for the number of spanning trees of the graph $K_n - G$ was proposed in [14], where G is a c-split graph.

Let G be a c-split graph on p vertices and let $V(G) = K + S$ be the partition of its vertex set. Then, from Theorem 4.1 we obtain that the number of spanning trees of the graphs $K_n^m \pm G$ is given by the following closed formula:

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-1}(mn + \alpha \cdot |K|)^{|S|-1}(mn + \alpha \cdot p)^{|K|},$$

where $p = |K| + |S|$ and $p \le n$. $\square$

5.2 Multi-star graphs

A *multi-star* graph, denoted by $K_r(b_1, b_2, \ldots, b_r)$, consists of a complete graph K_r with vertices labelled $v_1, v_2, \ldots, v_r$, and b_i vertices of degree one, which are incident with vertex v_i, $1 \le i \le r$ [6, 15, 21].

Let $G = K_r(b_1, b_2, \ldots, b_r)$ be a multi-star graph on $p = r + b_1 + b_2 + \cdots + b_r$ vertices. In [6, 15, 21], it has been proved that the number of spanning trees of the graph $K_n - G$ is given by the following closed formula:

$$\tau(K_n - G) = n^{n-p-2}(n-1)^{p-r}\left(1 + \sum_{i=1}^{r}\frac{1}{q_i - 1}\right) \cdot \prod_{i=1}^{r}(q_i - 1),$$

where $q_i = n - (r - 1 + b_i) - \frac{b_i}{n-1}$.

In this section, based on Theorem 3.1, we generalize the previous result by deriving a closed formula for the number of spanning trees of the graphs $K_n^m \pm G$, where G a multi-star on $p \le n$ vertices. Let K_r be the complete graph of the multi-star graph G and let $v_1, v_2, \ldots, v_r$ be its vertices. The vertex set consisting of the vertex v_i and the b_i vertices of degree one which are incident with vertex v_i induces a star on $b_i + 1$ vertices, $1 \le i \le r$. We construct a $(b_i + 1) \times (b_i + 1)$ matrix M_i which corresponds to the star with center vertex v_i; it has the following form:

$$M_i = \begin{bmatrix} mn + \alpha & & & -\alpha \\ & mn + \alpha & & -\alpha \\ & & & \vdots \\ -\alpha & -\alpha & \cdots & mn + \alpha \cdot (r - 1 + b_i) \end{bmatrix},$$

where $\alpha = \pm 1$.

In order to compute the determinant of the matrix M_i we first multiply the first row by -1 and add it to the next $b_i - 1$ rows. We then add the b_i columns to the first column. Finally, we multiply the first column by $\frac{\alpha}{mn+\alpha}$ and add it to the last column. Thus, by observing that $\alpha^2 = 1$, we obtain:

$$\det(M_i) = (mn + \alpha)^{b_i} \cdot \left(mn + \alpha \cdot (r - 1 + b_i) - \frac{\alpha^2 b_i}{mn+\alpha}\right)$$

$$= (mn + \alpha)^{b_i} \cdot \left(mn + \alpha \cdot (r - 1 + b_i) - \frac{b_i}{mn+\alpha}\right)$$

$$= (mn + \alpha)^{b_i} \cdot q_i,$$

where

$$q_i = mn + \alpha \cdot (r - 1 + b_i) - \frac{b_i}{mn+\alpha}, \quad 1 \le i \le r. \tag{4}$$

We are now in a position to compute the number of spanning trees $\tau(K_n^m \pm G)$ using Theorem 3.1. Thus, we have

