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Abstract

We examine the electromagnetic scattering of spherical waves by a buried spheroidal perfect conductor. The proposed analysis is based on the integral equation formalism of the problem and focuses on the establishment of a multi-parametric model describing analytically the scattering process under consideration. Both the theoretical and the numerical treatment are presented. The outcome of the analysis is the determination of the scattered field in the observation environment along with its multivariable on several physical and geometric parameters of the system.

Keywords: Electromagnetic scattering; Buried objects; Spheroidal wave functions

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1. Introduction

This paper addresses the three-dimensional direct scattering problem of electromagnetic spherical waves by a prolate spheroidal perfect conductor, which is embedded in a semi-infinite dielectric medium. The interference of electromagnetic waves with inaccessible scatterers (as the buried object case) constitutes an area of great scientific interest from the theoretical and the application point of view. Indeed, there exist numerous applications having as fundamental prerequisite the implementation of non-invasive techniques for the localization and reconstruction of non-accessible structures. In the past several researchers have studied the problem of the localization of buried metallic objects (ordnance, mines, etc) [1-8].

Within the framework of arbitrary scatterer's shape, the suggested techniques for handling the direct scattering problem belong almost totally to the numerical regime [9-12]. In fact, the validity of all these methods strongly depends on the electrical size of the scatterer. Nevertheless, in the case that we have a priori information concerning the geometric features of the scatterer, it is possible to develop analytical methods facing the scattering problem and leading to the establishment of multi parametric models describing the studied process. It is mentioned here the application of the localized nonlinear approximation [13], which constitutes a method for the investigation of the low-frequency behavior of the scattering problem from an ellipsoidal body in the half-space earth.

The present work is subsumed into the general framework of integral equation formalism [14,15], it is frequency independent and examines the case of the prolate spheroidal scatterer, which simulates perfectly a convex body that lacks symmetry in only one direction.

The paper is organized in five sections. Section 2 provides the mathematical formulation of the scattering problem under consideration. The main outcome of this section is the establishment of appropriate integral representations for the electric fields in the two half-space media. In section 3 we expand the electric field in the scatterer's region in terms of the spheroidal vector wave functions and force this expansion to satisfy the boundary condition on the scatterer's surface. Exploiting orthogonality arguments of the underlying spheroidal functions we obtain a set of algebraic equations with unknowns the electric field expansion coefficients. The handling of the integral representation for the electric field in the host medium via the aforementioned spectral decomposition in spheroidal coordinates is presented in section 4. A lot of analysis is dedicated to encounter the coexistence of the cylindrical with the spheroidal geometry in order to obtain analytical expressions for the above integral representation. Based on these expressions, we arrive at algebraic equations resulting from the investigation of the electric field in the asymptotic regime. These equations are combined with those originated from the boundary condition satisfaction and met in section 3 in order to form the final non-homogeneous algebraic system whose solution provides with the spectral decomposition coefficients. Actually, the knowledge of these coefficients is equivalent to the determination of the scattered electric field in the scattering region-above the interface-as it is explained in section 5. Finally, in section 6 we present some numerical results, which are based on the numerical solution of the above algebraic system.

The main outcome is the determination of the scattered field for representative values for some of the parameters entering the system. A more systematic exploitation of these results would serve as the basis for the development of a methodology towards the solution of the inverse scattering problem. This is mainly due to the fact that all the features of the process are incorporated in the method as specific parameters and therefore their influence to the scattering mechanism is expected to be recovered.

2. Formulation of the problem

We consider two separate subregions characterized by different electric permittivities $\varepsilon_i, i=(1,2)$, separated by a flat infinite interface S_δ on which suitable transmission conditions are satisfied (Fig. 1). The media occupying the two half-spaces $V_i, i=(1,2)$ are isotropic, homogeneous and non-magnetic, having magnetic permeability μ_0. The upper half-space is the region where we can stimulate or measure electromagnetic fields. In contrast, the lower half-space corresponds to the propagation environment, where we usually do not have access or we cannot make any measurements.

A prolate spheroidal scatterer, with surface S_{sph}, is embedded in subregion (2). The scatterer has semiaxes a_0, b_0 $(a_0 > b_0)$, focal distance a, it is orientated vertically with respect to the interface S_δ and its center is located at a distance δ from the boundary S_δ. The origin of the coordinate system coincides with the center of the spheroidal object.

A short electric dipole (i.e. a Hertzian dipole) emanating time-harmonic electromagnetic spherical waves is located in region (1), at a position O' with cylindrical coordinates (ρ_0, ϕ_0, z_0). The electromagnetic field generated by the current source constitutes the incident field, the interference of which with the interface S_δ along with the spheroidal surface S_{sph}, leads to the creation of the scattered wave. This interference depends on the boundary conditions on the scatterer's surface. For the problem under discussion we assume that the spheroidal body constitutes a perfect conductor. The scattered wave encodes all the information concerning the physical characteristics of the scatterer along with the relevant geometric features of the system. The total electromagnetic field is formed by the superposition of the incident field with the scattered one and is denoted by the vector pairs $(\overline{E}_i, \overline{H}_i)$, $i = (1,2)$ in each region, where $\overline{E}_i$ and $\overline{H}_i$ are the electric and magnetic field in region i, respectively. These fields satisfy the non-homogeneous Maxwell's equations

$$\nabla \times \overline{E}_i(\overline{r}) = i\omega\mu_0 \overline{H}_i(\overline{r}), \tag{1}$$

$$\nabla \times \overline{H}_i(\overline{r}) = \overline{J}_i(\overline{r}) - i\omega\varepsilon_i \overline{E}_i(\overline{r}), \tag{2}$$

$$\nabla \cdot \left(\varepsilon_i \overline{E}_i(\overline{r})\right) = 0, \tag{3}$$

$$\nabla \cdot \left(\mu_0 \overline{H}_i(\overline{r})\right) = 0, \quad \overline{r} \in V_i, i = (1,2), \tag{4}$$

where we have suppressed the harmonic time dependence $\exp\{-i\omega t\}$ with ω standing for the angular frequency of the current source. The current distribution $\overline{J}_i(\overline{r})$, which corresponds to the excitation of the short electric dipole located at $\overline{r} = \overline{r}_0$ and orientated in the direction of $\hat{a}$, is given by

$$\overline{J}_i(\overline{r}) = \begin{Bmatrix} \mathrm{I}a\hat{a}\delta(\overline{r} - \overline{r}_0) & i = 1 \\ \overline{0} & i = 2 \end{Bmatrix}, \tag{5}$$

where $\delta(\bar{r}-\bar{r}_0)$ denotes the three-dimensional delta function and Ia denotes the magnitude of the current moment of the source expressed in terms of its length a and its constant amplitude I [16], which is normalized such that $i\omega\mu_0 \mathrm{Ia} = 1$. Eliminating the magnetic fields from Maxwell's equations we infer that the electric fields satisfy the following vector wave equations

$$\nabla\times\nabla\times\bar{E}_1(\bar{r}) - k_1^2\bar{E}_1(\bar{r}) = i\omega\mu_0\bar{J}_1(\bar{r}), \quad \bar{r}\in V_1, \tag{6}$$

$$\nabla\times\nabla\times\bar{E}_2(\bar{r}) - k_2^2\bar{E}_2(\bar{r}) = \bar{0}, \quad \bar{r}\in V_2, \tag{7}$$

where $k_i = \dfrac{\omega}{c_i} = \omega\sqrt{\mu_0\varepsilon_i}$, $i=(1,2)$ are the wave numbers in the two media, respectively.

On the interface S_δ, the electric fields satisfy the following transmission conditions

$$\hat{z}\times\left[\bar{E}_1(\bar{r}) - \bar{E}_2(\bar{r})\right] = \bar{0}, \tag{8.a}$$

$$\hat{z}\times\left[\nabla\times\bar{E}_1(\bar{r}) - \nabla\times\bar{E}_2(\bar{r})\right] = \bar{0}, \quad \bar{r}\in S_\delta. \tag{8.b}$$

Furthermore, the tangential component of the total electric field on the spheroidal surface S_{sph} vanishes, i.e.

$$\hat{n}\times\bar{E}_2(\bar{r}) = \bar{0}, \quad \bar{r}\in S_{sph}, \tag{9}$$

where $\hat{n}$ is the outward unit normal vector to the scatterer's surface.

In addition, the electric fields satisfy the radiation conditions

$$\lim_{r\to\infty} r\left[\nabla\times\bar{E}_i(\bar{r}) - ik_i\hat{r}\times\bar{E}_i(\bar{r})\right] = \bar{0}, \quad \bar{r}\in V_i,\ i=(1,2). \tag{10}$$

The partial differential Eqs. (6)-(7) along with the boundary conditions (8)-(9) and the radiation conditions (10) constitute a well-posed boundary value problem. Its solution will be obtained following the integral equation formalism of the problem. In this framework we evoke the theory of dyadic

Green's functions, presented briefly in Appendix A, since the sought integral representations require dyadic kernel functions.

The free-space dyadic Green function $\overline{\overline{G}}_{e0}^{(1)}(\overline{r},\overline{r}')$ [Appendix A] must be exploited suitably in order to obtain dyads incorporating the boundary conditions on the interface S_δ. This is necessary in acquiring integral representations only on the scatterer's surface, avoiding the undesirable infinite flat surface.

To this end we first apply the scattering superposition method to solve the scattering problem of dyad incidence in the absence of the spheroidal scatterer obtaining

$$\overline{\overline{G}}_{e}^{(11)}(\overline{r},\overline{r}') = \overline{\overline{G}}_{e0}^{(1)}(\overline{r},\overline{r}') + \overline{\overline{G}}_{es}^{(11)}(\overline{r},\overline{r}'), \tag{11}$$

$$\overline{\overline{G}}_{e}^{(21)}(\overline{r},\overline{r}') = \overline{\overline{G}}_{es}^{(21)}(\overline{r},\overline{r}'), \tag{12}$$

where the double superscript notation (ij), $i, j = (1,2)$ denotes that the field point is located in region (i) and the source point is located in region (j) while the subscript is used to denote the scattered (s) or the incident (0) part of the corresponding dyadic field. Afterwards, we expand $\overline{\overline{G}}_{es}^{(11)}$, $\overline{\overline{G}}_{es}^{(21)}$ in terms of the cylindrical vector wave functions [Appendix B], imitating the construction of the free-space dyads met in Appendix A and taking into account the outgoing character of the waves, i.e.

$$\overline{\overline{G}}_{es}^{(11)}(\overline{r},\overline{r}') = \int_0^\infty d\lambda \sum_{m=0}^{\infty} C^{(1)}\left[a\overline{M}(h_1)\overline{M}'(h_1) + b\overline{N}(h_1)\overline{N}'(h_1)\right],\ z \lessgtr z', \tag{13}$$

$$\overline{\overline{G}}_{es}^{(21)}(\overline{r},\overline{r}') = \int_0^\infty d\lambda \sum_{m=0}^{\infty} C^{(1)}\left[c\overline{M}(-h_2)\overline{M}'(h_1) + d\overline{N}(-h_2)\overline{N}'(h_1)\right],\ z < z'. \tag{14}$$

The dyads satisfy similar boundary conditions on S_δ with those satisfied by the electromagnetic fields, which means that

$$\hat{z}\times\left[\overline{\overline{G}}_e^{(11)}\left(\overline{r},\overline{r}'\right)-\overline{\overline{G}}_e^{(21)}\left(\overline{r},\overline{r}'\right)\right]=\overline{\overline{0}}, \tag{15}$$

$$\hat{z}\times\left[\nabla\times\overline{\overline{G}}_e^{(11)}\left(\overline{r},\overline{r}'\right)-\nabla\times\overline{\overline{G}}_e^{(21)}\left(\overline{r},\overline{r}'\right)\right]=\overline{\overline{0}},\quad \overline{r}\in S_\delta . \tag{16}$$

Using the specific form of $\overline{\overline{G}}_{e0}^{(1)}$ (Eq. (A.7) for $z<z'$) and Eqs. (13)-(14) we conclude that Eqs. (15)-(16) are satisfied if the unknowns a,b,c,d are

$$a=\frac{h_1-h_2}{h_1+h_2}e^{-i2h_1\delta}, \tag{17.a}$$

$$b=\frac{n^2h_1-h_2}{n^2h_1+h_2}e^{-i2h_1\delta}, \tag{17.b}$$

$$c=\frac{2h_1}{h_1+h_2}e^{-i(h_1-h_2)\delta}, \tag{17.c}$$

$$d=\frac{2nh_1}{n^2h_1+h_2}e^{-i(h_1-h_2)\delta}, \tag{17.d}$$

where $n=\frac{k_2}{k_1}$ is the refractive index.

