SCATTERING OF A SPHERICAL ACOUSTIC FIELD FROM AN ECCENTRIC SPHEROIDAL STRUCTURE SIMULATING THE KIDNEY-STONE SYSTEM

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Scattering of a Spherical Acoustic Field from an Eccentric Spheroidal Structure Simulating the Kidney-stone System

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ABSTRACT

We consider the acoustic scattering of time-harmonic spherical waves from an eccentric non coaxial spheroidal structure simulating the kidney-stone system. The proposed analysis is based on the application of translational addition theorem for spheroidal wave functions. The resulting theoretical model is frequency-independent. Numerical results concerning the applicability of our approach are also presented.

1. Introduction

The investigation of exterior boundary value problems involving scattering processes shares nowadays considerable attention since it disposes several application branches. In particular, the exploitation of scattering methods to biomedical applications is one of the most interesting and challenging scientific areas. In this framework, we focus on the problem of kidney stones identification using sound scattering techniques. Although there exist ultrasound, x-rays or pyelogram techniques for the localization of kidney stones, it is of great importance to develop a model based on the scattering of a point source generated field in the resonance region, which could be easily interpreted by a physician suggesting a treatment method.

In previous works [1,2], the kidney-stone system has been simulated by a multilayered spheroidal structure and the theoretical and numerical implementation of the specific scattering problem has been developed, aiming at a parametric investigation of the system. The spheroidal geometry is proved to provide with a suitable and realistic configuration of the system, which also has the advantage of permitting the development of analytical methods for handling with the underlying scattering process. However, the results obtained are subject to the restrictive (although reasonable) assumption that confocal spheroids simulate the two components of the system, i.e. the kidney tissue and the stone. This requirement stems from the initial desire to have a common spheroidal system with respect to which all physical quantities entering scattering mechanism must be expressed via spectral analysis. This assumption cannot affront general situations in which the stone body is simulated by different spheroidal parameters and is not necessarily confocal or even coaxial with the kidney background. Actually the development of an efficient model describing uniformly the specific scattering problem and aiming at continuing the basis for the solution of the inverse scattering problem must allow general relevant positions of the components of the system.

This work aims at developing a model remedying this restriction, allowing independent configurations for the kidney and the stone. The two components are again simulated by spheroidal structures, which now are not forced to belong to the same system. The coexistence of two different spheroidal systems, along with the spherical system introduced by the point source, renders the investigation of the problem much more complicated. The involved secondary wave fields, produced by the scattering process, can be represented in terms of the spheroidal eigenvectors emerging from the two different spheroidal systems. Their determination consists of the specification of the spectral decomposition expansion coefficients. As commonly in boundary value problems, the boundary conditions on discontinuity surfaces provide with the necessary equations for expansion coefficients determination. However, in our case the discontinuity surfaces constitute coordinate surfaces of different systems and the implication of intricate addition theorems [3] is necessary to allow transition from one system to the other. This methodology is necessary though to remain in the analytical regime. The price to pay is the development of an extended and very demanding theoretical treatment, leading to the acquisition of a rather complicated infinite algebraic non-homogeneous system, whose solution provides with the aforementioned spectral expansions coefficients. A suitable truncation of the above system has been applied leading to the numerical implementation of the method for some special and indicative cases as far as the several parameters of the problem are concerned.

2. Formulation of the problem

Let us consider an eccentric spheroidal structure consisting of two non-coaxial spheroidal bodies. The inner spheroid with surface S_2 occupies region V_2 and is centered at the origin of a primed Cartesian coordinate system $O'(x', y', z')$. It has focal distance a', corresponds to the kidney stone and is impenetrable and rigid. The exterior region V_1 surrounded by surface S_1 corresponds to the kidney tissue and is characterized by its density ρ_1 and the speed of sound c_1. A Cartesian coordinate system $O(x, y, z)$ with its z-axis being parallel to the z'-axis is assumed to have its origin at the center of the outer spheroidal body, which has focal distance a and is penetrable. The spherical coordinates of O' with respect to this coordinate system are (r_1, θ_1, ϕ_1). The isotropic and homogeneous background medium occupying region V has density ρ_{ext} and velocity of sound propagation c. The geometry of the problem is given in Fig.1.

The kidney-stone system is illuminated by an acoustic spherical wave emanated by a source located at point M in region V with spheroidal coordinates $(\mu_0, \theta_0, \phi_0)$. The acoustic wave generated by the source constitutes the incident field. The interference of this field with the structure leads to the creation of the scattered field propagating outwards the scatterer as well as the creation of an acoustic wave penetrating the outer spheroidal surface but not the interior rigid body.

The time-independent part of the incident spherical wave is given by

$$u^{in}(\bar{r},\bar{r}_0)=\frac{e^{ik|\bar{r}-\bar{r}_0|}}{|\bar{r}-\bar{r}_0|}\quad,\ \bar{r}\in V\,, \tag{1}$$

where we have suppressed the harmonic time-dependence $\exp\{-i\omega t\}$, $k=\frac{\omega}{c}$ is the wave number in region V with ω standing for the angular frequency and $\bar{r}_0$ is the source position vector.

The scattered field satisfies the time-reduced Helmholtz equation

$$\Delta u^{sc}(\bar{r})+k^2u^{sc}(\bar{r})=0\quad,\ \bar{r}\in V, \tag{2}$$

and the Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty} r\left(\frac{\partial u^{sc}(\bar{r})}{\partial r}-iku^{sc}(\bar{r})\right)=0\quad,\ \bar{r}\in V. \tag{3}$$

Moreover, the interior field $u^{(1)}(\bar{r})$ satisfies also the Helmholtz equation

$$\Delta u^{(1)}(\bar{r})+k_1^2u^{(1)}(\bar{r})=0\quad,\ \bar{r}\in V_1, \tag{4}$$

where $k_1=\frac{\omega}{c_1}$ stands for the wave number in region V_1. These fields are connected through appropriate boundary conditions satisfied on the discontinuity surfaces S_1 and S_2. More precisely, a Newmann-type boundary condition must be satisfied on S_2, i.e.

$$\frac{\partial u^{(1)}(\bar{r})}{\partial n}=0\quad,\ \bar{r}\in S_2, \tag{5}$$

where $\frac{\partial}{\partial n}$ stands for the normal derivative operator. On S_1 the following transmission boundary conditions must be satisfied

$$\frac{\partial u^{in}(\bar{r})}{\partial n}+\frac{\partial u^{sc}(\bar{r})}{\partial n}=\frac{\partial u^{(1)}(\bar{r})}{\partial n}\ ,\ \bar{r}\in S_1, \tag{6}$$

$$\rho_{ext}(u^{in}(\bar{r})+u^{sc}(\bar{r}))=\rho_1\, u^{(1)}(\bar{r})\ ,\ \bar{r}\in S_1. \tag{7}$$

The rest of the paper is devoted to the solution of the well-posed boundary value problem described by Eqs. (2)-(7).

3. The Solution of the Direct Scattering Problem

The acoustic fields introduced in the formulation of the underlying scattering problem are expanded in terms of spheroidal wave functions, which constitute a complete basis in the space of scalar Helmholtz's equation solutions. These wave functions along with the necessary framework concerning the spheroidal geometry are briefly discussed in the Appendix. Furthermore, special attention must be paid on the fact that, as explained in the Appendix, we have reformulated slightly the adopted basis set, incorporating the complete range of the azimuthal separation constants in order to ensure a more convenient representation of the forthcoming addition formulae. The aforementioned expansions are expected to provide with the appropriate fields expressions fitting suitably to the boundary conditions imposed on the spheroidal surfaces.

