THE NUMBER OF SPANNING TREES IN QUASI-THRESHOLD GRAPHS

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The Number of Spanning Trees in Quasi-threshold Graphs

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Abstract: In this paper we examine the classes of graphs whose complements are tree graphs and quasi-threshold graphs and derive formulas for their number of spanning trees. More precisely, we derive formulas for the number of spanning trees of the graph $G = K_n - H$, where H is (i) a tree graph, and (ii) a quasi-threshold graph; G is defined to be the graph which results from the complete graph K_n after removing a set of edges that span H. Our proofs are based on the complement spanning-tree matrix theorem which expresses the number of spanning trees of a graph as a function of the determinant of a matrix that can be easily construct from the adjacency relation of the graph. Our results generalize previous results and extend the family of graphs of the form $K_n - H$ having formulas regarding the number of spanning trees.

Keywords: Spanning trees, complement spanning-tree matrix theorem, tree graphs, quasi-threshold graphs, combinatorial problems, networks.

1 Introduction

We consider finite undirected graphs with no loops nor multiple edges. Let G be such a graph on n vertices. A *spanning tree* of G is an acyclic $(n-1)$-edge subgraph. The complement $\overline{G}$ of the graph G is defined to be the n-vertex graph containing exactly the edges of K_n which are not in G; K_n denotes the complete graph on n vertices.

Let S be a set of edges that join pairs of vertices in K_n. The problem of calculating the number of spanning trees on the graph G that results from K_n after removing the edges of the set S, is an important, well-studied problem in graph theory. Deriving formulas for different types of graphs spanned by a set of edges can prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical sequences related to network reliability [2, 7, 17, 22].

Thus, for both theoretical and practical purposes, we are interested in deriving formulas for the number of spanning trees of classes of graphs of the form $K_n - H$, where such a graph results from the complete graph K_n after removing a set of edges that span the graph H. Many cases have been examined depending on the choice of H. For example there exist formulas for the cases where H is a pairwise disjoint set of edges [24], when it is a star [21], when it is a complete graph [1], when it is a chain of edges, that is, a path graph [6, 16], when it is a cycle [6], when it is a multi-star [5, 20, 26], and so on (see Berge [1] for an exposition of the main results).

The purpose of this paper is to study the above problem and derive formulas regarding the number of spanning tree of the graph $G = K_n - H$ in the cases where H is (i) a *tree* graph, and (ii) a *quasi-threshold* graph. A graph H on n vertices is called a tree graph if it is a connected $(n-1)$-edge graph; H is called a quasi-threshold graph if it contains no induced subgraph isomorphic to P_4 or

C_4 [9, 19, 25]. Our proofs are based on a classic result known as the *Complement Spanning-Tree Matrix* theorem [23], which expresses the number of spanning trees of a graph G as a function of the determinant of a matrix that can be easily construct from the adjacency relation (adjacency matrix, adjacency lists, ect) of the graph G. Calculating the determinant of the complement spanning-tree matrix seems to be a promising approach for computing the number of spanning trees of families of graphs of the form $K_n - H$, where H posses an inherent symmetry (see [1, 5, 6, 20, 26]). In our cases, since neither tree graphs nor quasi-threshold graphs posse any symmetry, we focus on their structural and algorithmic properties. Indeed, both tree and quasi-threshold graphs possess properties that allow us to efficiently use the Complement Spanning-Tree Matrix Theorem; tree graphs are characterized by simple structures and quasi-threshold graphs are characterized by a unique tree representation [13, 19] (see Section 2). We compute the number of spanning trees of these graphs using standard techniques from linear algebra and matrix theory on their complement spanning-tree matrices. Our ideas and techniques will be formalized and further clarified in the sequel.

It is well-known that various classes of graphs are subclasses of the class of the tree graphs; for example, the classes of path graphs, star graphs, ice-graphs are all subclasses of the class of tree graphs (an ice-graph is obtained from a multi-star graph $K_p(b_1, b_2, \ldots, b_p)$ by setting $K_p := K_1 + \overline{K}_p$ [20]). Moreover, the class of quasi-threshold graphs contains the classes of perfect graphs known as threshold graphs and complete split (or, c-split) graphs (a graph is defined to be a c-split if there is a partition of its vertex set into a stable set S and a complete set K and every vertex in S sees all the vertices in K) [8]; we note that the quasi-threshold graphs are also perfect graphs. Thus, the results of this paper generalize previous results and extent the family of graphs of the form $K_n - H$ having formulas regarding the number of spanning trees.