Similar results are obtained in the case when a hypothetic current source is located in region V_2. In fact, for $z>z'$, we have

$$\overline{\overline{G}}_e^{(22)}\left(\overline{r},\overline{r}'\right)=\overline{\overline{G}}_{e0}^{(2)}\left(\overline{r},\overline{r}'\right)+\overline{\overline{G}}_{es}^{(22)}\left(\overline{r},\overline{r}'\right), \tag{18.a}$$

$$\overline{\overline{G}}_e^{(12)}\left(\overline{r},\overline{r}'\right)=\overline{\overline{G}}_{es}^{(12)}\left(\overline{r},\overline{r}'\right), \tag{18.b}$$

$$\overline{\overline{G}}_{es}^{(22)}\left(\overline{r},\overline{r}'\right)=\int_0^\infty d\lambda\sum_{m=0}^{\infty}C^{(2)}\left[a'\overline{M}(-h_2)\overline{M}'(-h_2)+b'\overline{N}(-h_2)\overline{N}'(-h_2)\right],\ z\gtrless z', \tag{18.c}$$

$$\overline{\overline{G}}_{es}^{(12)}\left(\overline{r},\overline{r}'\right)=\int_0^\infty d\lambda\sum_{m=0}^{\infty}C^{(2)}\left[c'\overline{M}(h_1)\overline{M}'(-h_2)+d'\overline{N}(h_1)\overline{N}'(-h_2)\right],\ z>z', \tag{18.d}$$

where

$$a'=\frac{h_2-h_1}{h_1+h_2}e^{i2h_2\delta}, \tag{19.a}$$

$$b'=\frac{h_2-n^2h_1}{n^2h_1+h_2}e^{i2h_2\delta}, \tag{19.b}$$

$$c' = \frac{2h_2}{h_1 + h_2} e^{-i(h_1 - h_2)\delta}, \tag{19.c}$$

$$d' = \frac{2nh_2}{n^2 h_1 + h_2} e^{-i(h_1 - h_2)\delta}. \tag{19.d}$$

The integral representations are based on the application of the second vector dyadic Green's theorem according to which, for every vector $\overline{P}$ and dyad $\overline{\overline{Q}}$ we have

$$\iiint_V \left[\overline{P} \cdot \nabla \times \nabla \times \overline{\overline{Q}} - \left(\nabla \times \nabla \times \overline{P} \right) \cdot \overline{\overline{Q}} \right] dV = -\iint_S \left[\left(\hat{n} \times \nabla \times \overline{P} \right) \cdot \overline{\overline{Q}} + \left(\hat{n} \times \overline{P} \right) \cdot \nabla \times \overline{\overline{Q}} \right] dS. \tag{20}$$

We apply the above theorem for (i) $\overline{P} = \overline{E}_1(\overline{r})$, $\overline{\overline{Q}} = \overline{\overline{G}}_e^{(11)}(\overline{r}, \overline{r'})$ with $\overline{r} \in V_1$ and (ii) $\overline{P} = \overline{E}_2(\overline{r})$, $\overline{\overline{Q}} = \overline{\overline{G}}_e^{(21)}(\overline{r}, \overline{r'})$ with $\overline{r} \in V_2$. Using Eqs. (6)-(7), the corresponding partial differential dyadic equations valid for the underlying Green's functions [16], the boundary conditions (8)-(9), (15)-(16) and the radiation conditions (10) we infer that

$$\overline{E}_1(\overline{r'}) = i\omega\mu_0 \int_{V_1} \overline{J}_1(\overline{r}) \cdot \overline{\overline{G}}_e^{(11)}(\overline{r}, \overline{r'}) dV + \int_{S_{sph}} \left[\hat{n} \times \nabla \times \overline{E}_2(\overline{r}) \right] \cdot \overline{\overline{G}}_e^{(21)}(\overline{r}, \overline{r'}) dS_{sph}. \tag{21}$$

Similarly for the pairs $\left(\overline{P}, \overline{\overline{Q}} \right) = \left(\overline{E}_2(\overline{r}), \overline{\overline{G}}_e^{(22)}(\overline{r}, \overline{r'}) \right), \overline{r} \in V_2$ and $\left(\overline{E}_1(\overline{r}), \overline{\overline{G}}_e^{(12)}(\overline{r}, \overline{r'}) \right), \overline{r} \in V_1$, we obtain

$$\overline{E}_2(\overline{r'}) = i\omega\mu_0 \int_{V_1} \overline{J}_1(\overline{r}) \cdot \overline{\overline{G}}_e^{(12)}(\overline{r}, \overline{r'}) dV + \int_{S_{sph}} \left[\hat{n} \times \nabla \times \overline{E}_2(\overline{r}) \right] \cdot \overline{\overline{G}}_e^{(22)}(\overline{r}, \overline{r'}) dS_{sph}. \tag{22}$$

Exploiting reciprocity theorems concerning the aforementioned dyads we obtain the more appropriate final integral representations for the electric fields

$$\overline{E}_1(\overline{r}) = i\omega\mu_0 \int_{V_1} \overline{\overline{G}}_e^{(11)}(\overline{r}, \overline{r'}) \cdot \overline{J}_1(\overline{r'}) \, dV' + \int_{S_{sph}} \overline{\overline{G}}_e^{(12)}(\overline{r}, \overline{r'}) \cdot \left[\hat{n}' \times \nabla' \times \overline{E}_2(\overline{r'}) \right] dS'_{sph}, \quad \overline{r} \in V_1, \tag{23}$$

$$\overline{E}_2(\overline{r}) = i\omega\mu_0 \int_{V_1} \overline{\overline{G}}_e^{(21)}(\overline{r}, \overline{r'}) \cdot \overline{J}_1(\overline{r'}) \, dV' + \int_{S_{sph}} \overline{\overline{G}}_e^{(22)}(\overline{r}, \overline{r'}) \cdot \left[\hat{n}' \times \nabla' \times \overline{E}_2(\overline{r'}) \right] dS'_{sph}, \quad \overline{r} \in V_2. \tag{24}$$

In the sequel, our aim is the exploitation of the above integral representations in order to determine the electric fields from the knowledge of the physical and geometric characteristics of the scatterer.

3. Spectral decomposition in spheroidal geometry and boundary conditions investigation

The electric field in region (2) is expressed in terms of the spheroidal vector wave functions [Appendix B], which constitute a basis in the space of Maxwell's equations solutions. This spectral representation is expected to fit suitably to the boundary conditions imposed on the spheroid S_{sph}. Consequently, it holds that

$$\overline{E}_2(\overline{r})=\sum_{j=3}^{4}\sum_{m'=0}^{\infty}\sum_{n'=m'}^{\infty}\left[A^{(j)}_{\substack{e\\o}m'n'}\overline{M}^{(j)}_{\substack{e\\o}m'n'}(\overline{r})+B^{(j)}_{\substack{e\\o}m'n'}\overline{N}^{(j)}_{\substack{e\\o}m'n'}(\overline{r})\right],\ \overline{r}\in V_2. \qquad (25)$$

We force representation (25) to obey the boundary condition (9). Projecting the resulting vector equation on the tangential unit vectors $\hat{\theta},\hat{\phi}$ of the spheroidal system with μ_0 being the specific value of the coordinate μ, which defines the spheroidal surface we obtain the following scalar relations

$$\sum_{j=3}^{4}\sum_{m'=0}^{\infty}\sum_{n'=m'}^{\infty}\left[A^{(j)}_{\substack{e\\o}m'n'}\left[\hat{\phi}\cdot\overline{M}^{(j)}_{\substack{e\\o}m'n'}(\overline{r})\right]+B^{(j)}_{\substack{e\\o}m'n'}\left[\hat{\phi}\cdot\overline{N}^{(j)}_{\substack{e\\o}m'n'}(\overline{r})\right]\right]_{\mu=\mu_0}=0, \qquad (26)$$

$$\sum_{j=3}^{4}\sum_{m'=0}^{\infty}\sum_{n'=m'}^{\infty}\left[A^{(j)}_{\substack{e\\o}m'n'}\left[\hat{\theta}\cdot\overline{M}^{(j)}_{\substack{e\\o}m'n'}(\overline{r})\right]+B^{(j)}_{\substack{e\\o}m'n'}\left[\hat{\theta}\cdot\overline{N}^{(j)}_{\substack{e\\o}m'n'}(\overline{r})\right]\right]_{\mu=\mu_0}=0. \qquad (27)$$

In order to obtain fully algebraic equations, we project Eqs. (26)-(27) on the complete set of functions $P_n^m(\cos\theta)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$, $m=0,1,2,\ldots;n\geq m$ in the θ,ϕ-space with weight function $w=(\cosh^2\mu_0-\cos^2\theta)\sin\theta$. In this analysis specific ''inner'' products arise, i.e. the products $\langle\hat{a}\cdot\overline{\mathrm{A}},P_n^m(\cos\theta)\sin\theta\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}\rangle$ with $\hat{a}=(\hat{\theta},\hat{\phi})$; $\overline{\mathrm{A}}=\left(\overline{M}^{(j)}_{\substack{e\\o}m'n'},\overline{N}^{(j)}_{\substack{e\\o}m'n'}\right)$, which give birth to simple ϕ-inner products along with several θ-inner products denoted as $\Re^{i,m}_{n,n'}$, $i=1,\ldots18$. The latter quantities are derived and presented in [20]. Finally, we obtain the following set of algebraic equations for every $m=0,1,2,\ldots;n\geq m$, which is equivalent to the boundary condition on S_{sph}

$$\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A_{e,mn'}^{(j)}\left(\pi\hat{\delta}_{0m}\right)[\sinh\mu_0\cosh\mu_0 R_{mn'}^{(j)}(\cosh\mu_0;c_2)\Re_{n,n'}^{4,m}-\sinh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)\Re_{n,n'}^{6,m}]$$

$$+B_{o,mn'}^{(j)}\frac{2}{k_2 a}(m\pi)\{[\frac{(\sinh^2\mu_0-1)}{\sinh\mu_0}R_{mn'}^{(j)}(\cosh\mu_0;c_2)+\sinh\mu_0\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{1,m}$$

$$+\frac{1}{\sinh\mu_0}R_{mn'}^{(j)}(\cosh\mu_0;c_2)[2\Re_{n,n'}^{3,m}+\Re_{n,n'}^{5,m}+\Re_{n,n'}^{2,m}]\}\}=0, \tag{28}$$

$$\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A_{o,mn'}^{(j)}\left(\pi(1-\delta_0)\right)[\sinh\mu_0\cosh\mu_0 R_{mn'}^{(j)}(\cosh\mu_0;c_2)\Re_{n,n'}^{4,m}-\sinh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)\Re_{n,n'}^{6,m}]$$

$$+B_{e,mn'}^{(j)}\frac{2}{k_2 a}(-m\pi)\{[\frac{(\sinh^2\mu_0-1)}{\sinh\mu_0}R_{mn'}^{(j)}(\cosh\mu_0;c_2)+\sinh\mu_0\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{1,m}$$

$$+\frac{1}{\sinh\mu_0}R_{mn'}^{(j)}(\cosh\mu_0;c_2)[2\Re_{n,n'}^{3,m}+\Re_{n,n'}^{5,m}+\Re_{n,n'}^{2,m}]\}\}=0, \tag{29}$$

$$\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A_{o,mn'}^{(j)}(m\pi)(\cosh\mu_0 R_{mn'}^{(j)}(\cosh\mu_0;c_2)[\sinh^4\mu_0\Re_{n,n'}^{1,m}+2\sinh^2\mu_0\Re_{n,n'}^{3,m}+\Re_{n,n'}^{7,m}]\}$$

$$-B_{e,mn'}^{(j)}\frac{2\left(\pi\hat{\delta}_{0m}\right)}{k_2 a}\{[\sinh^2\mu_0\left(\sinh^2\mu_0-1\right)R_{mn'}^{(j)}(\cosh\mu_0;c_2)+\sinh^4\mu_0\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{4,m}$$

$$+\sinh^2\mu_0[4R_{mn'}^{(j)}(\cosh\mu_0;c_2)+\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{8,m}$$

$$+\sinh^2\mu_0[\left(\frac{k_2 a}{2}\right)^2\sinh^2\mu_0 R_{mn'}^{(j)}(\cosh\mu_0;c_2)+2\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{6,m}$$