More precisely, we adopt the following spheroidal representations

$$u^{(1)}(\overline{r'})=\sum_{j=1,3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\Lambda_{mn}(c')}a_{mn}^{1,j}\Psi_{mn}^{(j)}(\xi',\eta',\phi';c')\ ,\ \overline{r'}\in V_1, \tag{8}$$

$$u^{sc}(\bar{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\Lambda_{mn}(c^{ext})}a_{mn}^{ext}\Psi_{mn}^{(3)}(\xi,\eta,\phi;c^{ext}) \,,\ \bar{r}\in V. \tag{9}$$

In Eq. (8) the scalar solutions $\Psi_{mn}^{(j)}(\xi',\eta',\phi';c')$ are defined with respect to the primed spheroidal coordinate system associated to the enclosed spheroid. Moreover, we have introduced the parameters $c' = \frac{1}{2}k_1 a'$ and $c^{ext} = \frac{1}{2}ka$ which constitute a measure of the relation between the geometric characteristic dimensions of the system and the wavelengths. In addition, $\Lambda_{mn}(c_a), c_a = \{c', c^{ext}\}$ are related to the normalization constants of the angular functions $S_{mn}(\eta;c_a)$ as follows

$$\Lambda_{mn}(c_a) = \int_{-1}^{1}\left|S_{mn}(\eta;c_a)\right|^2 d\eta = \sum_{\kappa=0,1}^{\infty}{}' \left|d_{\kappa}^{mn}(c_a)\right|^2\left(\frac{2}{2\kappa+2m+1}\right) \times\begin{cases}\dfrac{(\kappa+2m)!}{\kappa!}, & n=0,1,2,\ldots;\ 0\le m\le n\\[2ex] \dfrac{\kappa!}{(\kappa+2m)!}, & n=0,1,2,\ldots;\ -n\le m<0,\end{cases} \tag{10}$$

where we have trivially used the same symbol for the $d_{\kappa}^{mn}(c_a)$ coefficients for both $m\ge 0$ and $m<0$ although they are determined through the solution of a specific eigenvalue problem depending on m being positive or negative as it is explained in the Appendix. Finally, we notice that the specific selection of the radial functions $R_{mn}^{(3)}(\xi;c^{ext})$ to express the scattered field incorporates the outgoing propagating character of the scattered wave.

Similarly to the secondary fields, the incident field has to be expanded in the same function basis. Indeed, we employ a slightly modified version of an addition theorem met in [4] concerning the spheroidal representation of a spherical wave

$$u^{inc}(r) = 2ik\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\Lambda_{mn}(c^{ext})}S_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0} \times \begin{cases}\Psi_{mn}^{(1)}(\xi,\eta,\phi;c^{ext})R_{mn}^{(3)}(\cosh\mu_0;c^{ext}), & \mu \le \mu_0\\ \Psi_{mn}^{(3)}(\xi,\eta,\phi;c^{ext})R_{mn}^{(1)}(\cosh\mu_0;c^{ext}), & \mu > \mu_0\ .\end{cases} \tag{11}$$

In view of Eq. (8), the boundary condition (5) can be immediately satisfied yielding

$$\sum_{j=1,3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{a_{mn}^{1,j}}{\Lambda_{mn}(c')}R_{mn}^{(j)'}(\cosh\mu'_{02};c')S_{mn}(\cos\theta';c')e^{im\phi'} = 0, \tag{12}$$

for $0 \le \theta' \le \pi$, $0 \le \phi' < 2\pi$, with μ'_{02} being the specific value of the coordinate μ' characterizing the spheroidal surface S_2. Exploiting orthogonality arguments of the underlying trigonometric and angular spheroidal functions we conclude that

$$a_{mn}^{1,1}R_{mn}^{(1)'}(\cosh\mu'_{02};c') + a_{mn}^{1,3}R_{mn}^{(3)'}(\cosh\mu'_{02};c') = 0,\ n \ge 0,\ m = -n,\ldots,n. \tag{13}$$

In order to satisfy for the boundary conditions (6)-(7) on S_1, we employ the following translational addition theorem for scalar spheroidal wave functions [3]

$$\Psi_{mn}^{(j)}(\xi',\eta',\phi';c') = \begin{cases}\displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}A_{mn}^{\mu\nu(j)}\Psi_{mn}^{(1)}(\xi,\eta,\phi;c), & r \le r_1\\ \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\ {\sum_{t=|m-\mu|,\,|m-\mu|+1}^{\infty}}{}' B_{mn}^{\mu\nu t}\Psi_{m-\mu,t}^{(j)}(\xi,\eta,\phi;c), & r > r_1\ ,\end{cases} \tag{14}$$

where $c = \frac{1}{2}ka$.

The coefficients $A_{mn}^{\mu\nu(j)}$ are defined as

$$A_{mn}^{\mu\nu(j)} = 2\frac{(-1)^{\mu}}{\Lambda_{\mu\nu}(c)}{\sum_{q=0,1}^{\infty}}{}'{\sum_{s=0,1}^{\infty}}{}'\sum_{p} i^{p+\nu-n}\frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!}d_q^{mn}(c')d_s^{\mu\nu}(c) \times a(m,|m|+q|-\mu,|\mu|+s|p)Z_p^{(j)}(kr_1)P_p^{m-\mu}(-\cos\theta_1)e^{i(m-\mu)(\pi+\phi_1)}\ , \tag{15}$$

where for the summation index p we have that $p = |m| + q + |\mu| + s$, $|m| + q + |\mu| + s - 2, \ldots, \big||m| + q - |\mu| - s\big|$ if $\big||m| + q - |\mu| - s\big| \geq |m - \mu|$. In the case that $\big||m| + q - |\mu| - s\big| < |m - \mu|$, the lower limit of p is replaced by $|m - \mu|$ or $|m - \mu| + 1$ according as $|m| + q + |\mu| + s + |m - \mu|$ is even or odd. Similarly the coefficients $B_{mn}^{\mu\nu t}$ are given as

$$B_{mn}^{\mu\nu t} = 2\frac{(-1)^{\mu}}{\Lambda_{m-\mu,t}(c)}(2\nu+1)\frac{(\nu+\mu)!}{(\nu-\mu)!}Z_{\nu}^{(1)}(kr_1)P_{\nu}^{\mu}(-\cos\theta_1)e^{i\mu(\pi+\phi_1)}$$
$$\times \sum_{q=0,1}^{\infty}{}' \sum_{p} \frac{i^{2p+\nu-n-t}}{(2p+1)}\frac{(p+m-\mu)!}{(p-m+\mu)!}d_q^{mn}(c')d_{p-|m-\mu|}^{m-\mu,t}(c)a(m,|m|+q|-\mu,\nu|p), \tag{16}$$

where $p = |m| + q + \nu, |m| + q + \nu - 2, \ldots, \big||m| + q - \nu\big|$ if $\big||m| + q - \nu\big| \geq |m - \mu|$. In the case that $\big||m| + q - \nu\big| < |m - \mu|$, the lower limit $\big||m| + q - \nu\big|$ of index p is replaced by $|m - \mu|$ or $|m - \mu| + 1$ according as $|m| + q + \nu + |m - \mu|$ is even or odd.