The paper is organized as follows. In Section 2 we establish the notation and related terminology and we present background results. In particular, we show structural properties for the class of quasi-threshold graphs and define a unique tree representation of such graphs. In Sections 3 and 4 we present the results obtained for the graphs $K_n - T$ and $K_n - Q$, respectively, where T is a tree graph and Q is a quasi-threshold graph. Finally, in Section 5 we conclude the paper and discuss possible future extensions.

2 Definitions and Background Results

Let G be a graph on n vertices and let $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. The neighbourhood of a vertex u is the set $N(u) = N_G(u)$ consisting of all the vertices of G which are adjacent with u. The closed neighbourhood of u is defined by $N[u] = N_G[u] := \{u\} \cup N(u)$. The subgraph of a graph G induced by a subset $S \subseteq V(G)$ is denoted by $G[S]$; a subset $T \subseteq E(G)$ spans a subgraph H, where $V(H) = \{u \in V(G) \mid u \text{ is an endpoint of some edges of } T\}$ and $E(H) = T$. A *spanning tree* of G is an acyclic subgraph H of G spanned by $n-1$ edges of G.

A graph G on n vertices is called *tree graph* if it is a connected graph and has $n-1$ edges. A graph G is called a *quasi-threshold* graph, or *QT-graph* for short, if and only if G has no induced subgraph isomorphic to P_4 or C_4 [9, 15, 19, 25].

Let H be an undirected graph on p vertices and let $K_n - H$ be the graph that results from the complete graph K_n after removing a set of edges that span H; recall that K_n denotes the complete graph on n vertices. The purposes of this work is to derive formulas for the number of spanning trees of the graphs $K_n - T$ and $K_n - Q$, where T is a tree graph on k vertices and Q a quasi-threshold graph on p vertices; obviously, $k \leq n$ and $p \leq n$. To this end, we use the *Complement Spanning-Tree Matrix* theorem [23] (hereafter, *CSTM* theorem); it expresses the number of spanning trees of a graph $G = K_n - H$ on n vertices as a function of the determinant of an $n \times n$ matrix that is called *complement* *spanning-tree matrix* of G. The complement spanning-tree matrix A of a graph G is defined as follows:

$$A_{i,j} = \begin{cases} 1 - \frac{d_i}{n} & \text{if } i = j, \\ \frac{1}{n} & \text{if } i \neq j \text{ and } (i,j) \in \overline{E}, \\ 0 & \text{otherwise,} \end{cases}$$

where d_i is the number of edges incident to vertex u_i in the complement of G; that is, d_i is the degree of the vertex u_i in $\overline{G}$. It has been shown [23], that the number of spanning trees $\tau(G)$ of G is given by

$$\tau(G) = n^{n-2} \det(A).$$

In the case where $G = K_n$, we have that $\det(A) = 1$; the *Cayley's tree formula* [11] states that $\tau(K_n) = n^{n-2}$.

We next provide characterizations and structural properties of QT-graphs and show that such a graph has a unique tree representation. The following lemma follows immediately from the fact that for every subset $S \subset V(G)$ and for a vertex $u \in S$, we have $N_{G[S]}[u] = N[u] \cap S$ and that $G[V(G) - S]$ is an induced subgraph.

Lemma 2.1 ([13, 19]). *If G is a QT-graph, then for every subset $S \subset V(G)$, both $G[S]$ and $G[V(G) - S]$ are also QT-graphs.*

The following theorem provides important properties for the class of QT-graphs. For convenience, we define

$$cent(G) = \{x \in V(G) \mid N[x] = V(G)\}.$$

Theorem 2.1 ([13, 19]). *The following three statements hold.*

(i) A graph G is a QT-graph if and only if every connected induced subgraph $G[S], S \subseteq V(G)$, satisfies $cent(G[S]) \neq \emptyset$.

(ii) A graph G is a QT-graph if and only if $G[V(G) - cent(G)]$ is a QT-graph.