$$+\left(\frac{k_2 a}{2}\right)^2 R_{mn'}^{(j)}(\cosh\mu_0;c_2)[2\sinh^2\mu_0\Re_{n,n'}^{13,m}+\Re_{n,n'}^{15,m}]$$

$$+R_{mn'}^{(j)}(\cosh\mu_0;c_2)[\sinh^2\mu_0\Re_{n,n'}^{9,m}+\Re_{n,n'}^{16,m}+2\Re_{n,n'}^{14,m}-\sinh^2\mu_0\Re_{n,n'}^{10,m}]\}\}=0, \tag{30}$$

$$\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A_{e,mn'}^{(j)}(-m\pi)(\cosh\mu_0 R_{mn'}^{(j)}(\cosh\mu_0;c_2)[\sinh^4\mu_0\Re_{n,n'}^{1,m}+2\sinh^2\mu_0\Re_{n,n'}^{3,m}+\Re_{n,n'}^{7,m}]\}-$$

$$B_{o,mn'}^{(j)}\frac{2\pi(1-\delta_0)}{k_2 a}\{[\sinh^2\mu_0\left(\sinh^2\mu_0-1\right)R_{mn'}^{(j)}(\cosh\mu_0;c_2)+\sinh^4\mu_0\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{4,m}$$

$$+\sinh^2\mu_0[4R_{mn'}^{(j)}(\cosh\mu_0;c_2)+\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{8,m}$$

$$+\sinh^2\mu_0[\left(\frac{k_2 a}{2}\right)^2\sinh^2\mu_0 R_{mn'}^{(j)}(\cosh\mu_0;c_2)+2\cosh\mu_0 R_{mn'}^{(j)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{6,m}$$

$$+\left(\frac{k_2 a}{2}\right)^2 R_{mn'}^{(j)}(\cosh\mu_0;c_2)[2\sinh^2\mu_0\Re_{n,n'}^{13,m}+\Re_{n,n'}^{15,m}]$$

$$+R_{mn'}^{(j)}(\cosh\mu_0;c_2)[\sinh^2\mu_0\Re_{n,n'}^{9,m}+\Re_{n,n'}^{16,m}+2\Re_{n,n'}^{14,m}-\sinh^2\mu_0\Re_{n,n'}^{10,m}]\}\}=0, \tag{31}$$

where $\hat{\delta}_{0m}=2$ if $m=0$ and $\hat{\delta}_{0m}=1$ otherwise.

These equations will be combined with the corresponding non-homogeneous ones resulting from the integral representation (24), after being amenable to asymptotic analysis in the far-field region.

4. Determination of the electric field in the host environment

4.1. Investigation of the integral representation in the scatterer's region

The integral representation (24) is characterized by the fact that its surface integral involves functions expressed in different geometric coordinate systems. As a matter of fact, the electric field "lives" in spheroidal geometry, since it is expressed via the spectral decomposition (25), while the kernel dyadic $\overline{\overline{G}}_e^{(22)}\left(\overline{r},\overline{r}'\right)$ is expressed in cylindrical coordinates. It is then apparent that a lot of techniques based on addition theorems, referring to these two geometries, have to be applied in order for this integral representation to be handled.

More precisely, in view of Eqs. (18.a), (18.c) and (A.8) (for $z < z'$), the surface integral of representation (24), denoted as I, can be written as

$$I = \int_0^{\infty} d\lambda \sum_{m=0}^{\infty} C^{(2)} \left[\overline{M}(-h_2) I_1 + \overline{N}(-h_2) I_2\right], \tag{32}$$

where

$$I_1 = \int_{S_{sph}} \left[\overline{M'}(h_2) + a'\overline{M'}(-h_2)\right] \cdot \left[\hat{n}' \times \nabla' \times \overline{E}_2\left(\overline{r'}\right)\right] dS'_{sph}, \tag{33}$$

$$I_2 = \int_{S_{sph}} \left[\overline{N'}(h_2) + b'\overline{N'}(-h_2)\right] \cdot \left[\hat{n}' \times \nabla' \times \overline{E}_2\left(\overline{r'}\right)\right] dS'_{sph}. \tag{34}$$

The substitution of the spectral decomposition (25) of the electric field $\overline{E}_2\left(\overline{r'}\right)$ (with unprimed argument there) in the surface integrals I_1, I_2 leads to corresponding surface integrals, incorporating simultaneously vector wave functions in the spheroidal and the cylindrical coordinate system. Extended use of several vector analysis arguments helps in manipulating the integrands of the aforementioned integrals. Following these arguments and performing the azimuthal integration we transform Eqs. (33)-(34) into the following form

$$I_1 = I_1\left(A^{(j)}_{\substack{e\\o}mn'}, B^{(j)}_{\substack{e\\o}mn'}, h_2 \right) = \sum_{j=3}^{4} \sum_{n'=m}^{\infty} \{A^{(j)}_{\substack{e\\o}mn'} I^{A}_{1,\substack{e\\o}\, jmn'}(h_2) + B^{(j)}_{\substack{e\\o}mn'} I^{B}_{1,\substack{e\\o}\, jmn'}(h_2)\}, \tag{35}$$

$$I_2 = I_2\left(A^{(j)}_{\substack{e\\o}mn'}, B^{(j)}_{\substack{e\\o}mn'}, h_2 \right) = \sum_{j=3}^{4} \sum_{n'=m}^{\infty} \{A^{(j)}_{\substack{e\\o}mn'} I^{A}_{2,\substack{e\\o}\, jmn'}(h_2) + B^{(j)}_{\substack{e\\o}mn'} I^{B}_{2,\substack{e\\o}\, jmn'}(h_2)\}. \tag{36}$$

The appeared θ-integrals $I^{(A,B)}_{(1,2),\substack{e\\o}\, jmn'}$ in Eqs. (35)-(36) can be decomposed as

$$I^{A}_{1,\substack{e\\o}\, jmn'}(h_2) = k_2\left[J_{1,1}(h_2) + a' J_{1,1}(-h_2)\right], \tag{37}$$

$$I^{B}_{1,\substack{e\\o}\, jmn'}(h_2) = k_2\left[J_{1,2}(h_2) + a' J_{1,2}(-h_2)\right], \tag{38}$$

$$I^{A}_{2,\substack{e\\o}\, jmn'}(h_2) = k_2\left[J_{2,1}(h_2) + b' J_{2,1}(-h_2)\right], \tag{39}$$

$$I^{B}_{2,\substack{e\\o}\, jmn'}(h_2) = k_2\left[J_{2,2}(h_2) + b' J_{2,2}(-h_2)\right]. \tag{40}$$

The terms $J_{i,j}(h), i, j = (1,2)$ are expressed via a plethora (38 terms) of structural θ-integrals by the following relations

$$J_{1,2}(h) = J^{1,1}_{1,2}(h) + J^{1,2,1}_{1,2}(h) - J^{1,2,2}_{1,2}(h) - J^{2}_{1,2}(h), \tag{41}$$

$$\begin{aligned} J_{1,1}(h) = \frac{1}{k_2}[& J^{1,1}_{1,1}(h) + J^{1,2}_{1,1}(h) + J^{2,1,1}_{1,1}(h) + J^{2,1,2,1}_{1,1}(h) + J^{2,1,2,2.a}_{1,1}(h) + J^{2,1,2,2.b}_{1,1}(h) \\ & + J^{2,1,2,2.c}_{1,1}(h) + J^{2,2,1}_{1,1}(h) + J^{2,2,2}_{1,1}(h) + J^{2,3,1}_{1,1}(h) + J^{2,3,2}_{1,1}(h) + J^{3}_{1,1}(h)], \end{aligned} \tag{42}$$

$$J_{2,2}(h) = J^{1,1,1}_{2,2}(h) + J^{1,1,2}_{2,2}(h) + J^{1,2}_{2,2}(h) + J^{1,3,1}_{2,2}(h) + J^{1,3,2}_{2,2}(h) - J^{2,1}_{2,2}(h) - J^{2,2}_{2,2}(h), \tag{43}$$

$$\begin{aligned} J_{2,1}(h) = \frac{1}{k_2}[& J^{1,1}_{2,1}(h) + J^{1,2}_{2,1}(h) + J^{1,3}_{2,1}(h) + J^{2,1,1}_{2,1}(h) + J^{2,1,2}_{2,1}(h) + J^{2,2,1}_{2,1}(h) + J^{2,2,2,1}_{2,1}(h) + J^{2,2,2,2.a}_{2,1}(h) \\ & + J^{2,2,2,2.b}_{2,1}(h) + J^{2,2,2,2.c}_{2,1}(h) + J^{2,3,1}_{2,1}(h) + J^{2,3,2}_{2,1}(h) + J^{2,4,1}_{2,1}(h) + J^{2,4,2}_{2,1}(h) + J^{3}_{2,1}(h)]. \end{aligned} \tag{44}$$

These integrals are of "mixed-type'' in the sense that their integrands constitute products of scalar functions expressed in different coordinate systems; the spheroidal and the cylindrical one. The analytical treatment of these θ-integrals is based on the investigation and handling of suitable addition theorems connecting the two geometries and actually constitutes the most demanding part of the analytical burden of this work. The exact values of all these terms are given in [20]. Here we discuss the treatment of one representative of them. In particular, the first one of the terms is defined by

$$J_{1,2}^{1,1}(h_2) = -\left(\frac{a}{2}\right)^2 \sinh\mu_0 \cosh\mu_0 (m\pi) D_{o \xrightarrow{(+1)} e}^{e \xrightarrow{(-1)} o} R_{mn'}^{(j)}(\cosh\mu_0; c_2)$$
$$\cdot \int_0^\pi \{\frac{1}{\sin\theta'} S_{mn'}(\cos\theta'; c_2) \frac{\partial}{\partial\rho'}(J_m(\lambda\rho')) e^{ih_2 z'} \sin\theta' d\theta'\}, \tag{45}$$

where the appeared quantity D, steaming from azimuthal integration, is assigned the values (± 1) depending on the chosen branch while it vanishes for any other choice. Using Bessel functions properties we obtain

$$J_{1,2}^{1,1}(h_2) = -\frac{\lambda}{2}\left(\frac{a}{2}\right)^2 \sinh\mu_0 \cosh\mu_0 (m\pi) D_{o \xrightarrow{(+1)} e}^{e \xrightarrow{(-1)} o} R_{mn'}^{(j)}(\cosh\mu_0; c_2)$$
$$\cdot \int_0^\pi \{\frac{1}{\sin\theta'} S_{mn'}(\cos\theta'; c_2) [J_{m-1}(\lambda\rho') - J_{m+1}(\lambda\rho')] e^{ih_2 z'} \sin\theta' d\theta'\}. \tag{46}$$

Crucial role to the treatment of the above integral plays the following addition theorem [17]

$$\int_0^\pi e^{ih_2 z'} J_m(\lambda\rho') P_n^m(\cos\theta') \sin\theta' d\theta' = \frac{2}{i^{m-n}} P_n^m(\cos\theta_0) j_n(kr_0), \tag{47}$$

where

$$\cos\theta_0 = \frac{h_2 \cosh\mu_0}{\sqrt{\lambda^2 \sinh^2\mu_0 + h_2^2 \cosh^2\mu_0}}, \tag{48.a}$$

$$kr_0 = \frac{a}{2}\sqrt{\lambda^2 \sinh^2\mu_0 + h_2^2 \cosh^2\mu_0}. \tag{48.b}$$

The basic ingredient in the analysis of all the aforementioned integrals is the effort to render equal the order of the cylindrical Bessel functions with the azimuthal order of the associated Legendre functions. Thereby, mobilizing suitable recurrence relations [20] to fulfill the indices coincidence we obtain

$$J_{1,2}^{1,1}(h_2) = -\frac{\lambda}{2}\left(\frac{a}{2}\right)^2 \sinh\mu_0 \cosh\mu_0 (m\pi) D_{o \xrightarrow{(+1)} e}^{e \xrightarrow{(-1)} o} R_{mn'}^{(j)}(\cosh\mu_0; c_2) \sum_{k=0,1}{}' d_k^{mn'}(c_2)$$
$$\cdot \{\sum_{l=0}^{\left[\frac{k}{2}\right]} \{(2m+2k-4l-1) \int_0^\pi P_{m+k-2l-1}^{m-1}(\cos\theta') J_{m-1}(\lambda\rho') e^{ih_2 z'} \sin\theta' d\theta'\}$$
$$+ \sum_{l=0}^{\left[\frac{k}{2}\right]-1} \{\frac{(2m+k)!(k-2l-2)!}{(2m+k-2l)!k!}(2m+2k-4l-1) \int_0^\pi P_{m+k-2l-1}^{m+1}(\cos\theta') J_{m+1}(\lambda\rho') e^{ih_2 z'} \sin\theta' d\theta'\}$$
$$- \frac{(2m+k)!}{\left(2m+k-2\left[\frac{k}{2}\right]\right)!k!} \int_0^\pi \frac{1}{\sin\theta} P_{m+k-2\left[\frac{k}{2}\right]}^m(\cos\theta') J_{m+1}(\lambda\rho') e^{ih_2 z'} \sin\theta' d\theta'\}. \tag{49}$$