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-2} \cdot \det(B(G)) \tag{5}$$

where

$$B(G) = \begin{bmatrix} M_{1,1} & & & & -\alpha & & & & \\ & M_{2,2} & & & & -\alpha & & & \\ & & \ddots & & & & \ddots & & \\ & & & M_{r,r} & & & & & -\alpha \\ -\alpha & & & & mn+\alpha\cdot d_G(v_1) & -\alpha & \cdots & & -\alpha \\ & -\alpha & & & -\alpha & mn+\alpha\cdot d_G(v_2) & \cdots & & -\alpha \\ & & \ddots & & \vdots & \vdots & \ddots & & \vdots \\ & & & -\alpha & -\alpha & -\alpha & \cdots & & mn+\alpha\cdot d_G(v_r) \end{bmatrix}$$

is a $p \times p$ matrix and $M_{i,i}$ is a submatrix which is obtained from M_i by deleting its last row and its last column, $1 \le i \le r$. The degrees of the vertex v_i of K_r is equal to $d_G(v_i) = r - 1 + b_i$, $1 \le i \le r$. It is now suffices to compute the determinant of the matrix $B(G)$. Following a procedure similar to that we applied to the matrix M_i, we obtain:

$$\det\left(B(G)\right) = (mn+\alpha)^{p-r} \cdot \begin{vmatrix} q_1 & -\alpha & \cdots & -\alpha \\ -\alpha & q_2 & \cdots & -\alpha \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha & -\alpha & \cdots & q_r \end{vmatrix}$$

$$= (mn+\alpha)^{p-r} \cdot \det\left(D\right).$$

Recall that, $q_i = mn + \alpha \cdot (r - 1 + b_i) - \frac{b_i}{mn+\alpha}$; see Equation (4). In order to compute the determinant of the $r \times r$ matrix D we first multiply the first row of D by -1 and add it to the $r - 1$ rows. Then, we multiply column i by $\frac{q_1+\alpha}{q_i+\alpha}$, $2 \le i \le r$, and add it to the first column. Expanding in terms of the rows of matrix D, we have that

$$\det\left(D\right) = \left(1 - \alpha\sum_{i=1}^{r}\frac{1}{q_i+\alpha}\right)\cdot\prod_{i=1}^{r}(q_i+\alpha).$$

Thus, substituting the value of $\det\left(D\right)$ into Equation (5), we obtain the following theorem.

Theorem 5.2. *Let $G = K_r(b_1, b_2, \ldots, b_r)$ be a multi-star graph on $p = r + b_1 + b_2 + \cdots + b_r$ vertices. Then,*

$$\tau(K_n^m \pm G) = m \cdot (mn)^{n-p-2}(mn+\alpha)^{p-r}\left(1 - \alpha\cdot\sum_{i=1}^{r}\frac{1}{q_i+\alpha}\right)\cdot\prod_{i=1}^{r}(q_i+\alpha),$$

where $p \le n$, $q_i = mn + \alpha\cdot(r - 1 + b_i) - \frac{b_i}{mn+\alpha}$ and $\alpha = \pm 1$ according to $K_n^m \pm G$.

6 Concluding remarks

In this paper we derived determinant based formulas for the number of spanning trees of the family of graphs of the form $K_n^m \pm G$, and also for the more general family of graphs $K_n^{(m)} \pm G$, where K_n^m (resp. $K_n^{(m)}$) is the complete multigraph on n vertices with exactly (resp. at least) m edges joining every pair of vertices and G is a multigraph spanned by a set of edges of K_n^m (resp. $K_n^{(m)}$). Based on these determinant based formulas, we prove closed formulas for the number of spanning trees $\tau(K_n^m \pm G)$, in the case where G is (i) a complete multipartite graph, and (ii) a multi-star graph.

In light of our results, it would be interesting to consider the problem of proving closed formulas for the number of spanning tree $\tau(K_n^m \pm G)$ in the cases where G belongs to other classes of simple graphs or multigraphs. We pose it as an open problem.

The problem of maximizing the number of spanning trees was solved for some families of graphs of the form $K_n - G$, where G is a multi-star graph, a union of paths and cycles, etc. (see [6, 8, 15, 18]). Thus, an interesting open problem is that of maximizing the number of spanning trees of graphs of the form $K_n^m \pm G$.

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