(iii) Let G be a connected QT-graph. If $V(G) - cent(G[S]) \neq \emptyset$, then $G[V(G) - cent(G)]$ contains at least two connected components.

Let G be a connected QT-graph. Then $V_1 := cent(G)$ is not an empty set by Theorem 2.1. Put $G_1 := G$, and $G[V(G) - V_1] = G_2 \cup G_3 \cup \ldots \cup G_r$, where each G_i is a connected component of $G[V(G) - V_1]$ and $r \geq 3$. Then since each G_i is an induced subgraph of G, G_i is also a QT-graph, and so let $V_i := cent(G_i) \neq \emptyset$ for $2 \leq i \leq r$. Since each connected component of $G_i[V(G_i) - cent(G_i)]$ is also a QT-graph, we can continue this procedure until we get an empty graph. Then we finally obtain the following partition of $V(G)$.

$$V(G) = V_1 + V_2 + \ldots + V_k, \quad \text{where } V_i = cent(G_i).$$

Moreover we can define a partial order $\leq$ on $\{V_1, V_2, \ldots, V_k\}$ as follows:

$$V_i \leq V_j \ \text{ if } \ V_i = cent(G_i) \ \text{ and } \ V_j \subseteq V(G_i).$$

It is easy to see that the above partition of $V(G)$ possesses the following properties.

Theorem 2.2 ([13, 19]). *Let G be a connected QT-graph, and let $V(G) = V_1 + V_2 + \ldots + V_k$ be the partition defined above; in particular, $V_1 := cent(G)$. Then this partition and the partially ordered set $(\{V_i\}, \leq)$ have the following properties:*

(P1) If $V_i \leq V_j$, then every vertex of V_i and every vertex of V_j are joined by an edge of G.

(P2) For every $V_j, cent(G[\{\cup V_i \mid V_i \leq V_j\}]) = V_j$.

(P3) For every two V_s and V_t such that $V_s \leq V_t, G[\{\cup V_i \mid V_s \leq V_i \leq V_t\}]$ is a complete graph. Moreover, for every maximal element V_t of $(\{V_i\}, \leq), G[\{\cup V_i \mid V_1 \leq V_i \leq V_t\}]$ is a maximal complete subgraph of G.

(P4) Every edge with both endpoints in V_i is a free edge; an edge (x, y) is called free if $N[x] = N[y]$.

(P5) Every edge with one endpoint in V_i and the other endpoint in V_j, where $V_i \neq V_j$, is a semi-free edge; an edge (x, y) is called semi-free if either $N[x] \subset N[y]$ or $N[x] \supset N[y]$.

The results of Theorem 2.2 provide structural properties for the class of QT-graphs. We shall refer to the structure that meets the properties of Theorem 2.2 as *cent-tree* of the graph G and denote it by $T_c(G)$. The cent-tree is a rooted tree with root V_1; every node V_i of the tree $T_c(G)$ is either a leaf or has at least two children. Moreover, $V_s \leq V_t$ if and only if V_s is an ancestor of V_t. Thus, we can state the following result.

Corollary 2.1. A graph G is a QT-graph if and only if G has a cent-tree $T_c(G)$.

If V_i and V_j are disjoint vertex sets of the above defined partition of the vertex set of a QT-graph G and $V_i \leq V_j$ or $V_j \leq V_i$, we say that V_i and V_j are *clique-adjacent* and denote $V_i \approx V_j$.

3 Tree Graphs

Let T be a tree graph on k vertices and let T' be a rooted tree of T rooted at vertex $r \in V$. We partition the vertex set of the graph T, with respect to the rooted tree T', in the following manner:

We set $T'_1 := T'$ and let $leaves(T'_1)$ be the set of leaves of the tree T'_1. Then $V_1 := leaves(T'_1)$ is not an empty set. We delete the leaves of the tree T'_1 and let T'_2 be the resulting tree. We set $V_2 := leaves(T'_2)$ and we continue this procedure until we get an empty tree. Then, we finally obtain the following partition of $V(T)$:

$$V(T) = V_1 + V_2 + \ldots + V_h, \text{where } V_i = leaves(T'_i),\ T'_{i+1} = T'_i - leaves(T'_i) \text{ and } T'_1 = T'.$$

We call the above defined partition *st-partition* of the tree graph T or, equivalently, *st-partition* of the rooted tree T'.