In Eqs. (15)-(16), we also meet the spherical functions $Z_l^{(j)}$ introduced in (A.7)-(A.10) and the coefficients $a(m,n|\mu,\nu|p)$, which can be identified through the relation

$$a(m,n|\mu,\nu|p) = (-1)^{m+\mu}(2p+1)\left[\frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m-\mu)!}\right]^{1/2}$$
$$\times \begin{bmatrix} n & \nu & p \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n & \nu & p \\ m & \mu & -m-\mu \end{bmatrix}, \tag{17}$$

where the last two factors are the Wigner 3-j symbols [5].

Using the addition theorem (14), the representation (8) of the interior field $u^{(1)}$ is transformed into the following one, expressed in the $O(x,y,z)$ system

$$u^{(1)}(\bar{r}) = \sum_{j=1,3}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{a_{mn}^{1,j}}{\Lambda_{mn}(c')} \begin{cases} \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu} A_{mn}^{\mu\nu(j)} \Psi_{\mu\nu}^{(1)}(\xi,\eta,\phi;c) \ , \ r \le r_1 \\ \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{\substack{t=|m-\mu|,\\ t|m-\mu|+1}}^{\infty}{}' B_{mn}^{\mu\nu t} \Psi_{m-\mu,t}^{(j)}(\xi,\eta,\phi;c) \ , \ r > r_1, \end{cases} \tag{18}$$

which has the suitable form for handling the boundary conditions (6)-(7). Hence, in view of Eqs. (9), (11) and (18), the boundary condition (6) after a suitable relabelling of the summation indices and a functional projection on the basis of the azimuthal functions is written as

$$\begin{aligned} &\sum_{n=|m|}^{\infty} a_{mn}^{ext} R_{mn}^{(3)\prime}(\cosh\mu_{01};c^{ext}) \frac{S_{mn}(\cos\theta;c^{ext})}{\Lambda_{mn}(c^{ext})} \\ &+2ik\sum_{n=|m|}^{\infty} \frac{S_{mn}(\cos\theta_0;c^{ext})}{\Lambda_{mn}(c^{ext})} e^{-im\phi_0} R_{mn}^{(3)}(\cosh\mu_0;c^{ext}) R_{mn}^{(1)\prime}(\cosh\mu_{01};c^{ext}) S_{mn}(\cos\theta;c^{ext}) = \\ &\sum_{n=|m|}^{\infty}\left[\sum_{m'=-\infty}^{\infty}\sum_{n'=|m'|}^{\infty} \frac{a_{m'n'}^{1,1}}{\Lambda_{m'n'}(c')} A_{m'n'}^{mn(1)}\right] R_{mn}^{(1)\prime}(\cosh\mu_{01};c) S_{mn}(\cos\theta;c) \\ &+\sum_{m'=-\infty}^{\infty}\sum_{n'=|m'|}^{\infty}\sum_{n''=|m'+m|}^{\infty}\sum_{\substack{t=|m|,\\ |m|+1}}^{\infty}{}' \frac{a_{m+m',n''}^{1,3}}{\Lambda_{m+m',n''}(c')} B_{m+m',n''}^{m'n't} R_{mt}^{(3)\prime}(\cosh\mu_{01};c) S_{mt}(\cos\theta;c) \ . \end{aligned} \tag{19}$$

Projecting then Eq. (19) on the complete and orthogonal set of functions $S_{mn'}(\cos\theta;c^{ext})$ and again relabelling the summation indices, we obtain the following algebraic relation for every pair (m,n), $n \ge 0; |m| \le n$

$$a_{mn}^{ext} R_{mn}^{(3)'}(\cosh\mu_{01};c^{ext}) + 2ikS_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0} R_{mn}^{(3)}(\cosh\mu_0;c^{ext}) R_{mn}^{(1)'}(\cosh\mu_{01};c^{ext}) =$$

$$\sum_{n'=|m|}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\frac{a_{m'l}^{1,1}}{\Lambda_{m'l}(c')}A_{m'l}^{mn'(1)}\right]R_{mn'}^{(1)'}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle$$

$$+\sum_{n'=|m|,|m|+1}^{\infty}{}' \left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\sum_{n''=|m'+m|}^{\infty}\frac{a_{m+m',n''}^{1,3}}{\Lambda_{m+m',n''}(c')}B_{m+m',n''}^{m',l,n'}\right]$$

$$\times R_{mn'}^{(3)'}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle, \tag{20}$$

where the brackets $\langle S_{mn'}, S_{mn}\rangle$ indicate the mixed "inner" products of angular functions

$$\langle S_{mn'}, S_{mn}\rangle = \int_{-1}^{1} S_{mn'}(\eta;c) S_{mn}(\eta;c^{ext})d\eta =$$

$$\sum_{\kappa=0,1}^{\infty}{}'\sum_{\kappa'=0,1}^{\infty}{}' d_{\kappa'}^{mn'}(c)d_{\kappa}^{mn}(c^{ext})\int_{-1}^{1}P_{m+\kappa'}^{m}(\eta)P_{m+\kappa}^{m}(\eta)d\eta = \tag{21}$$

$$\sum_{\kappa=0,1}^{\infty}{}' d_{\kappa}^{mn'}(c)d_{\kappa}^{mn}(c^{ext})\frac{2}{2\kappa+2m+1}\begin{cases}\dfrac{(\kappa+2m)!}{\kappa!}, & n=0,1,2,...;0\le m\le n\\ \dfrac{\kappa!}{(\kappa+2m)!}, & n=0,1,2,...;-n\le m<0,\end{cases}$$

which do not share mutual orthogonality, i.e., they would be diagonal only in the special case $c = c^{ext}$.

Following similar manipulations, the boundary condition (7) assumes the form

$$\rho_{ext}[a_{mn}^{ext} R_{mn}^{(3)}(\cosh\mu_{01};c^{ext}) + 2ikS_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0} R_{mn}^{(3)}(\cosh\mu_0;c^{ext}) R_{mn}^{(1)}(\cosh\mu_{01};c^{ext})] =$$

$$\rho_1\sum_{n'=|m|}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\frac{a_{m'l}^{1,1}}{\Lambda_{m'l}(c')}A_{m'l}^{mn'(1)}\right]R_{mn'}^{(1)}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle$$

$$+\rho_1\sum_{n'=|m|,|m|+1}^{\infty}{}' \left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\sum_{n''=|m'+m|}^{\infty}\frac{a_{m+m',n''}^{1,3}}{\Lambda_{m+m',n''}(c')}B_{m+m',n''}^{m',l,n'}\right]$$

$$\times R_{mn'}^{(3)}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle, \tag{22}$$

which is also valid for every specific pair of (m,n) with $(m,n), n\geq 0; |m|\leq n$. Eqs. (20), (22) may be rewritten as follows

$$a_{mn}^{ext} R_{mn}^{(3)'}(\cosh\mu_{01};c^{ext}) + 2ikS_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0}R_{mn}^{(3)}(\cosh\mu_0;c^{ext})R_{mn}^{(1)'}(\cosh\mu_{01};c^{ext}) =$$
$$\sum_{n'=|m|}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\frac{a_{m'l}^{1,1}}{\Lambda_{m'l}(c')}A_{m'l}^{mn'(1)}\right]R_{mn'}^{(1)'}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle$$
$$+\sum_{n'=|m|}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\sum_{n''=|m'+m|}^{\infty}\frac{a_{m+m',n''}^{1,3}}{\Lambda_{m+m',n''}(c')}B_{m+m',n''}^{m',l,n'}\right]$$
$$\times R_{mn'}^{(3)'}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle, \tag{23}$$