The first two integrals are treated using the addition theorem (47), while the third one disposes a peculiar recurrence expansion, as it is explained in [20], and therefore is treated separately. Finally, we obtain that

$$
\begin{aligned}
J_{1,2}^{1,1}(h_2) = & -\frac{\lambda}{2}\left(\frac{a}{2}\right)^2 \sinh\mu_0 \cosh\mu_0 (m\pi) D^{e \xrightarrow{(-1)} o}_{o \xrightarrow{(+1)} e} R_{mn'}^{(j)}(\cosh\mu_0; c_2) \sum_{k=0,1}{}' d_k^{mn'}(c_2) \\
& \cdot \{ \sum_{l=0}^{\left[\frac{k}{2}\right]} \{ (2m+2k-4l-1) \frac{1}{i^{2l-k}} P_{m+k-2l-1}^{m-1}(\cos\theta_0) j_{m+k-2l-1}(kr_0) \} \\
& + \sum_{l=0}^{\left[\frac{k}{2}\right]-1} \{ \frac{(2m+k)!(k-2l-2)!}{(2m+k-2l)!k!} (2m+2k-4l-1) \frac{1}{i^{2l-k}} P_{m+k-2l-1}^{m+1}(\cos\theta_0) j_{m+k-2l-1}(kr_0) \} \\
& - \frac{(2m+k)!}{\left(2m+k-2\left[\frac{k}{2}\right]\right)!k!} L_{(1)} \},
\end{aligned}
\tag{50.a}
$$

where

$$
L_{(1)} = \begin{cases}
\left\{ \begin{array}{l} -\sum_{l=2}^{m+1} \{ \left(\frac{2}{u}\right)^{l-2} \frac{m!}{(m-l+2)!} \frac{\gamma_m}{\gamma_{m-l+1}} P_{m-l+1}^{m-l+1}(\cos\theta_0) j_{m-l+1}(kr_0) \} \\ + \frac{2^m m!}{u^m} \frac{\gamma_m}{\gamma_0} \{ P_1^1(\cos\theta_0) j_1(kr_0) + \frac{1}{u} [\cos(\zeta) - j_0(kr_0) + \zeta\, P_1^0(\cos\theta_0) j_1(kr_0)] \} \end{array} \right\}, & k = \text{even} \\
(2m+1) \left\{ \begin{array}{l} -\sum_{l=2}^{m+1} \{ \left(\frac{2}{u}\right)^{l-2} \frac{m!}{(m-l+2)!} \frac{\gamma_m}{\gamma_{m-l+1}} \frac{i}{(2m-2l+3)} P_{m-l+2}^{m-l+1}(\cos\theta_0) j_{m-l+2}(kr_0) \} \\ + \frac{2^m m!}{u^{m+1}} \frac{\gamma_m}{\gamma_0} [i\sin(\zeta) - i\zeta\, j_0(kr_0)] \end{array} \right\}, & k = \text{odd}
\end{cases},
\tag{50.b}
$$

and we have set for simplicity $u = \lambda \frac{a}{2} \sinh\mu_0$, $\zeta = h_2 \frac{a}{2} \cosh\mu_0$ and $\gamma_m = \frac{(2m)!}{2^m m!}$.

Coming back to the integral representation (24) and observing that the volume integral is handled easily since it includes the "Dirac" current distribution (5), we obtain the final form of this representation

$$
\begin{aligned}
&\sum_{j=3}^{4}\sum_{m'=0}^{\infty}\sum_{n'=m'}^{\infty}\{A^{(j)}_{\substack{e\\o}m'n'}\overline{M}^{(j)}_{\substack{e\\o}m'n'}(\bar{r})+B^{(j)}_{\substack{e\\o}m'n'}\overline{N}^{(j)}_{\substack{e\\o}m'n'}(\bar{r})\}\\
&-\int_0^{\infty}d\lambda\sum_{m=0}^{\infty}C^{(2)}\{\overline{M}(-h_2)\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A^{(j)}_{\substack{e\\o}mn'}I^{A}_{1,\substack{e\\o}jmn'}(h_2)+B^{(j)}_{\substack{e\\o}mn'}I^{B}_{1,\substack{e\\o}jmn'}(h_2)\}\\
&\quad+\overline{N}(-h_2)\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A^{(j)}_{\substack{e\\o}mn'}I^{A}_{2,\substack{e\\o}jmn'}(h_2)+B^{(j)}_{\substack{e\\o}mn'}I^{B}_{2,\substack{e\\o}jmn'}(h_2)\}\\
&=i\omega\mu_0(\mathrm{Ia})\int_0^{\infty}d\lambda\sum_{m=0}^{\infty}C^{(1)}\{c\overline{M}(-h_2)\overline{M}^{(0)}(h_1)\cdot\hat{\mathrm{a}}+d\overline{N}(-h_2)\overline{N}^{(0)}(h_1)\cdot\hat{\mathrm{a}}\},\ z<\min\{z_{sph}\},
\end{aligned}
\tag{51}
$$

where the superscript (0) indicates reference to the source position vector $\bar{r}_0$. We notice that Eq. (51) is valid for $z<\min\{z_{sph}\}$, given that we have used the corresponding expansion of the free-space Green's dyad. The motivation for this hypothesis stems from the necessity to examine the asymptotic behavior of the above representation in the far-field region of the host environment. We have then the opportunity to apply stationary phase arguments and thus to avoid the λ-integral, which owes its existence to the implication of the cylindrical geometry.

Indeed, applying stationary phase techniques [16] and using the asymptotic behavior of Bessel functions we obtain

$$
\begin{aligned}
&\sum_{j=3}^{4}\sum_{m'=0}^{\infty}\sum_{n'=m'}^{\infty}\left\{A^{(j)}_{\substack{e\\o}m'n'}\overline{M}^{(j)}_{\substack{e\\o}m'n'}(\bar{r})+B^{(j)}_{\substack{e\\o}m'n'}\overline{N}^{(j)}_{\substack{e\\o}m'n'}(\bar{r})\right\}_{k_2r>>1}\\
&+\frac{e^{ik_2r}}{4\pi k_2 r\sin\theta}\sum_{m=0}^{\infty}(2-\delta_0)(-i)^{m+1}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}\\
&\cdot\left\{\begin{aligned}&i\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A^{(j)}_{\substack{e\\o}mn'}I^{A}_{1,\substack{e\\o}jmn'}(h_2,\lambda_0)+B^{(j)}_{\substack{e\\o}mn'}I^{B}_{1,\substack{e\\o}jmn'}(h_2,\lambda_0)\}\hat{\phi}\\&+\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A^{(j)}_{\substack{e\\o}mn'}I^{A}_{2,\substack{e\\o}jmn'}(h_2,\lambda_0)+B^{(j)}_{\substack{e\\o}mn'}I^{B}_{2,\substack{e\\o}jmn'}(h_2,\lambda_0)\}\hat{\theta}\end{aligned}\right\}_{\substack{\lambda_0=k_2\sin\theta\\ h_2=-k_2\cos\theta}}\\
&=i\omega\mu_0(\mathrm{Ia})\frac{e^{ik_2r}}{4\pi r\sin\theta\sqrt{k_1^2-k_2^2\sin^2\theta}}\cos\theta\sum_{m=0}^{\infty}(2-\delta_0)(-i)^{m+1}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}\\
&\cdot\left\{i\,c(\theta)\left[\overline{M}^{(0)}(h_1)\cdot\hat{\mathrm{a}}\right]\hat{\phi}+d(\theta)\left[\overline{N}^{(0)}(h_1)\cdot\hat{\mathrm{a}}\right]\hat{\theta}\right\}_{h_1=\sqrt{k_1^2-k_2^2\sin^2\theta}},\quad z<\min\{z_{sph}\},
\end{aligned}
\tag{52}
$$

where

$$
c(\theta)=\frac{2\sqrt{1-n^2\sin^2\theta}}{\sqrt{1-n^2\sin^2\theta}-n\cos\theta}\,e^{-ik_1\left(\sqrt{1-n^2\sin^2\theta}+n\cos\theta\right)\delta},
\tag{53.a}
$$

$$d(\theta)=\frac{2\sqrt{1-n^2\sin^2\theta}}{n\sqrt{1-n^2\sin^2\theta}-\cos\theta}e^{-ik_1\left(\sqrt{1-n^2\sin^2\theta}+n\cos\theta\right)\delta}, \tag{53.b}$$

which results from Eqs. (17.c)-(17.d) with

$$\left.\begin{aligned} h_1 &= \sqrt{k_1^2-k_2^2\sin^2\theta} \\ h_2 &= -k_2\cos\theta \end{aligned}\right\}. \tag{54}$$

Moreover, in view of Eqs. (37)-(40), it is readily seen that similar expressions are valid for the coefficients a,b defined in Eqs. (17.a)-(17.b). We remark that asymptotic analysis reveals as primitive coordinates the spherical ones, i.e. (r,θ,ϕ). It is also noticeable that the restriction $z<\min\{z_{sph}\}$ implies that $\theta\in(\frac{\pi}{2},\pi]$.

Expression (52) must be in accordance with the spectral representation (25) of the electric field in the asymptotic realm. To assure their coincidence we need to acquire first the asymptotic behavior of the spheroidal vector wave functions for large arguments [Appendix B] as it results from the corresponding behavior of the spheroidal "radial" functions $R_{m'n'}^{(j)}(\cosh\mu;c_2)$ and their derivatives. Indeed, we obtain

$$\begin{aligned}\overline{M}_{\substack{e\\o}m'n'}^{(j)}(\overline{r})\cong{}&\frac{1}{k_2r}e^{i\left[k_2r-\frac{\pi}{2}(n'+1)\right]\mathrm{sgn}(j+1)}\\ &\cdot\{\frac{S_{m'n'}(\cos\theta;c_2)}{\sin\theta}\begin{Bmatrix}-m'\sin(m'\phi)\\ m'\cos(m'\phi)\end{Bmatrix}\hat{\theta}+\sin\theta S'_{m'n'}(\cos\theta;c_2)\begin{Bmatrix}\cos(m'\phi)\\ \sin(m'\phi)\end{Bmatrix}\hat{\phi}\}+O\left(\frac{1}{r^2}\right),\end{aligned} \tag{55.a}$$

$$\begin{aligned}\overline{N}_{\substack{e\\o}m'n'}^{(j)}(\overline{r})\cong{}&\frac{1}{k_2r}e^{i\left[k_2r-\frac{\pi}{2}(n'+1)\right]\mathrm{sgn}(j+1)}(i\,\mathrm{sgn}(j+1))\\ &\cdot\{-\sin\theta S'_{m'n'}(\cos\theta;c_2)\begin{Bmatrix}\cos(m'\phi)\\ \sin(m'\phi)\end{Bmatrix}\hat{\theta}+\frac{S_{m'n'}(\cos\theta;c_2)}{\sin\theta}\begin{Bmatrix}-m'\sin(m'\phi)\\ m'\cos(m'\phi)\end{Bmatrix}\hat{\phi}\}+O\left(\frac{1}{r^2}\right),\end{aligned} \tag{55.b}$$

where $\mathrm{sgn}(j+1)=1$ if $j=3$ and $\mathrm{sgn}(j+1)=-1$ if $j=4$.