Figure 1 depicts a tree graph T on ten vertices; we can also view the graph T as a tree, say, T', rooted at vertex u_{10}. The vertex sets of the st-partition of the tree graph T, with respect to the rooted tree T', are the following: $V_1 = (u_2, u_1, u_6, u_5, u_4, u_3), V_2 = (u_7, u_8), V_3 = (u_9)$ and $V_4 = (u_{10})$.

We consider the vertex sets $V_1, V_2, \ldots, V_h$ of the st-partition of a graph T, with respect to a rooted tree T', as ordered sets; we here adopt the left-to-right ordering of the leaves of the tree T'. Notice that $V_i^{-1}(u_j)$ denotes the position of the vertex u_j in the ordered set V_i. For example, in the partition of the vertex set of the tree of Figure 1 we have $V_1^{-1}(u_6) = 3, V_2^{-1}(u_7) = 1$, etc.

We label the vertices of the graph T from 1 to k in the order that they appear in the ordered sets $V_1, V_2, \ldots, V_h$. More precisely, if l_i and l_j denote the labels of the vertices u_i and u_j, respectively, then $l_i < l_j$ if and only if either both vertices u_i and u_j belong to the same vertex set V_p and $V_p^{-1}(u_i) < V_p^{-1}(u_j)$ or vertices u_i and u_j belong to deferent vertex sets V_p and V_q, respevtively, and $p < q$. This labeling defines a vertex ordering of the tree graph T; we call it *st-labeling* of the tree graph T or, equivalently, *st-labeling* of the rooted tree T'.

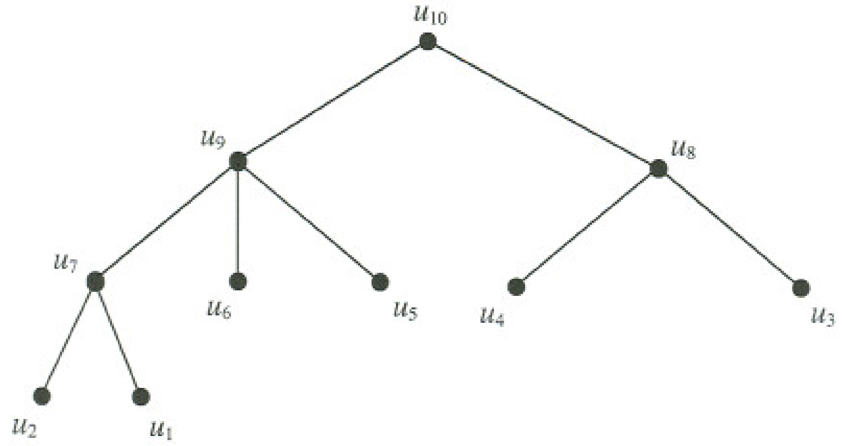

Figure 1: A tree graph T on ten vertices.

Let T be a tree graph on k vertices and let $(l_1, l_2, \ldots, l_k)$ be the labels taken by the *st*-labeling of the graph T with respect to a rooted tree T'. For every vertex u_i of T, we define the vertex set $ch(i) \subseteq V(T)$ as follows:

$$ch(i) = \{u_j \in V(T) \mid u_j \in N(u_i) \text{ and } l_i > l_j\}.$$

Hereafter, we shall also use i to denote the vertex u_i of the graph T, $1 \leq i \leq k$. We call *leaf* a vertex $i \in V(T)$, if $ch(i)$ is an empty set. Given a rooted tree T' of a tree graph T, we define the following function L on $V(T)$:

$$L(i) = \begin{cases} a_i & \text{if } i \text{ is a leaf,} \\ \\ a_i - b^2 \displaystyle\sum_{j \in ch(i)} \frac{1}{L(j)} & \text{otherwise,} \end{cases}$$

where, $a_i = 1 - d_i b$ and $b = 1/n$; recall that $n \geq k$ and d_i is the degree of the vertex i in the graph T.

We call the function L *st-function* of the tree graph T or, equivalently, *st-function* of the rooted tree T'; hereafter, we use L_i to denote $L(i)$, $1 \leq i \leq k$.