$$\rho_{ext}[a_{mn}^{ext} R_{mn}^{(3)}(\cosh\mu_{01};c^{ext}) + 2ikS_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0}R_{mn}^{(3)}(\cosh\mu_0;c^{ext})R_{mn}^{(1)}(\cosh\mu_{01};c^{ext})] =$$
$$\rho_1\sum_{n'=|m|}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\frac{a_{m'l}^{1,1}}{\Lambda_{m'l}(c')}A_{m'l}^{mn'(1)}\right]R_{mn'}^{(1)}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle$$
$$+\rho_1\sum_{n'=|m|}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{m'=-l}^{l}\sum_{n''=|m'+m|}^{\infty}\frac{a_{m+m',n''}^{1,3}}{\Lambda_{m+m',n''}(c')}B_{m+m',n''}^{m',l,n'}\right]$$
$$\times R_{mn'}^{(3)}(\cosh\mu_{01};c)\left\langle S_{mn'}(\cos\theta;c), S_{mn}(\cos\theta;c^{ext})\right\rangle, \tag{24}$$

where again $(m,n), n\geq 0; |m|\leq n$ and the "primed" summation over n' has been combined with the summation running over the index p in the definition (16) of the coefficient $B_{mn}^{\mu\nu t}$ yielding the modification of this definition by a multiplicative parity factor $\Xi(p,t)=0$ or 1 depending if $p-t$ is odd or even.

Finally, the algebraic equations (23)-(24), which are equivalent to the boundary conditions on S_2, are written in the condensed form

$$a_{mn}^{ext} R_{mn}^{(3)'}(\cosh\mu_{01};c^{ext}) + 2ikS_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0} R_{mn}^{(3)}(\cosh\mu_0;c^{ext}) R_{mn}^{(1)'}(\cosh\mu_{01};c^{ext})$$
$$= \sum_{n''=0}^{\infty}\sum_{m''=-n''}^{n''} \frac{a_{m''n''}^{1,1}}{\Lambda_{m''n''}(c')} Z_{m'',n'',m,n} + \sum_{n''=0}^{\infty}\sum_{m''=-n''}^{n''} \frac{a_{m''n''}^{1,3}}{\Lambda_{m''n''}(c')} D_{m''-m,n'',m,n},$$
$$n \geq 0,\ m = -n,...,n, \tag{25}$$

$$\rho_{ext}[a_{mn}^{ext} R_{mn}^{(3)}(\cosh\mu_{01};c^{ext}) + 2ikS_{mn}(\cos\theta_0;c^{ext})e^{-im\phi_0} R_{mn}^{(3)}(\cosh\mu_0;c^{ext}) R_{mn}^{(1)}(\cosh\mu_{01};c^{ext})]$$
$$= \rho_1 \sum_{n''=0}^{\infty}\sum_{m''=-n''}^{n''} \frac{a_{m''n''}^{1,1}}{\Lambda_{m''n''}(c')} C_{m'',n'',m,n} + \rho_1 \sum_{n''=0}^{\infty}\sum_{m''=-n''}^{n''} \frac{a_{m''n''}^{1,3}}{\Lambda_{m''n''}(c')} E_{m''-m,n'',m,n},$$
$$n \geq 0,\ m = -n,...,n, \tag{26}$$

where we have introduced the notations

$$Z_{m',l,m,n} = \sum_{n'=|m|}^{\infty} A_{m'l}^{mn'(1)} R_{mn'}^{(1)'}(\cosh\mu_{01};c) \left\langle S_{mn'}(c), S_{mn}(c^{ext}) \right\rangle, \tag{27}$$

$$C_{m',l,m,n} = \sum_{n'=|m|}^{\infty} A_{m'l}^{mn'(1)} R_{mn'}^{(1)}(\cosh\mu_{01};c) \left\langle S_{mn'}(c), S_{mn}(c^{ext}) \right\rangle, \tag{28}$$

$$D_{m',n'',m,n} = \sum_{n'=|m|}^{\infty}\sum_{l=|m'|}^{\infty} B_{m+m',n''}^{m',l,n'} R_{mn'}^{(3)'}(\cosh\mu_{01};c) \left\langle S_{mn'}(c), S_{mn}(c^{ext}) \right\rangle, \tag{29}$$

$$E_{m',n'',m,n} = \sum_{n'=|m|}^{\infty}\sum_{l=|m'|}^{\infty} B_{m+m',n''}^{m',l,n'} R_{mn'}^{(3)}(\cosh\mu_{01};c) \left\langle S_{mn'}(c), S_{mn}(c^{ext}) \right\rangle. \tag{30}$$

Eqs. (13), (25), (26) corresponding to the boundary conditions (5), (6), (7), respectively can be organized in matrix form by first defining the matrices

$$\mathbf{B}_{m,n}^{m'',n''} = \begin{bmatrix} \mathbf{0} & 0 \\ -\mathbf{B}_1^{m,n,m'',n''} & \mathbf{0}^{\mathrm{T}} \end{bmatrix}, \tag{31}$$

$$\mathbf{B}_1^{m,n,m'',n''} = \begin{bmatrix} \dfrac{Z_{m'',n'',m,n}}{\Lambda_{m''n''}(c')} & \dfrac{D_{m''-m,n'',m,n}}{\Lambda_{m''n''}(c')} \\ \rho_1 \dfrac{C_{m'',n'',m,n}}{\Lambda_{m''n''}(c')} & \rho_1 \dfrac{E_{m''-m,n'',m,n}}{\Lambda_{m''n''}(c')} \end{bmatrix}, \tag{32}$$

when $m'' \neq m$ or $n'' \neq n$ and

$$\mathbf{B}_{m,n}^{m,n} = \begin{bmatrix} \mathbf{w}_{mn} & 0 \\ -\mathbf{B}_1^{m,n,m,n} & \mathbf{y}_{mn} \end{bmatrix}, \tag{33}$$

$$\mathbf{B}_1^{m,n,m,n} = \begin{bmatrix} \dfrac{Z_{m,n,m,n}}{\Lambda_{mn}(c')} & \dfrac{D_{m-m,n,m,n}}{\Lambda_{mn}(c')} \\ \rho_1 \dfrac{C_{m,n,m,n}}{\Lambda_{mn}(c')} & \rho_1 \dfrac{E_{m-m,n,m,n}}{\Lambda_{mn}(c')} \end{bmatrix}, \tag{34}$$

when $m'' = m$ and $n'' = n$ where

$$\mathbf{w}_{mn} = \left[R_{mn}^{(1)'}(\cosh \mu'_{02}; c'), \quad R_{mn}^{(3)'}(\cosh \mu'_{02}; c') \right], \tag{35}$$

$$\mathbf{y}_{mn} = \left[R_{mn}^{(3)'}(\cosh \mu_{01}; c^{ext}), \quad \rho_{ext} R_{mn}^{(3)}(\cosh \mu_{01}; c^{ext}) \right]^{\mathrm{T}}, \tag{36}$$

and $\mathbf{0} = [0 \quad 0]$. Hence the resulting non-homogenous system of equations is