Inserting the asymptotic expressions (55.a)-(55.b) in Eq. (52) we conclude that

$$
\sum_{j=3}^{4} \frac{1}{k_2 r} e^{ik_2 r \operatorname{sgn}(j+1)} \sum_{m'=0}^{\infty}\sum_{n'=m'}^{\infty} e^{-i\frac{\pi}{2}(n'+1)\operatorname{sgn}(j+1)}
$$

$$
\cdot\{A^{(j)}_{\substack{e\\o}m'n'}[\frac{S_{m'n'}(\cos\theta;c_2)}{\sin\theta}\begin{Bmatrix}-m'\sin(m'\phi)\\ m'\cos(m'\phi)\end{Bmatrix}\hat{\theta}+\sin\theta S'_{m'n'}(\cos\theta;c_2)\begin{Bmatrix}\cos(m'\phi)\\ \sin(m'\phi)\end{Bmatrix}\hat{\phi}]
$$

$$
+i\operatorname{sgn}(j+1)B^{(j)}_{\substack{e\\o}m'n'}[-\sin\theta S'_{m'n'}(\cos\theta;c_2)\begin{Bmatrix}\cos(m'\phi)\\ \sin(m'\phi)\end{Bmatrix}\hat{\theta}+\frac{S_{m'n'}(\cos\theta;c_2)}{\sin\theta}\begin{Bmatrix}-m'\sin(m'\phi)\\ m'\cos(m'\phi)\end{Bmatrix}\hat{\phi}]\}
$$

$$
+\frac{e^{ik_2 r}}{4\pi k_2 r\sin\theta}\sum_{m=0}^{\infty}(2-\delta_0)(-i)^{m+1}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}
$$

$$
\cdot\left\{\begin{matrix} i\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A^{(j)}_{\substack{e\\o}mn'}I^{A}_{1\substack{e\\o}jmn'}(h_2,\lambda_0)+B^{(j)}_{\substack{e\\o}mn'}I^{B}_{1\substack{e\\o}jmn'}(h_2,\lambda_0)\}\hat{\phi} \\ +\sum_{j=3}^{4}\sum_{n'=m}^{\infty}\{A^{(j)}_{\substack{e\\o}mn'}I^{A}_{2\substack{e\\o}jmn'}(h_2,\lambda_0)+B^{(j)}_{\substack{e\\o}mn'}I^{B}_{2\substack{e\\o}jmn'}(h_2,\lambda_0)\}\hat{\theta}\end{matrix}\right\}_{\substack{\lambda_0=k_2\sin\theta\\ h_2=-k_2\cos\theta}}
$$

$$
=i\omega\mu_0(\mathrm{Ia})\frac{e^{ik_2 r}}{4\pi r\sin\theta\sqrt{k_1^2-k_2^2\sin^2\theta}}\cos\theta\sum_{m=0}^{\infty}(2-\delta_0)(-i)^{m+1}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}
$$

$$
\cdot\left\{i\,c(\theta)\left[\overline{M}^{(0)}(h_1)\cdot\hat{a}\right]\hat{\phi}+d(\theta)\left[\overline{N}^{(0)}(h_1)\cdot\hat{a}\right]\hat{\theta}\right\}_{h_1=\sqrt{k_1^2-k_2^2\sin^2\theta}},\quad z<\min\{z_{sph}\} \tag{56}
$$

It can be proved that the case $j=4$ is not allowed in our spectral representation given that, in the framework of harmonic time dependence $\exp\{-i\omega t\}$, all the waves corresponding to the above choice constitute incoming propagation waves. Asymptotic analysis of the spherical Hankel functions of the second kind ($j=4$) proves that these incoming modes behave like $\frac{e^{ik_2 r}}{r}$ on a large sphere of radius r. Hence, the total incoming energy of these terms would be of order 4π no matter how large the radius r was. However, this incoming energy owes its existence to the secondary sources located on the interface S_δ, which remain outside the large sphere. In other words, the inward propagating energy would remain, if $j=4$ was accepted, constant although the responsible sources would diminish further and further. Thus, we restrict our analysis to the case $j=3$.

Taking now the projections of Eq. (56) on the unit vectors $\hat{\theta},\hat{\phi}$ and projecting then functionally on the azimuthal functions $\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$, $m=0,1,2,\ldots$, we obtain, for every index m, four scalar equations, which depend on the coordinate $\theta\in(\frac{\pi}{2},\pi]$. Our aim is to combine these equations with those originated from the boundary condition satisfaction on S_{sph}, i.e. Eqs. (28)-(31). However, the new equations are functions of θ and this is a qualitative difference. It would be desirable to project them on a complete set of functions in the interval $\frac{\pi}{2} < \theta \le \pi$. Nevertheless, we realized that this increases significantly the technical complexity due to the very cumbersome form of the integrated functions. Thus, we suggest an alternative "projection" resembling the "discrete analogous" of the aforementioned integration multiplying with basis θ-functions and averaging over a dense partition of the interval $(\frac{\pi}{2}, \pi]$. Consequently, for every pair of indices (m,n) with $m = 0,1,2,\ldots; n \ge m$, we obtain

$$\sum_{n'=m}^{\infty} e^{-i\frac{\pi}{2}(n'+1)}\left\{A^{(3)}_{o,mn'}(m\pi)\mathrm{K}^{1,m}_{n,n',N} - iB^{(3)}_{e,mn'}\left(\pi\hat{\delta}_{0m}\right)\mathrm{K}^{2,m}_{n,n',N}\right\} + \sum_{n'=m}^{\infty} \frac{1}{2}(-i)^{m+1}\left\{A^{(3)}_{e,mn'}\mathrm{K}^{3,m}_{e,n,n',N} + B^{(3)}_{e,mn'}\mathrm{K}^{4,m}_{e,n,n',N}\right\} = i\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(-i)^{m+1}\mathrm{J}^{1,m}_{e,n,N}(\bar{\mathrm{a}}), \tag{57}$$

$$\sum_{n'=m}^{\infty} e^{-i\frac{\pi}{2}(n'+1)}\left\{A^{(3)}_{e,mn'}(-m\pi)\mathrm{K}^{1,m}_{n,n',N} - iB^{(3)}_{o,mn'}\left(\pi(1-\delta_0)\right)\mathrm{K}^{2,m}_{n,n',N}\right\} + \sum_{n'=m}^{\infty} \frac{1}{2}(1-\delta_0)(-i)^{m+1}\left\{A^{(3)}_{o,mn'}\mathrm{K}^{3,m}_{o,n,n',N} + B^{(3)}_{o,mn'}\mathrm{K}^{4,m}_{o,n,n',N}\right\} = i\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(1-\delta_0)(-i)^{m+1}\mathrm{J}^{1,m}_{o,n,N}(\bar{\mathrm{a}}), \tag{58}$$

$$\sum_{n'=m}^{\infty} e^{-i\frac{\pi}{2}(n'+1)}\left\{A^{(3)}_{e,mn'}\left(\pi\hat{\delta}_{0m}\right)\mathrm{K}^{2,m}_{n,n',N} + iB^{(3)}_{o,mn'}(m\pi)\mathrm{K}^{1,m}_{n,n',N}\right\} + \sum_{n'=m}^{\infty} \frac{1}{2}(-i)^{m}\left\{A^{(3)}_{e,mn'}\mathrm{K}^{5,m}_{e,n,n',N} + B^{(3)}_{e,mn'}\mathrm{K}^{6,m}_{e,n,n',N}\right\} = i\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(-i)^{m}\mathrm{J}^{2,m}_{e,n,N}(\bar{\mathrm{a}}), \tag{59}$$

$$\sum_{n'=m}^{\infty} e^{-i\frac{\pi}{2}(n'+1)}\left\{A^{(3)}_{o,mn'}\left(\pi(1-\delta_0)\right)\mathrm{K}^{2,m}_{n,n',N} + iB^{(3)}_{e,mn'}(-m\pi)\mathrm{K}^{1,m}_{n,n',N}\right\} + \sum_{n'=m}^{\infty} \frac{1}{2}(1-\delta_0)(-i)^{m}\left\{A^{(3)}_{o,mn'}\mathrm{K}^{5,m}_{o,n,n',N} + B^{(3)}_{o,mn'}\mathrm{K}^{6,m}_{o,n,n',N}\right\} = i\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(1-\delta_0)(-i)^{m}\mathrm{J}^{2,m}_{o,n,N}(\bar{\mathrm{a}}), \tag{60}$$

where

$$\mathrm{K}^{1,m}_{n,n',N} = \frac{1}{N}\sum_{i=1}^{N}\left\{S_{mn'}(\cos\theta_i; c_2)P_n^m(\cos\theta_i)\right\}, \tag{61}$$

$$\mathrm{K}^{2,m}_{n,n',N} = \frac{1}{N}\sum_{i=1}^{N} \left\{ sin^2\,\theta_i S'_{mn'}(cos\,\theta_i; c_2) P_n^m(cos\,\theta_i) \right\}, \tag{62}$$

$$\mathrm{K}^{3,m}_{\substack{e\\o},n,n',N} = \frac{1}{N}\sum_{i=1}^{N} \left\{ I^A_{2,\substack{e\\o}(3)mn'}(h_2,\lambda_0) P_n^m(cos\,\theta_i) \right\}_{\substack{\lambda_0 = k_2\,sin\,\theta_i \\ h_2 = -k_2\,cos\,\theta_i}}, \tag{63}$$

$$\mathrm{K}^{4,m}_{\substack{e\\o},n,n',N} = \frac{1}{N}\sum_{i=1}^{N} \left\{ I^B_{2,\substack{e\\o}(3)mn'}(h_2,\lambda_0) P_n^m(cos\,\theta_i) \right\}_{\substack{\lambda_0 = k_2\,sin\,\theta_i \\ h_2 = -k_2\,cos\,\theta_i}}, \tag{64}$$

$$\mathrm{K}^{5,m}_{\substack{e\\o},n,n',N} = \frac{1}{N}\sum_{i=1}^{N} \left\{ I^A_{1,\substack{e\\o}(3)mn'}(h_2,\lambda_0) P_n^m(cos\,\theta_i) \right\}_{\substack{\lambda_0 = k_2\,sin\,\theta_i \\ h_2 = -k_2\,cos\,\theta_i}}, \tag{65}$$

$$\mathrm{K}^{6,m}_{\substack{e\\o},n,n',N} = \frac{1}{N}\sum_{i=1}^{N} \left\{ I^B_{1,\substack{e\\o}(3)mn'}(h_2,\lambda_0) P_n^m(cos\,\theta_i) \right\}_{\substack{\lambda_0 = k_2\,sin\,\theta_i \\ h_2 = -k_2\,cos\,\theta_i}}, \tag{66}$$

$$\mathrm{J}^{1,m}_{\substack{e\\o},n,N}(\hat{\mathrm{a}}) = \frac{1}{N}\sum_{i=1}^{N} \left\{ \frac{cos\,\theta_i\, d(\theta_i)}{\sqrt{k_1^2 - k_2^2\, sin^2\,\theta_i}} \left[\overline{N}^{(0)}_{\substack{e\\o}m}(h_1)\cdot \hat{\mathrm{a}} \right] P_n^m(cos\,\theta_i) \right\}_{h_1 = \sqrt{k_1^2 - k_2^2\, sin^2\,\theta_i}}, \tag{67}$$

$$\mathrm{J}^{2,m}_{\substack{e\\o},n,N}(\hat{\mathrm{a}}) = \frac{1}{N}\sum_{i=1}^{N} \left\{ \frac{cos\,\theta_i\, c(\theta_i)}{\sqrt{k_1^2 - k_2^2\, sin^2\,\theta_i}} \left[\overline{M}^{(0)}_{\substack{e\\o}m}(h_1)\cdot \hat{\mathrm{a}} \right] P_n^m(cos\,\theta_i) \right\}_{h_1 = \sqrt{k_1^2 - k_2^2\, sin^2\,\theta_i}}. \tag{68}$$

4.2. Formulation of the linear algebraic spectral expansion coefficients system

In this section, our goal is the construction of the linear algebraic non-homogeneous system whose solution are the expansion coefficients of the spectral decomposition (25). We have already commented that the arising equations have two origins: the boundary condition on S_{sph} and the asymptotic treatment of the integral representation in V_2. What we have mainly accomplished is the "projection"-continuous or discrete according to the comments of the previous section-of these equations on the complete set of functions $P_n^m(\cos\theta)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$, $m = 0,1,2,\ldots; n \geq m$. This remark implies the optimal sorting of these equations in order to maximize the acquired information when a specific truncation is imposed for the numerical treatment of the resulting system. More precisely, we have to project all the considered equations to a specific member of the function basis, before proceeding to the projection on the next basis component.