$$\mathbf{Dx} = \mathbf{b}\,, \tag{37}$$

where the supermatrix $\mathbf{D}$ is given as

$$\mathbf{D} = \begin{bmatrix} \mathbf{B}_{0,0}^{0,0} & \mathbf{B}_{0,0}^{-1,1} & \mathbf{B}_{0,0}^{0,1} & \mathbf{B}_{0,0}^{1,1} & \cdots & \mathbf{B}_{0,0}^{-\kappa,\kappa} & \cdots & \mathbf{B}_{0,0}^{\kappa,\kappa} & \cdots \\ \mathbf{B}_{-1,1}^{0,0} & \mathbf{B}_{-1,1}^{-1,1} & \mathbf{B}_{-1,1}^{0,1} & \mathbf{B}_{-1,1}^{1,1} & \cdots & \mathbf{B}_{-1,1}^{-\kappa,\kappa} & \cdots & \mathbf{B}_{-1,1}^{\kappa,\kappa} & \cdots \\ \mathbf{B}_{0,1}^{0,0} & \mathbf{B}_{0,1}^{-1,1} & \mathbf{B}_{0,1}^{0,1} & \mathbf{B}_{0,1}^{1,1} & \cdots & \mathbf{B}_{0,1}^{-\kappa,\kappa} & \cdots & \mathbf{B}_{0,1}^{\kappa,\kappa} & \cdots \\ \mathbf{B}_{1,1}^{0,0} & \mathbf{B}_{1,1}^{-1,1} & \mathbf{B}_{1,1}^{0,1} & \mathbf{B}_{1,1}^{1,1} & \cdots & \mathbf{B}_{1,1}^{-\kappa,\kappa} & \cdots & \mathbf{B}_{1,1}^{\kappa,\kappa} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ \mathbf{B}_{-\kappa,\kappa}^{0,0} & \mathbf{B}_{-\kappa,\kappa}^{-1,1} & \mathbf{B}_{-\kappa,\kappa}^{0,1} & \mathbf{B}_{-\kappa,\kappa}^{1,1} & \cdots & \mathbf{B}_{-\kappa,\kappa}^{-\kappa,\kappa} & \cdots & \mathbf{B}_{-\kappa,\kappa}^{\kappa,\kappa} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ \mathbf{B}_{\kappa,\kappa}^{0,0} & \mathbf{B}_{\kappa,\kappa}^{-1,1} & \mathbf{B}_{\kappa,\kappa}^{0,1} & \mathbf{B}_{\kappa,\kappa}^{1,1} & \cdots & \mathbf{B}_{\kappa,\kappa}^{-\kappa,\kappa} & \cdots & \mathbf{B}_{\kappa,\kappa}^{\kappa,\kappa} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}, \tag{38}$$

and

$$\mathbf{x} = \left[a_{0,0}^{1,1} \quad a_{0,0}^{1,3} \quad a_{0,0}^{ext} \;\middle|\; a_{-1,1}^{1,1} \quad a_{-1,1}^{1,3} \quad a_{-1,1}^{ext} \quad a_{0,1}^{1,1} \quad a_{0,1}^{1,3} \quad a_{0,1}^{ext} \quad a_{1,1}^{1,1} \quad a_{1,1}^{1,3} \quad a_{1,1}^{ext} \;\middle|\; \cdots \right.$$
$$\left. \cdots \;\middle|\; a_{-\kappa,\kappa}^{1,1} \quad a_{-\kappa,\kappa}^{1,3} \quad a_{-\kappa,\kappa}^{ext} \quad \cdots \quad a_{\kappa,\kappa}^{1,1} \quad a_{\kappa,\kappa}^{1,3} \quad a_{\kappa,\kappa}^{ext} \;\middle|\; \cdots \right]^{\mathrm{T}}, \tag{39}$$

$$\mathbf{b} = \left[\mathbf{b}_{0,0} \middle| \ \mathbf{b}_{-1,1} \ \ \mathbf{b}_{0,1} \ \ \mathbf{b}_{1,1} \middle| \ \cdots \ \middle| \mathbf{b}_{-\kappa,\kappa} \ \cdots \ \mathbf{b}_{\kappa,\kappa} \middle| \ \cdots \right]^{\mathrm{T}}, \tag{40}$$

with

$$\mathbf{b}_{l,j} = \Big[0, \ \ -2ikS_{lj}(\cos\theta_0; c^{ext})e^{-il\phi_0}R_{lj}^{(3)}(\cosh\mu_0; c^{ext})R_{lj}^{(1)'}(\cosh\mu_{01}; c^{ext}),$$
$$-2ik\rho_{ext}S_{lj}(\cos\theta_0; c^{ext})e^{-il\phi_0}R_{lj}^{(3)}(\cosh\mu_0; c^{ext})R_{lj}^{(1)}(\cosh\mu_{01}; c^{ext}) \Big]. \tag{41}$$

4. Numerical Solution-Results

The determination of the scattered field expansion coefficients is accomplished through the solution of the non-homogeneous system (37). Obviously, a suitable truncation procedure needs to be imposed for this system to be solved numerically. In particular, the infinite system (37) gives place to the finite truncated replica of it

$$\mathbf{D}^{(\kappa)}\mathbf{x}^{(\kappa)} = \mathbf{b}^{(\kappa)}, \ \ \kappa = 0,1,2,\ldots \ , \tag{42}$$

where $\mathbf{D}^{(\kappa)}$ is a square matrix of dimension $3(\kappa+1)^2 \times 3(\kappa+1)^2$ as imposed by the structure of matrix $\mathbf{D}$ while the matrices $\mathbf{x}^{(\kappa)}$, $\mathbf{b}^{(\kappa)}$ are of dimension $3(\kappa+1)^2$. The parameter κ naturally corresponds to the truncation level of the summation index n'' appearing in Eqs. (25,26). Truncation level N, i.e. $\kappa = N$ in (42), is chosen in such a way that ensures the convergence of $\left|u^{sc}\right|$ computed using Eq. (9), to the desired accuracy. This is achieved by repeating calculations for successively larger values of N and fixed values of the several parameters entering the problem, until $\left| \left|u^{sc}\right|_N - \left|u^{sc}\right|_{N+1} \right| \approx O(10^{-3})$.

In our computations the scattered field is computed on the minimum sphere containing the kidney external surface and for a frequency of incident wave 20 kHz. A suitable transformation is adopted to represent the obtained results in spherical geometry. In addition, we have used the following properties for the kidney material and the surrounding medium

$$\rho_1 = 1,022\, kg/m^3, c_1 = 1,533\, m/\sec$$

$$\rho_{ext} = 1,000\, kg/m^3, c = 1,493\, m/\sec.$$

Moreover, we have considered the case when the point source is located on the symmetry axis of the spheroidal kidney at distance $0.01m$. Finally, the spheroidal kidney stone is assumed to have a specific ratio of semi-axis $\alpha_{02}/\beta_{02} = 1.5$. Having in mind realistic configurations and aiming at settling a model which assures stability as the confocal geometry gives place to the eccentric one, we have adopted that the spherical coordinates of the stone center are r_1 (parameter), $\theta_1 = 0.1^0$ and $\phi_1 = 0^0$. Our first result motivated by this discussion is the computation of the scattered field for some special cases to check both the validity and accuracy of the results. We consider a perturbed confocal structure consisting of two spheroids with focal distances $a_1 = a_2 = 0.015$. We examine three indicative cases in which the spheroidal systems centers distance r_1 takes the values $r_1 = 10^{-5}, 10^{-4}$ and $10^{-3}m$ respectively. We remark that for the first case, there is an excellent agreement with the result obtained in [2] where the confocal case is examined. In addition, the distance r_1 affects the scattered field as Fig.2 indicates, though the influence is rather slight. More precisely for the case $r_1 = 10^{-5}m$, the average relative deviation of the scattered field from the confocal case is up to 3.17×10^{-4}, for $r_1 = 10^{-4}m$ we have an average relative deviation of order 3.2×10^{-3} while for the case $r_1 = 10^{-3}m$ this quantity is up to 3.23×10^{-2}.

The sensitivity of the measurement of the scattered field as the distance r_0 of the (diagnostic equipment) point source varies is shown in Fig.3 (where r_1 is selected to be $10^{-4} m$). It is clear that an increase of the source distance results in a considerable decrease of the scattered field computed. Fig. 4 shows the shape dependence of the scattered field as the spheroidal kidney deviates from the spherical geometry $(\alpha_{01} / \beta_{01} \approx 1)$. This indicates that the scattered wave can provide with information on the spheroidal ratio of radii.

5. Conclusions

We have developed a theoretical model for the acoustic scattering of time – harmonic spherical waves from an eccentric non – coaxial spheroidal structure. Our approach is frequency independent and it is based on transitional addition theorem on the spheroidal wave functions. We have the model to provide with numerical results for the kidney – stone system.

To our knowledge other researchers have addressed similar problems using numerical treatment such as boundary elements etc. The importance of our work is twofold. First it provides with a theoretical model for a complicated problem. In the system under consideration two spheroidal systems along with a spherical system are involved. The additional difficulty appears in the treatment of the boundary conditions on the discontinuity surfaces, which constitute coordinate surfaces of different systems. Second the numerical results indicate that acoustic scattering can be used as diagnostic tool which permits the identification of stones in kidneys and their characteristics (e.g. size, orientation, position, etc.). In this direction further results must be obtained for different system parameter values in order to construct an expert system which can used for the inverse scattering problem.

6. Appendix

Let us present briefly the geometry under investigation, which is the spheroidal one as we have already stated.

The connection between Cartesian and spheroidal coordinates as well as the scalar factors are given by the relations

$$x = \frac{a}{2}\sinh\mu\sin\theta\cos\phi\,,\; y = \frac{a}{2}\sinh\mu\sin\theta\sin\phi\,,\; z = \frac{a}{2}\cosh\mu\cos\theta\,, \qquad (A.1)$$

$$h_\mu = h_\theta = \frac{1}{2}\alpha\sqrt{\cosh\mu^2 - \cos^2\theta}\,,\; h_\phi = \frac{1}{2}\alpha\sinh\mu\sin\theta\,, \qquad (A.2)$$

where the spheroidal coordinates range over the intervals $\mu \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$. The case $\mu = 0$ corresponds to the line interval connecting the two foci of the spheroidal system located at $z = -\frac{1}{a}$ and $z = \frac{1}{a}$.

The solvability of Helmholtz equation

$$\Delta\Psi + k^2\Psi = 0\,, \qquad (A.3)$$

is based on separation of variable techniques adapted to the spheroidal coordinate system. In [4], the scalar spheroidal wave functions are presented under the hypothesis of real azimuthal dependence, which is equivalent to the positive sign of the azimuthal separation of variables parameter m. However, the uniform applicability of addition formulae requires the adoption of the following spheroidal wave functions

$$\Psi_{mn}^{(j)}(\xi,\eta,\theta;c) = R_{mn}^{(j)}(\xi;c)\,S_{mn}(\eta;c)\,e^{im\phi}\,, \eta = \cos\theta, \xi = \cosh\mu\,, \qquad (A.4)$$

$$m = 0, \pm 1, \pm 2, \ldots; n = |m|, |m|+1, |m|+2, \ldots.$$

where the azimuthal dependence is complex and incorporates the full range of parameter m.

It is proved [4] that the functions R and S for $m = 0,1,\ldots; n \geq m$ are given by

$$S_{mn}(\eta;c) = \sum_{\kappa=0,1}^{\infty}{}' d_{\kappa}^{mn}(c) P_{m+\kappa}^{m}(\eta) = \begin{cases} \sum_{\kappa=0}^{\infty} d_{2\kappa}^{mn}(c) P_{m+2\kappa}^{m}(\eta), & n-m = \text{even} \\ \sum_{\kappa=0}^{\infty} d_{2\kappa+1}^{mn}(c) P_{m+2\kappa+1}^{m}(\eta), & n-m = \text{odd}, \end{cases} \tag{A.5}$$

$$R_{mn}^{(j)}(\xi;c) = \left[\sum_{\kappa=0,1}^{\infty}{}' \frac{(2m+\kappa)!}{\kappa!} d_{\kappa}^{mn}(c) \right]^{-1} (1-\frac{1}{\xi^2})^{m/2} \sum_{\kappa=0,1}^{\infty}{}' i^{\kappa+m-n} \frac{(2m+\kappa)!}{\kappa!} d_{\kappa}^{mn}(c) Z_{m+\kappa}^{(j)}(c\xi), \tag{A.6}$$

where four alternatives for the spherical Bessel functions $Z_n^{(j)}$ exist

$$Z_n^{(1)}(z) = j_n(z), \tag{A7}$$

$$Z_n^{(2)}(z) = y_n(z), \tag{A.8}$$

$$Z_n^{(3)}(z) = h_n^{(1)}(z) = (j_n(z) + iy_n(z)), \tag{A.9}$$

$$Z_n^{(4)}(z) = h_n^{(2)}(z) = (j_n(z) - iy_n(z)), \tag{A.10}$$

while $P_n^m(\eta)$ denotes the associated Legendre functions of the first kind. In addition, the symbol $\sum_{\kappa=0,1}^{\infty}{}'$, as it is clear from Eq. (A.5), indicates summation over even or odd indices, depending on the starting index. Given that we have used the complete range of m, i.e. $m = 0, \pm 1, \pm 2, \ldots$, in the definition (A.7) of the scalar spheroidal wave functions then Eqs. (A.5) and (A.6), must be extended to

$$S_{mn}(\eta;c)=\sum_{\kappa=0,1}^{\infty}{}' d_{\kappa}^{mn}(c)P_{|m|+\kappa}^{m}(\eta)=\begin{cases}\sum_{\kappa=0}^{\infty} d_{2\kappa}^{mn}(c)P_{|m|+2\kappa}^{m}(\eta), & n-m=\text{even}\\ \sum_{\kappa=0}^{\infty} d_{2\kappa+1}^{mn}(c)P_{|m|+2\kappa+1}^{m}(\eta), & n-m=\text{odd},\end{cases} \tag{A.11}$$

$$R_{mn}^{(j)}(\xi;c)=R_{-m,n}^{(j)}(\xi;c)=$$

$$\left[\sum_{\kappa=0,1}^{\infty}{}' \frac{(2m+\kappa)!}{\kappa!}d_{\kappa}^{mn}(c)\right]^{-1}(1-\frac{1}{\xi^2})^{m/2}\sum_{\kappa=0,1}^{\infty}{}' i^{\kappa+m-n}\frac{(2m+\kappa)!}{\kappa!}d_{\kappa}^{mn}(c)Z_{m+\kappa}^{(j)}(c\xi), \tag{A.12}$$