The above remarks lead to the following classification of Eqs. (28)-(31), (57)-(60), for every particular pair (m,n) with $m = 0,1,2,\ldots; n \ge m$

$$\sum_{n'=m}^{L}\{A_{e,mn'}^{(3)}\left(\pi\hat{\delta}_{0m}\right)[\sinh\mu_0\cosh\mu_0 R_{mn'}^{(3)}(\cosh\mu_0;c_2)\Re_{n,n'}^{4,m} - \sinh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)\Re_{n,n'}^{6,m}]$$
$$+B_{o,mn'}^{(3)}\frac{2(m\pi)}{k_2 a}\{[\frac{(\sinh^2\mu_0-1)}{\sinh\mu_0}R_{mn'}^{(3)}(\cosh\mu_0;c_2)+\sinh\mu_0\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{1,m} \quad (69)$$
$$+\frac{1}{\sinh\mu_0}R_{mn'}^{(3)}(\cosh\mu_0;c_2)[2\Re_{n,n'}^{3,m}+\Re_{n,n'}^{5,m}+\Re_{n,n'}^{2,m}]\}\} = 0,$$

$$\sum_{n'=m}^{L}\{A_{o,mn'}^{(3)}(\pi(1-\delta_0))[\sinh\mu_0\cosh\mu_0 R_{mn'}^{(3)}(\cosh\mu_0;c_2)\Re_{n,n'}^{4,m} - \sinh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)\Re_{n,n'}^{6,m}]$$
$$+B_{e,mn'}^{(3)}\frac{2(-m\pi)}{k_2 a}\{[\frac{(\sinh^2\mu_0-1)}{\sinh\mu_0}R_{mn'}^{(3)}(\cosh\mu_0;c_2)+\sinh\mu_0\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{1,m} \quad (70)$$
$$+\frac{1}{\sinh\mu_0}R_{mn'}^{(3)}(\cosh\mu_0;c_2)[2\Re_{n,n'}^{3,m}+\Re_{n,n'}^{5,m}+\Re_{n,n'}^{2,m}]\}\} = 0,$$

$$\sum_{n'=m}^{L}\{A_{o,mn'}^{(3)}(m\pi)[\cosh\mu_0 R_{mn'}^{(3)}(\cosh\mu_0;c_2)[\sinh^4\mu_0\Re_{n,n'}^{1,m}+2\sinh^2\mu_0\Re_{n,n'}^{3,m}+\Re_{n,n'}^{7,m}]]$$
$$-B_{e,mn'}^{(3)}\frac{2\left(\pi\hat{\delta}_{0m}\right)}{k_2 a}\{[\sinh^2\mu_0\left(\sinh^2\mu_0-1\right)R_{mn'}^{(3)}(\cosh\mu_0;c_2)+\sinh^4\mu_0\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{4,m}$$
$$+\sinh^2\mu_0[4R_{mn'}^{(3)}(\cosh\mu_0;c_2)+\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{8,m}$$
$$+\sinh^2\mu_0[\left(\frac{k_2 a}{2}\right)^2\sinh^2\mu_0 R_{mn'}^{(3)}(\cosh\mu_0;c_2)+2\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{6,m}$$
$$+\left(\frac{k_2 a}{2}\right)^2 R_{mn'}^{(3)}(\cosh\mu_0;c_2)[2\sinh^2\mu_0\Re_{n,n'}^{13,m}+\Re_{n,n'}^{15,m}]$$
$$+R_{mn'}^{(3)}(\cosh\mu_0;c_2)[\sinh^2\mu_0\Re_{n,n'}^{9,m}+\Re_{n,n'}^{16,m}+2\Re_{n,n'}^{14,m}-\sinh^2\mu_0\Re_{n,n'}^{10,m}]\}\} = 0,$$
(71)

$$\sum_{n'=m}^{L}\{A_{e,mn'}^{(3)}(-m\pi)[\cosh\mu_0 R_{mn'}^{(3)}(\cosh\mu_0;c_2)[\sinh^4\mu_0\Re_{n,n'}^{1,m}+2\sinh^2\mu_0\Re_{n,n'}^{3,m}+\Re_{n,n'}^{7,m}]]$$
$$-B_{o,mn'}^{(3)}\frac{2\pi(1-\delta_0)}{k_2 a}\{[\sinh^2\mu_0\left(\sinh^2\mu_0-1\right)R_{mn'}^{(3)}(\cosh\mu_0;c_2)+\sinh^4\mu_0\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{4,m}$$
$$+\sinh^2\mu_0[4R_{mn'}^{(3)}(\cosh\mu_0;c_2)+\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{8,m}$$
$$+\sinh^2\mu_0[\left(\frac{k_2 a}{2}\right)^2\sinh^2\mu_0 R_{mn'}^{(3)}(\cosh\mu_0;c_2)+2\cosh\mu_0 R_{mn'}^{(3)'}(\cosh\mu_0;c_2)]\Re_{n,n'}^{6,m}$$
$$+\left(\frac{k_2 a}{2}\right)^2 R_{mn'}^{(3)}(\cosh\mu_0;c_2)[2\sinh^2\mu_0\Re_{n,n'}^{13,m}+\Re_{n,n'}^{15,m}]$$
$$+R_{mn'}^{(3)}(\cosh\mu_0;c_2)[\sinh^2\mu_0\Re_{n,n'}^{9,m}+\Re_{n,n'}^{16,m}+2\Re_{n,n'}^{14,m}-\sinh^2\mu_0\Re_{n,n'}^{10,m}]\}\} = 0,$$
(72)

$$\sum_{n'=m}^{L}\{A^{(3)}_{e,mn'}[\frac{1}{2}(-i)^{m+1}\,\mathrm{K}^{3,m}_{e,n,n',N}]+A^{(3)}_{o,mn'}(-i)^{n'+1}(m\pi)\mathrm{K}^{1,m}_{n,n',N}]$$
$$+B^{(3)}_{e,mn'}[\frac{1}{2}(-i)^{m+1}\mathrm{K}^{4,m}_{e,n,n',N}-(-i)^{n'}(\pi\hat{\delta}_{0m})\mathrm{K}^{2,m}_{n,n',N}]\} \tag{73}$$
$$=\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(-i)^m\,\mathrm{J}^{1,m}_{e,n,N}(\hat{\mathrm{a}}),$$

$$\sum_{n'=m}^{L}\{A^{(3)}_{e,mn'}[(-i)^{n'+1}(-m\pi)\mathrm{K}^{1,m}_{n,n',N}]+A^{(3)}_{o,mn'}[\frac{1}{2}(-i)^{m+1}(1-\delta_0)\mathrm{K}^{3,m}_{o,n,n',N}]$$
$$+B^{(3)}_{o,mn'}[\frac{1}{2}(-i)^{m+1}(1-\delta_0)\mathrm{K}^{4,m}_{o,n,n',N}-(-i)^{n'}(\pi(1-\delta_0))\mathrm{K}^{2,m}_{n,n',N}]\} \tag{74}$$
$$=\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(-i)^m(1-\delta_0)\mathrm{J}^{1,m}_{o,n,N}(\hat{\mathrm{a}}),$$

$$\sum_{n'=m}^{L}\{A^{(3)}_{e,mn'}[(-i)^{n'+1}(\pi\hat{\delta}_{0m})\mathrm{K}^{2,m}_{n,n',N}+\frac{1}{2}(-i)^m\,\mathrm{K}^{5,m}_{e,n,n',N}]$$
$$+B^{(3)}_{e,mn'}[\frac{1}{2}(-i)^m\,\mathrm{K}^{6,m}_{e,n,n',N}]+B^{(3)}_{o,mn'}[(-i)^{n'}(m\pi)\mathrm{K}^{1,m}_{n,n',N}]\} \tag{75}$$
$$=-\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(-i)^{m+1}\mathrm{J}^{2,m}_{e,n,N}(\hat{\mathrm{a}}),$$

$$\sum_{n'=m}^{L}\{A^{(3)}_{o,mn'}[(-i)^{n'+1}(\pi(1-\delta_0))\mathrm{K}^{2,m}_{n,n',N}+\frac{1}{2}(-i)^m(1-\delta_0)\mathrm{K}^{5,m}_{o,n,n',N}]$$
$$+B^{(3)}_{e,mn'}[(-i)^{n'}(-m\pi)\mathrm{K}^{1,m}_{n,n',N}]+B^{(3)}_{o,mn'}[\frac{1}{2}(-i)^m(1-\delta_0)\mathrm{K}^{6,m}_{o,n,n',N}]\} \tag{76}$$
$$=-\omega\mu_0(\mathrm{Ia})\frac{k_2}{2}(-i)^{m+1}(1-\delta_0)\mathrm{J}^{2,m}_{o,n,N}(\hat{\mathrm{a}}).$$

Consequently, for every pair (m,n) we obtain eight equations involving all the unknown coefficients corresponding to the specific azimuthal parameter m. We remark that, for every pair (m,n'), $n'=m,m+1,\ldots$, four unknown coefficients must be determined, i.e. $A^{(3)}_{t,mn'}$, $B^{(3)}_{t,mn'}$; $t=(e,o)$. In other words the coefficients are grouped in sequences of four, labeled by the parameter n', for specific azimuthal dependence. We deduce that after selecting the truncation level L in the appeared summations, we need to consider k blocks of type (69)-(76) (corresponding to $n=1,2,\ldots k$) so that $4(L-m+1)=8k$. This last relation assures the acquisition of a finite number of linear equations with equal number of unknown coefficients.

In the sequel, the notation $D^{(i),s}_{mn',n}$; $i=1,2,\ldots 8$; $s=(A_e,A_o,B_e,B_o)$ is used for the coefficients of the unknowns $A^{(3)}_{e,mn'}, A^{(3)}_{o,mn'}, B^{(3)}_{e,mn'}, B^{(3)}_{o,mn'}$ in Eqs. (69)-(76) while $d^{(i)}_{m,n}$, $i=1,2,\ldots 8$, denotes the right-hand sides of these equations. Hence, Eqs. (69)-(76) can be written as

$$\sum_{n'=m}^{m+2k-1} \underline{E}^{(i)}_{mn',n}\,\underline{c}_{mn'} = d^{(i)}_{m,n}, \quad {}_{n=m,m+1,\ldots,m+k-1}^{i=1,2,\ldots 8}, \tag{77}$$

where

$$\underline{E}^{(i)}_{mn',n} = \left[D^{(i),A_e}_{mn',n} \quad D^{(i),A_o}_{mn',n} \quad D^{(i),B_e}_{mn',n} \quad D^{(i),B_o}_{mn',n}\right], \tag{78}$$

$$D^{(i),s}_{mn',n} = \begin{cases} 0, & (i,s) = (1,A_o),(1,B_e),(4,A_o),(4,B_e) \\ 0, & (i,s) = (2,A_e),(2,B_o),(3,A_e),(3,B_o) \\ 0, & (i,s) = (5,B_o),(6,B_e),(7,A_o),(8,A_e) \end{cases}, \tag{79}$$

and

$$\underline{c}_{mn'} = \left[A^{(3)}_{e,mn'} \quad A^{(3)}_{o,mn'} \quad B^{(3)}_{e,mn'} \quad B^{(3)}_{o,mn'}\right]^T. \tag{80}$$

The resulting non-homogeneous linear system of equations is

$$\mathbf{A}^{(m)}_k \; \mathbf{c}^{(m)}_k = \mathbf{d}^{(m)}_k, \; k = 1,2,3,\ldots, \tag{81}$$

where

$$\mathbf{A}^{(m)}_k = \begin{bmatrix} \mathbf{E}_{m,m,m} & \mathbf{E}_{m,m+1,m} & \mathbf{E}_{m,m+2,m} & \cdots & \mathbf{E}_{m,m+2k-1,m} \\ \mathbf{E}_{m,m,m+1} & \mathbf{E}_{m,m+1,m+1} & \mathbf{E}_{m,m+2,m+1} & \cdots & \mathbf{E}_{m,m+2k-1,m+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{E}_{m,m,m+k-1} & \mathbf{E}_{m,m+1,m+k-1} & \mathbf{E}_{m,m+2,m+k-1} & \cdots & \mathbf{E}_{m,m+2k-1,m+k-1} \end{bmatrix}, \tag{82}$$

$$\mathbf{E}_{m,n',n} = \left[\underline{E}^{(1)\,T}_{mn',n}, \underline{E}^{(4)\,T}_{mn',n}, \underline{E}^{(2)\,T}_{mn',n}, \underline{E}^{(3)\,T}_{mn',n}, \underline{E}^{(8)\,T}_{mn',n}, \underline{E}^{(7)\,T}_{mn',n}, \underline{E}^{(6)\,T}_{mn',n}, \underline{E}^{(5)\,T}_{mn',n}\right]^T, \tag{83}$$

$$\mathbf{c}^{(m)}_k = \left[\underline{c}^T_{m,m} \quad \underline{c}^T_{m,m+1} \quad \underline{c}^T_{m,m+2} \quad \cdots \quad \underline{c}^T_{m,m+2k-1}\right]^T, \tag{84}$$

and

$$\mathbf{d}^{(m)}_k = \left[\underline{d}^T_{m,m} \quad \underline{d}^T_{m,m+1} \quad \underline{d}^T_{m,m+2} \quad \cdots \quad \underline{d}^T_{m,m+k-1}\right]^T, \tag{85}$$

where

$$\underline{d}_{m,n} = \left[0 \quad 0 \quad 0 \quad 0 \quad d^{(8)}_{m,n} \quad d^{(7)}_{m,n} \quad d^{(6)}_{m,n} \quad d^{(5)}_{m,n}\right]^T. \tag{86}$$

5. Determination of the electric field in the scattering region

Once the electric field $\overline{E}_2(\overline{r})$ is determined (through the calculation of the spheroidal expansion coefficients), the determination of the electric field $\overline{E}_1(\overline{r})$ in the scattering region is possible through the integral representation (23). Notice first that the volume integral of this representation constitutes the incident wave and is independent of the specific scatterer. Thus we focus on the surface integral of representation (23), which stands for the scattered wave $\overline{E}_1^{sc}(\overline{r})$ and encodes all the information concerning the spheroidal scatterer.