Crucial role to the numerical calculation of the spheroidal functions play the coefficients $d_{\kappa}^{mn}(c)$ [6]. Their full determination is accomplished through the solution of a specific eigenvalue problem (which has a different formulation depending on m being positive or negative) provided that a normalization condition is imposed. More precisely, exploiting the ordinary differential equations satisfied by the $S_{mn}(\eta;c)$ and $P_{|m|+\kappa}^{m}(\eta)$ functions as well as suitable recurrence relations of the latter functions we obtain the following recursive scheme

$$\begin{aligned}&\frac{(|m|+m+\kappa+2)(|m|+m+\kappa+1)}{(2|m|+2\kappa+5)(2|m|+2\kappa+3)}c^2 d_{\kappa+2}^{mn}(c)\\ &+[(|m|+\kappa)(|m|+\kappa+1)-\lambda_{mn}(c)+\frac{2(|m|+\kappa)(|m|+\kappa+1)-2m^2-1}{(2|m|+2\kappa-1)(2|m|+2\kappa+3)}c^2]d_{\kappa}^{mn}(c)\\ &+\frac{(|m|-m+\kappa-1)(|m|-m+\kappa)}{(2|m|+2\kappa-3)(2|m|+2\kappa-1)}c^2 d_{\kappa-2}^{mn}(c)=0\,.\end{aligned} \tag{A.14}$$

Formula (A.12) gives rise to the following eigenvalue problems

$$\begin{bmatrix}\beta_0 & \alpha_0 & 0 & 0 & \cdots & \cdots\\ \gamma_2 & \beta_2 & \alpha_2 & 0 & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \gamma_{2\kappa} & \beta_{2\kappa} & \alpha_{2\kappa}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}d_0^{mn}(c)\\ d_2^{mn}(c)\\ d_4^{mn}(c)\\ \cdots\\ d_{2\kappa}^{mn}(c)\\ \cdots\end{bmatrix}=\lambda_{mn}(c)\begin{bmatrix}d_0^{mn}(c)\\ d_2^{mn}(c)\\ d_4^{mn}(c)\\ \cdots\\ d_{2\kappa}^{mn}(c)\\ \cdots\end{bmatrix}, \tag{A.15}$$

for $(n-|m|)$ even and

$$\begin{bmatrix} \beta_1 & \alpha_1 & 0 & 0 & .... & .... \\ \gamma_3 & \beta_3 & \alpha_3 & 0 & .... & .... \\ .... & .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... & .... \\ 0 & .... & .... & \gamma_{2\kappa+1} & \beta_{2\kappa+1} & \alpha_{2\kappa+1} \\ .... & .... & .... & .... & .... & .... \end{bmatrix} \begin{bmatrix} d_1^{mn}(c) \\ d_3^{mn}(c) \\ d_5^{mn}(c) \\ .... \\ d_{2\kappa+1}^{mn}(c) \\ .... \end{bmatrix} = \lambda_{mn}(c) \begin{bmatrix} d_1^{mn}(c) \\ d_3^{mn}(c) \\ d_5^{mn}(c) \\ .... \\ d_{2\kappa+1}^{mn}(c) \\ .... \end{bmatrix}. \quad \text{(A.16)}$$

for $(n-|m|)$ odd, where the coefficients α_κ, β_κ and γ_κ are determined through the relations

$$\alpha_\kappa = \frac{(2m+\kappa+2)(2m+\kappa+1)}{(2m+\kappa+5)(2m+2\kappa+3)} c^2, \quad \text{(A.17)}$$

$$\beta_\kappa = (m+\kappa)(m+\kappa+1) + \frac{2(m+\kappa)(m+\kappa+1)-2m^2-1}{(2m+2\kappa-1)(2m+2\kappa+3)} c^2, \quad \text{(A.18)}$$

$$\gamma_\kappa = \frac{\kappa(\kappa-1)}{(2m+2\kappa-3)(2m+2\kappa-1)} c^2, \quad \text{(A.19)}$$

in the case that $m>0$ while for $m<0$ it holds that

$$a_\kappa = \frac{(\kappa+2)(\kappa+1)}{(2|m|+2\kappa+5)(2|m|+2\kappa+3)} c^2, \quad \text{(A.20)}$$

$$\beta_\kappa = (\kappa+m)(m+\kappa+1) + \frac{2(m+\kappa)(m+\kappa+1)-2m^2-1}{(2|m|+2\kappa-1)(2|m|+2\kappa+3)} c^2, \quad \text{(A.21)}$$

$$\gamma_\kappa = \frac{(2|m|+\kappa-1)(2|m|+\kappa)}{(2|m|+2\kappa-3)(2|m|+2\kappa-1)} c^2. \quad \text{(A.22)}$$

Hence Eqs. (A.15) and (A.16) provide with the eigenvalues $\lambda_{m,|m|}(c)$, $\lambda_{m,|m|+2}(c)$,... and $\lambda_{m,|m|+1}(c)$,$\lambda_{m,|m|+3}(c)$,...,respectively. For every $\lambda_{mn}(c)$ obtained above, the $d_\kappa^{mn}(c)$ coefficients are determined modulo a multiplicative constant. As we have already mentioned, a normalization condition must be imposed in order for these coefficients to be uniquely determined [4, 6]).

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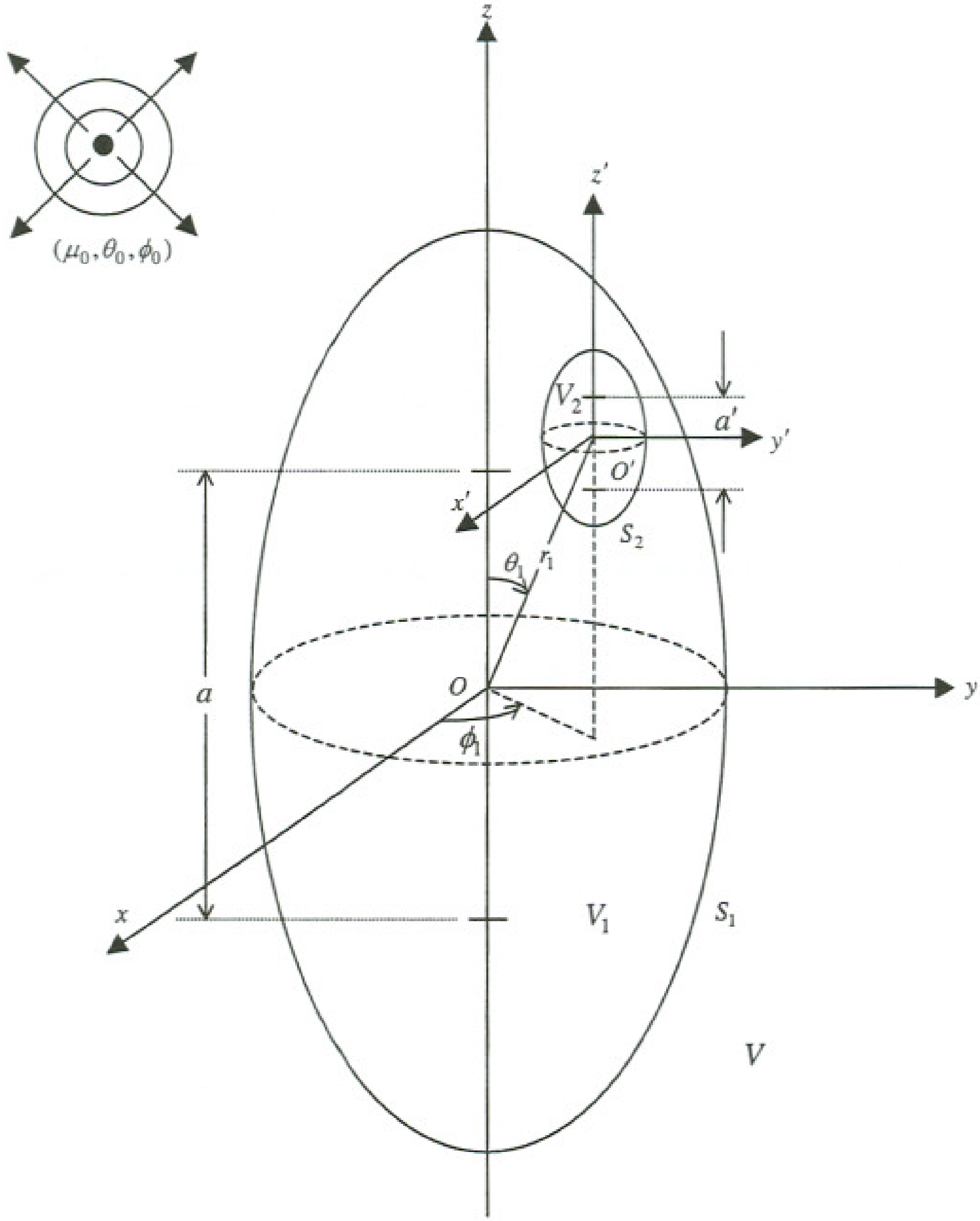