Substituting the known spectral spheroidal expansion of the electric field $\overline{E}_2(\overline{r})$ in the surface integral of (23) and exploiting the dyadic relations (18.b), (18.d) we obtain that

$$\begin{aligned}\overline{E}_1^{sc}(\overline{r}) = \int_0^\infty d\lambda \sum_{m=0}^{\infty} C^{(2)}\{c'\overline{M}(h_1) \int_{S_{sph}} \overline{M}'(-h_2)\cdot\left[\hat{n}'\times\nabla'\times\overline{E}_2(\overline{r}')\right] dS'_{sph} \\ + d'\overline{N}(h_1) \int_{S_{sph}} \overline{N}'(-h_2)\cdot\left[\hat{n}'\times\nabla'\times\overline{E}_2(\overline{r}')\right] dS'_{sph}\}.\end{aligned} \tag{87}$$

In Eq. (87) we recognize spheroidal surface integrals of the same kind with those encountered in the treatment of the electric field $\overline{E}_2(\overline{r})$. Following similar manipulations we conclude that

$$\begin{aligned}\overline{E}_1^{sc}(\overline{r}) = \int_0^\infty d\lambda \sum_{m=0}^{\infty} C^{(2)}\{c'\overline{M}(h_1) \sum_{n'=m}^{\infty}\{A^{(3)}_{{}^e_o mn'} I'^{A}_{1,{}^e_o(3)mn'}(-h_2) + B^{(3)}_{{}^e_o mn'} I'^{B}_{1,{}^e_o(3)mn'}(-h_2)\} \\ + d'\overline{N}(h_1) \sum_{n'=m}^{\infty}\{A^{(3)}_{{}^e_o mn'} I'^{A}_{2,{}^e_o(3)mn'}(-h_2) + B^{(3)}_{{}^e_o mn'} I'^{B}_{2,{}^e_o(3)mn'}(-h_2)\}\},\end{aligned} \tag{88}$$

where

$$I'^{A}_{1,{}^e_o(3)mn'}(-h_2) = k_2 J_{1,1}(-h_2), \tag{89}$$

$$I'^{B}_{1,{}^e_o(3)mn'}(-h_2) = k_2 J_{1,2}(-h_2), \tag{90}$$

$$I'^{A}_{2,{}^e_o(3)mn'}(-h_2) = k_2 J_{2,1}(-h_2), \tag{91}$$

$$I'^{B}_{2,{}^e_o(3)mn'}(-h_2) = k_2 J_{2,2}(-h_2). \tag{92}$$

When $k_1 r >> 1$, we are in a position to apply stationary phase arguments [16], which lead to the following representation of the scattered field in spherical coordinates

$$\overline{E}_1^{sc}(\overline{r}) = -\frac{e^{ik_1 r}}{4\pi r \sin\theta\sqrt{k_2^2 - k_1^2\sin^2\theta}}\cos\theta\sum_{m=0}^{\infty}(2-\delta_0)(-i)^{m+1}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$$\cdot\left\{\begin{aligned} & i\,c'(\theta)\sum_{n'=m}^{\infty}\{A^{(3)}_{{}^{e}_{o}mn'} I'^{A}_{1,{}^{e}_{o}(3)mn'}(-h_2,\lambda_1) + B^{(3)}_{{}^{e}_{o}mn'} I'^{B}_{1,{}^{e}_{o}(3)mn'}(-h_2,\lambda_1)\}\,\hat{\phi} \\ & + d'(\theta)\sum_{n'=m}^{\infty}\{A^{(3)}_{{}^{e}_{o}mn'} I'^{A}_{2,{}^{e}_{o}(3)mn'}(-h_2,\lambda_1) + B^{(3)}_{{}^{e}_{o}mn'} I'^{B}_{2,{}^{e}_{o}(3)mn'}(-h_2,\lambda_1)\}\,\hat{\theta}\end{aligned}\right\}_{\substack{\lambda_1 = k_1\sin\theta \\ h_2 = \sqrt{k_2^2 - k_1^2\sin^2\theta}}}, \qquad (93)$$

where $\theta \in (0, \frac{\pi}{2}]$ and $c'(\theta), d'(\theta)$ are given by Eqs. (19.c)-(19.d) with

$$\left.\begin{aligned} h_1 &= k_1\cos\theta \\ h_2 &= \sqrt{k_2^2 - k_1^2\sin^2\theta}\end{aligned}\right\}. \qquad (94)$$

Equation (93) is now of the form

$$\overline{E}_1^{sc}(\overline{r}) = \frac{e^{ik_1 r}}{r}\overline{f}(\theta,\phi), \qquad (95)$$

where $\overline{f}(\theta,\phi)$ is the vector scattering amplitude and therefore it can be used to compute the differential scattering cross section, which is defined by

$$\sigma(\theta,\phi) = r^2\left|\frac{\overline{E}_1^{sc}}{E_0}\right|^2, \qquad (96)$$

where E_0 denotes (in general) the amplitude of the incident wave.

6. Numerical solution-results

The numerical solution of the system of equations is a time consuming numerical task due to the structure of the multiple infinite summations appeared. We have obtained the angular distribution of the differential scattering cross section introduced in Eq. (96) for two indicative cases: (a) the perturbed sphere and (b) the spheroidal scatterer. In both cases we consider that the electric dipole is orientated towards the positive z-axis ($\hat{a} = \hat{z}$) and is located on the symmetry axis of the perfect conducting object at $z_0 = \delta + \lambda_2$ where δ is the depth and λ_2 is the wavelength in region 2. The refractive index is $n = 1.9$. Fig. 2 shows the differential scattering cross section $\sigma(\theta,0)$ for a perturbed sphere ($a_0/b_0 = 1.0001$) with $k_2a_0 = 3$ and $\delta = 4.5\lambda_2$. Fig. 3 shows the differential scattering cross section for a spheroidal scatterer ($a_0/b_0 = 1.5$) with $k_2a_0 = 3$ located at $\delta = 1.5\lambda_2$.

7. Concluding remarks

Wave propagation and scattering processes in a half-space environment constitute scientific areas of great importance and applicability. The complexity introduced by the medium interface gives rise to significant difficulties in the analytical and the numerical treatment of the problem. Given also that the imbedded scatterer has a complicated geometric structure, such as the spheroidal one, the problem becomes even harder.

In this work we have followed a rigorous analytical approach, avoiding any restrictive assumptions on the geometric or physical characteristics of the problem. We have implemented a systematic approach for the solution of the direct scattering problem. Numerical results have also been presented. It is noted that the numerical treatment is a rather slow process due the involvement of multiple infinite summations in the developed theoretical model, requiring a considerable computation time. The generalization of these results and the development of a parametric analysis constituting the basis of the inverse scattering problem are in due course.

Appendix A: Dyadic formulation

We consider an isotropic homogeneous medium with electric permittivity ε and magnetic permeability μ. The free-space solutions to the Maxwell's equations in dyadic form are the free-space electric and magnetic dyadic Green functions [16], given by

$$\overline{\overline{G}}_{e0}\left(\overline{r},\overline{r'}\right)=\left(\overline{\overline{I}}+\frac{1}{k^2}\nabla\nabla\right)G_0\left(\overline{r},\overline{r'}\right), \tag{A.1}$$

$$\overline{\overline{G}}_{m0}\left(\overline{r},\overline{r'}\right)=\nabla\times\left[\overline{\overline{I}}G_0\left(\overline{r},\overline{r'}\right)\right]=\nabla G_0\left(\overline{r},\overline{r'}\right)\times\overline{\overline{I}}, \tag{A.2}$$

where

$$G_0\left(\overline{r},\overline{r'}\right)=\frac{e^{ik\left|\overline{r}-\overline{r'}\right|}}{4\pi\left|\overline{r}-\overline{r'}\right|}, \tag{A.3}$$

is the free-space Green function of the scalar Helmholtz equation, $k=\omega\sqrt{\mu\varepsilon}$ is the wave number of the medium with ω standing for the angular frequency and $\overline{\overline{I}}$ is the three-dimensional identity dyadic. Using methods described in [16] we find an eigenfunction expansion for $\overline{\overline{G}}_{m0}$ through the cylindrical solenoidal vector wave functions [Appendix B], as follows

$$\overline{\overline{G}}_{m0}\left(\overline{r},\overline{r'}\right)=\int_0^{\infty}d\lambda\int_{-\infty}^{\infty}dh\sum_{m=0}^{\infty}\left\{\frac{(2-\delta_0)\kappa}{4\pi^2\lambda\left(\kappa^2-k^2\right)}\left[\overline{N}(h)\overline{M}'(-h)+\overline{M}(h)\overline{N}'(-h)\right]\right\}, \tag{A.4}$$

where $\kappa^2=\lambda^2+h^2$ with λ,h being two continuous eigenvalues and the primed functions are defined with respect to (ρ',ϕ',z') of the position vector $\overline{r'}$. Furthermore, we have used a condensed notation for the four dyadic pairs contained in the brackets of (A.4), for example $\overline{M}(h)=\overline{M}_{\substack{e\\o}m}(h)$.

For a problem involving two isotropic media separated by a flat infinite interface, the Fourier integral in (A.4) can be evaluated with the aid of the residue theorem in the h-plane. The integral representation of $\overline{\overline{G}}_{e0}$ in the upper half-space (1) is obtained using the dyadic Maxwell equation

$$\nabla\times\overline{\overline{G}}_{m0}\left(\overline{r},\overline{r'}\right)=\overline{\overline{I}}\delta\left(\overline{r}-\overline{r'}\right)+k^2\overline{\overline{G}}_{e0}\left(\overline{r},\overline{r'}\right), \tag{A.5}$$

and the dyadic boundary condition that describes the discontinuity of the magnetic dyadic Green function at $z=z'$ plane, considering that a current source is located at $\overline{r'}$ in medium (1),

$$\hat{n}\times\left(\overline{\overline{G}}_{m0}^{(1)+}\left(\overline{r},\overline{r'}\right)-\overline{\overline{G}}_{m0}^{(1)-}\left(\overline{r},\overline{r'}\right)\right)=\overline{\overline{I}}_s\,\delta(\overline{\rho}-\overline{\rho'}), \tag{A.6}$$

where $\overline{\overline{G}}_{m0}^{(1)+}$ is for $z>z'$, $\overline{\overline{G}}_{m0}^{(1)-}$ is for $z<z'$, $\overline{\overline{I}}_s$ denotes the two-dimensional identity dyadic defined by $\overline{\overline{I}}_s=\overline{\overline{I}}-\hat{z}\hat{z}$ and $\delta(\overline{\rho}-\overline{\rho'})$ is the two-dimensional delta function.

Hence, the expression for $\overline{\overline{G}}_{e0}^{(1)}$ can be written as

$$\overline{\overline{G}}_{e0}^{(1)}\left(\overline{r},\overline{r'}\right)=-\frac{1}{k_1^2}\hat{z}\hat{z}\delta\left(\overline{r}-\overline{r'}\right)+\int_0^\infty d\lambda\sum_{m=0}^{\infty}\{C^{(1)}[\overline{M}(\pm h_1)\overline{M}'(\mp h_1)+\overline{N}(\pm h_1)\overline{N}'(\mp h_1)]\},\quad z\gtrless z', \tag{A.7}$$

where the superscript in $\overline{\overline{G}}_{e0}^{(1)}$ indicates that the function is defined with respect to the propagation constant $k_1=\sqrt{\lambda^2+h_1^2}$ in medium (1) and $C^{(1)}=(2-\delta_0)/4\pi\lambda h_1$ with $\delta_0=1$ if $m=0$ and $\delta_0=0$ if $m\neq 0$. Similar results are obtained in the case of a current source located at $\overline{r'}$ in medium (2). It holds

$$\overline{\overline{G}}_{e0}^{(2)}\left(\overline{r},\overline{r'}\right)=-\frac{1}{k_2^2}\hat{z}\hat{z}\delta\left(\overline{r}-\overline{r'}\right)+\int_0^\infty d\lambda\sum_{m=0}^{\infty}\{C^{(2)}[\overline{M}(\pm h_2)\overline{M}'(\mp h_2)+\overline{N}(\pm h_2)\overline{N}'(\mp h_2)]\},\quad z\gtrless z', \tag{A.8}$$

where $k_2=\sqrt{\lambda^2+h_2^2}$ and $C^{(2)}=(2-\delta_0)/4\pi\lambda h_2$.

Appendix B: Cylindrical and spheroidal wave functions

Cylindrical geometry-cylindrical wave functions

Let us consider the scalar Helmholtz equation

$$\nabla^2\Phi + k^2\Phi = 0\,. \tag{B.1}$$

The method of separation of variables for the cylindrical coordinate system (ρ,ϕ,z) leads to the following scalar solutions

$$\Phi_{\substack{e\\o}m}(\rho,\phi,z) = J_m(\lambda\rho)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}e^{ihz}\,. \tag{B.2}$$

The corresponding solenoidal vector wave functions that satisfy the vector Helmholtz equation are given by

$$\begin{aligned}\overline{M}_{\substack{e\\o}m}(h) &= \nabla\times[\Phi_{\substack{e\\o}m}(\rho,\phi,z)\hat{z}]\\ &= \frac{1}{\rho}J_m(\lambda\rho)\begin{Bmatrix}-m\sin(m\phi)\\ m\cos(m\phi)\end{Bmatrix}e^{ihz}\hat{\rho} - \lambda J'_m(\lambda\rho)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}e^{ihz}\hat{\phi},\end{aligned} \tag{B.3}$$

$$\begin{aligned}\overline{N}_{\substack{e\\o}m}(h) &= \frac{1}{k}\nabla\times\nabla\times[\Phi_{\substack{e\\o}m}(\rho,\phi,z)\hat{z}]\\ &= \frac{ih\lambda}{k}J'_m(\lambda\rho)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}e^{ihz}\hat{\rho} + \frac{ih}{k}\frac{1}{\rho}J_m(\lambda\rho)\begin{Bmatrix}-m\sin(m\phi)\\ m\cos(m\phi)\end{Bmatrix}e^{ihz}\hat{\phi} + \frac{\lambda^2}{k}J_m(\lambda\rho)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}e^{ihz}\hat{z},\end{aligned} \tag{B.4}$$

where the prime in the functions above denotes derivative with respect to their argument and $k=\sqrt{\lambda^2+h^2}$ is the wave number.

Spheroidal geometry-spheroidal wave functions

The connection between cartesian and prolate spheroidal coordinates and the metric coefficients of the spheroidal system are given by the relations

$$\begin{pmatrix}x\\ y\end{pmatrix} = \frac{1}{2}a\sinh\mu\sin\theta\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \qquad z = \frac{1}{2}a\cosh\mu\cos\theta\,, \tag{B.5}$$

$$h_\mu = h_\theta = \frac{1}{2}a\sqrt{\cosh^2\mu - \cos^2\theta}\,, \qquad h_\phi = \frac{1}{2}a\sinh\mu\sin\theta\,, \tag{B.6}$$

where the spheroidal coordinates range over the intervals $\mu \geq 0,\ 0 \leq \theta < \pi,\ 0 \leq \phi \leq 2\pi$.

The case $\mu = 0$ corresponds to the line interval connecting the two foci of the spheroidal system located at $z = \frac{1}{2}a$ and $z = -\frac{1}{2}a$.

Applying separation of variables techniques to the scalar Helmholtz equation for the spheroidal coordinate system (μ, θ, ϕ) we conclude that

$$\Psi^{(j)}_{\substack{e\\o}mn} = R^{(j)}_{mn}(\xi;c) S_{mn}(\eta;c) \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix}, \quad \eta = \cos\theta,\ \xi = \cosh\mu. \tag{B.7}$$

It is proved [17], that the functions S, R are given by the relations

$$S_{mn}(\eta;c) = \sum_{k=0,1}^{\infty}{}' d^{mn}_k(c) P^m_{m+k}(\eta) = \begin{Bmatrix} \sum_{k=0}^{\infty} d^{mn}_{2k}(c) P^m_{m+2k}(\eta), & n = m, m+2, \ldots \\ \sum_{k=0}^{\infty} d^{mn}_{2k+1}(c) P^m_{m+2k+1}(\eta), & n = m+1, m+3, \ldots \end{Bmatrix}, \tag{B.8}$$

$$R^{(j)}_{mn}(\xi;c) = \frac{(n-m)!}{(n+m)!}\left(1 - \frac{1}{\xi^2}\right)^{m/2} \sum_{k=0,1}^{\infty}{}' i^{k+m-n} \frac{(2m+k)!}{k!} d^{mn}_k(c) Z^{(j)}_{m+k}(c\xi). \tag{B.9}$$

where four alternatives for the spherical Bessel functions $Z^{(j)}_{m+k}$ exist

$$Z^{(j)}_{m+k}(c\xi) = \begin{Bmatrix} j_{m+k}(c\xi), & j=1 \\ y_{m+k}(c\xi), & j=2 \\ h^{(1)}_{m+k}(c\xi) = j_{m+k}(c\xi) + i y_{m+k}(c\xi), & j=3 \\ h^{(2)}_{m+k}(c\xi) = j_{m+k}(c\xi) - i y_{m+k}(c\xi), & j=4 \end{Bmatrix},$$

with $c = \frac{1}{2}ka$. $P^m_n(\eta)$ are the associated Legendre functions of the first kind and the symbol $\sum_{k=0,1}^{\infty}{}'$ indicates summation over even or odd indices, depending on the starting index, while the coefficients $d^{mn}_k(c)$ satisfy suitable recurrence schemes [18] and play a vital role in the numerical treatment of the spheroidal functions.

The construction of the spheroidal vector wave functions $\overline{L}^{(j)}_{\substack{e\\o}mn}$, $\overline{M}^{(j)}_{\substack{e\\o}mn}$, $\overline{N}^{(j)}_{\substack{e\\o}mn}$, which satisfy the vector Helmholtz equation is based on the following definitions

$$\overline{L}^{(j)}_{\substack{e\\o}mn} = \nabla \Psi^{(j)}_{\substack{e\\o}mn}, \tag{B.10}$$

$$\overline{M}^{(j)}_{\substack{e\\o}mn} = \nabla \Psi^{(j)}_{\substack{e\\o}mn} \times \overline{r}, \tag{B.11}$$

$$\overline{N}^{(j)}_{\substack{e\\o}mn} = \frac{1}{k}\left(2\overline{L}^{(j)}_{\substack{e\\o}mn} + \left(\overline{r} \cdot \nabla\right)\overline{L}^{(j)}_{\substack{e\\o}mn} + k^2 \Psi^{(j)}_{\substack{e\\o}mn}\, \overline{r} \right), \tag{B.12}$$

and their representations in spheroidal coordinates are given in [19].

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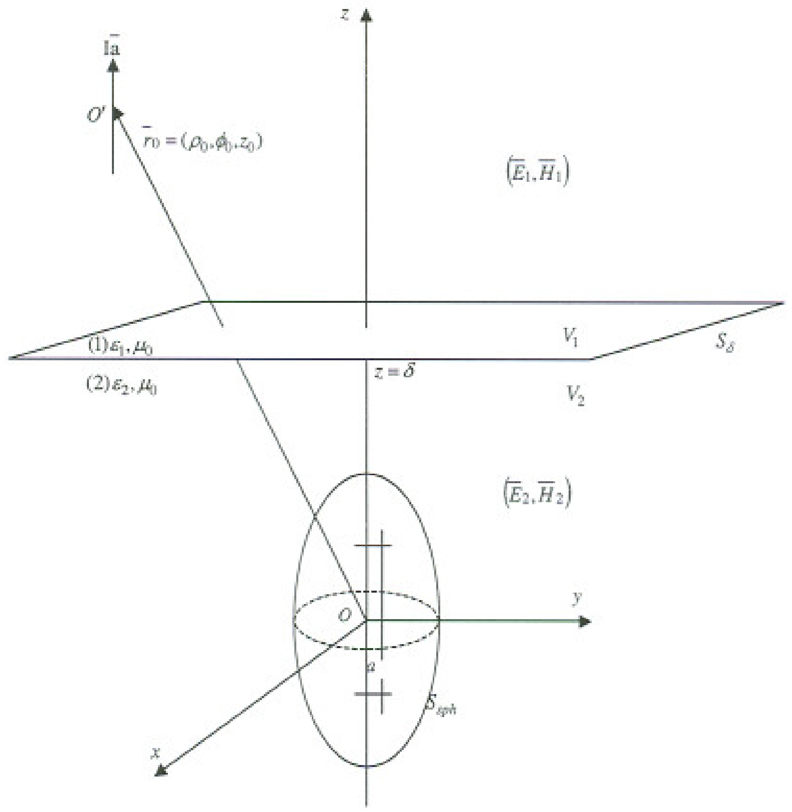

Figure 1: The problem geometry

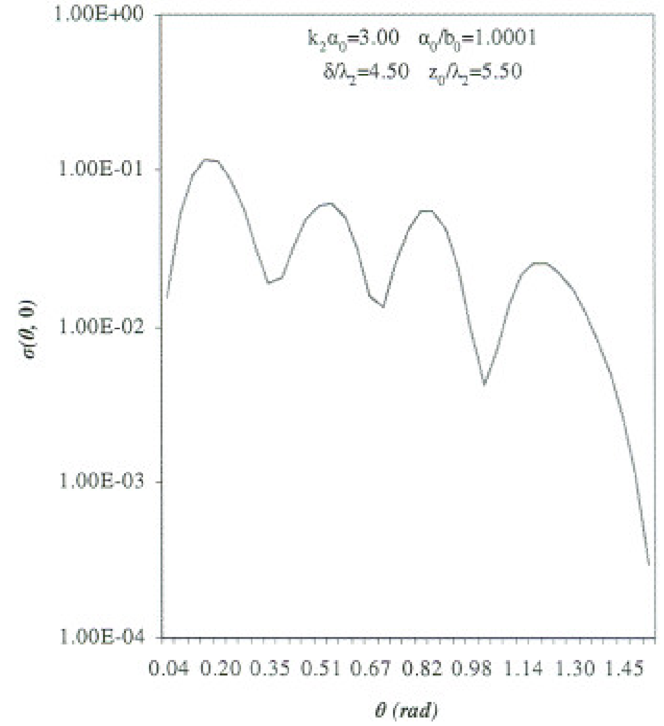

Figure 2: Differential scattering cross section $\sigma(\theta,0)$ for a perfectly conducting perturbed sphere

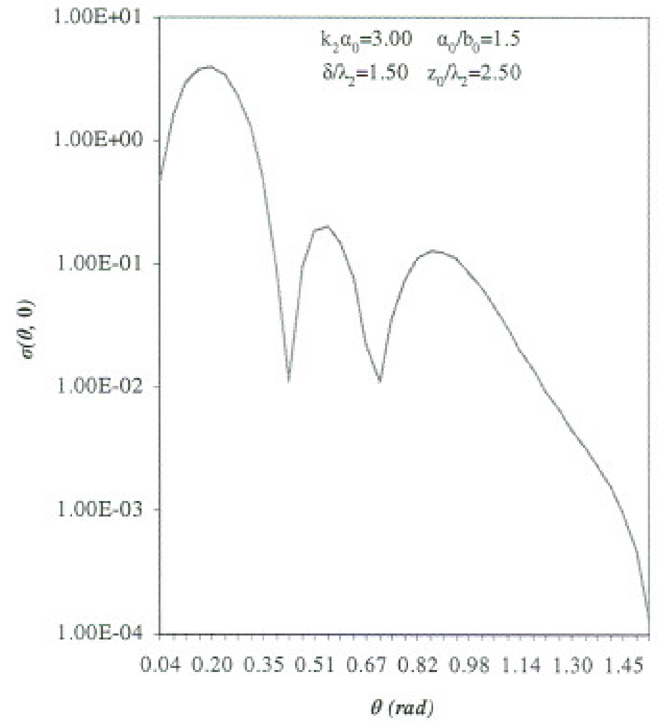

Figure 3: Differential scattering cross section $\sigma(\theta,0)$ for a perfectly conducting spheroid