Figure 1: The problem geometry

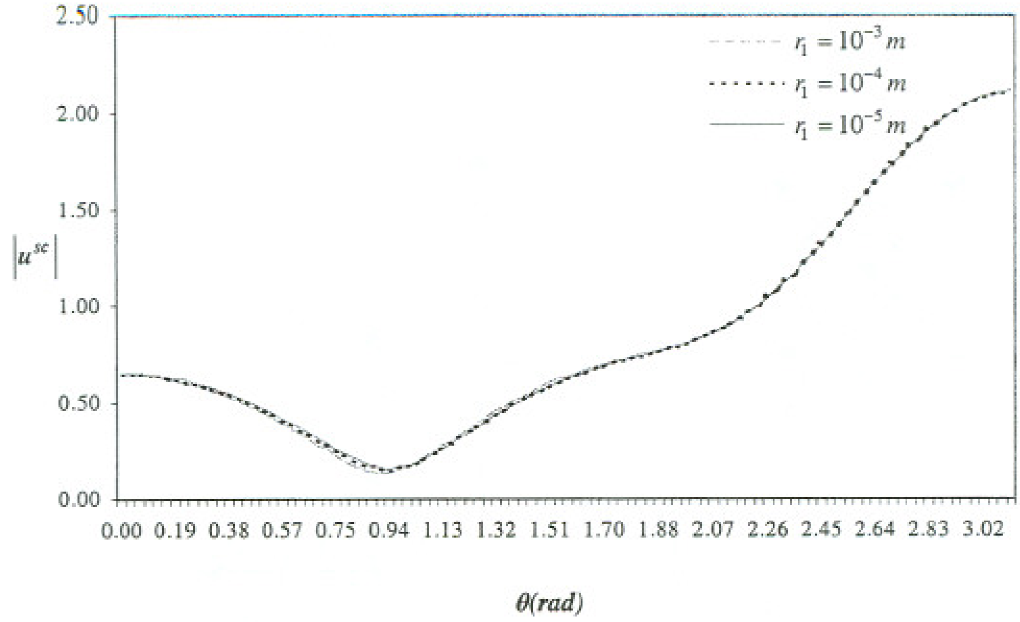

Figure 2: $|u^{sc}|$ as a function of θ for varying distance r_1 for a perturbed confocal case

$(a_1 = a_2 = 0.015)$.

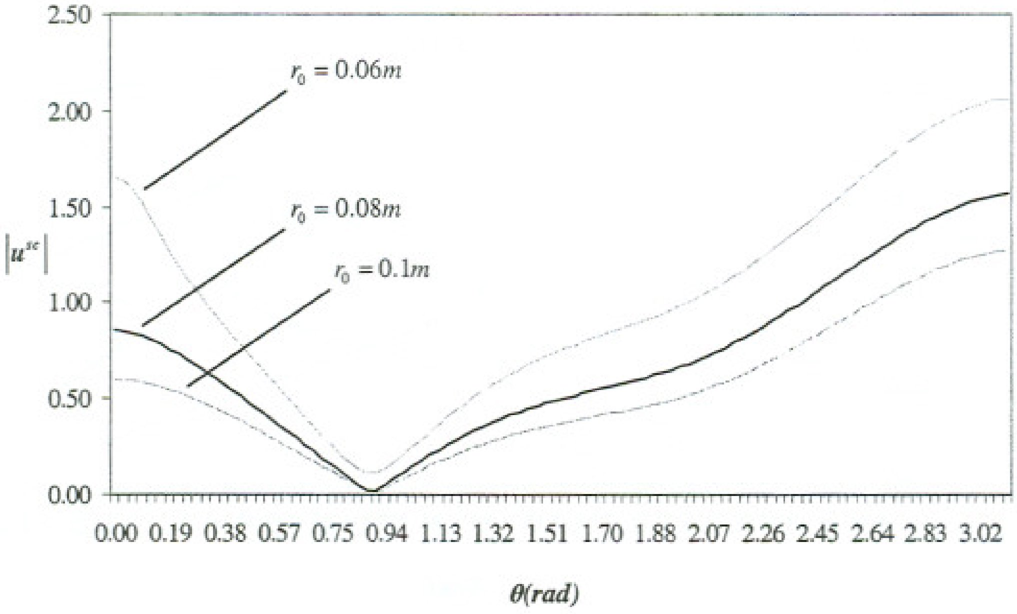

Figure 3: $|u^{sc}|$ as a function of θ for varying distance r_0

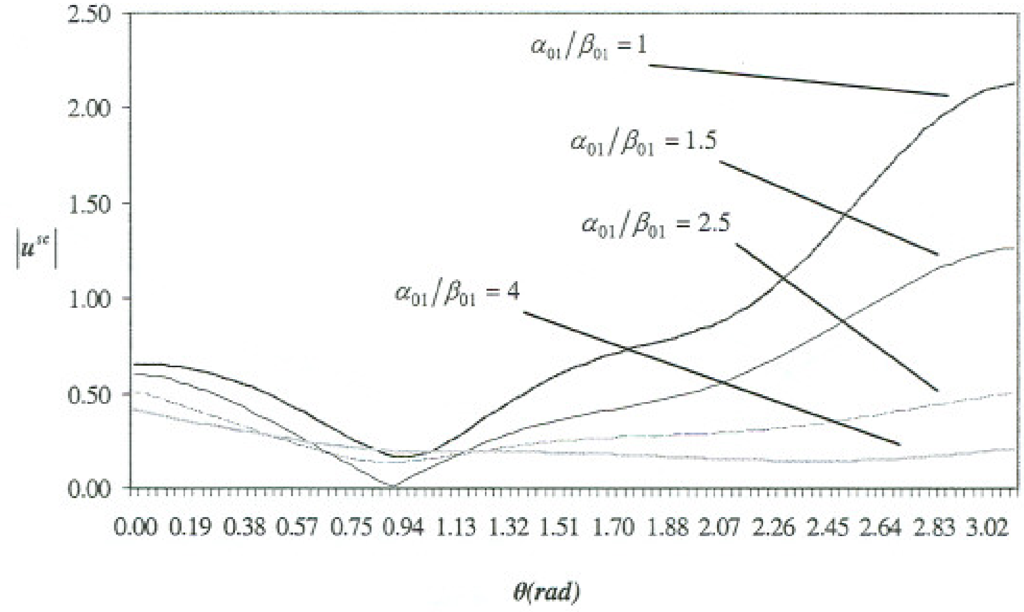

Figure 4: $|u^{sc}|$ as a function of θ for varying ratio α_{01}/β_{01}