We consider the graph $G = K_n - T$, where T is a tree graph on k vertices. We first assign to each vertex of the graph G a label from 1 to n so that the vertices with degree $n-1$ obtain the smallest labels; that is, we label the vertices with degree $n-1$ from 1 to $n-k$. We label all the other vertices with degree less than $n-1$ from $n-k+1$ to n according to an *st*-labeling of the tree graph T. Notice that the vertices with degree less than $n-1$ induce the graph $\overline{T}$. Then, we form the complement spanning-tree matrix A of the graph G; it has the following form:

$$A = \begin{bmatrix} I_{n-k} & \\ & B \end{bmatrix},$$

where the submatrix B concerns those vertices of the graph $K_n - T$ that have degree less than $n-1$; throughout the paper, empty entries in matrices or determinants represent $0's$. Let

$V_1 = (u_1, u_2, \ldots, u_l)$,

$V_2 = (u_{l+1}, u_{l+2}, \ldots, u_s)$,

$$
\begin{aligned}
V_3 &= (u_{s+1}, u_{s+2}, \ldots, u_r),\\
&\ \vdots\\
V_h &= (u_k)
\end{aligned}
$$

be the vertex sets of an st-partition of the graph T; recall that the vertices $u_1, u_2, \ldots, u_k$ of $K_n - T$ have degrees less than $n-1$. Thus, B is an $k \times k$ matrix having the following structure:

$$
B = \begin{bmatrix}
a_1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & a_l & & & & & & & \\
& & & a_{l+1} & & & (b)_{j,i} & & & \\
& & & & \ddots & & & & & \\
& & & & & a_s & & & & \\
& & & & & & a_{s+1} & & & \\
& & & (b)_{i,j} & & & & \ddots & & \\
& & & & & & & & a_r & \\
& & & & & & & & & \ddots \\
& & & & & & & & & & a_k
\end{bmatrix}, \tag{1}
$$

where, according to the definition of the complement spanning-tree matrix, $a_i = 1 - d_i b$, and the entry $(b)_{i,j}$ of the off-diagonal position (i,j) is b if $j \in ch(i)$ and 0 otherwise, $1 \leq i,\ j \leq k$. Note that $b = 1/n$ and d_i is the degree of the vertex i in the graph T.

Starting from the upper left part of the matrix, the first l rows of the matrix correspond to the l vertices of the set V_1; the next $r - l$ rows correspond to the vertices of the set V_2, and so forth. The last row corresponds to the root of the tree T'. For example, when we span the tree graph of Figure 1 from a complete graph K_n the matrix B has the following form:

$$
B = \begin{bmatrix}
a_2 & & & & & & b & & & \\
& a_1 & & & & & b & & & \\
& & a_6 & & & & & & b & \\
& & & a_5 & & & & & b & \\
& & & & a_4 & & & b & & \\
& & & & & a_3 & & b & & \\
b & b & & & & & a_7 & & b & \\
& & & & b & b & & a_8 & & b \\
& & b & b & & & b & & a_9 & b \\
& & & & & & & b & b & a_{10}
\end{bmatrix},
$$

where $a_2 = a_1 = a_6 = a_5 = a_4 = a_3 = 1 - b$, $a_7 = a_8 = 1 - 3b$, $a_9 = 1 - 4b$ and $a_{10} = 1 - 2b$, and $b = 1/n$.

From the matrix A, we obtain that $\det(A) = \det(B)$. Thus, we focus on the computation of the determinant of matrix B.

In order to compute the determinant $\det(B)$, we start by multiplying each column i, $1 \leq i \leq l$, of the matrix B by $-b/a_i$ and adding it to the column j if $(b)_{i,j} = b$ $(i < j \leq k)$. This, makes all the strictly upper-diagonal entries $(b)_{i,j}$, that is, $i < j$, into zeros. Now expand in terms of the $1, 2, \ldots, l$ rows, getting

$$\det(B) = \prod_{i=1}^{l} L_i \begin{vmatrix} f_{l+1}^l & & & & & & & \\ & \ddots & & & & & & \\ & & f_s^l & & & (b)_{j,i} & & \\ & & & f_{s+1}^l & & & & \\ & & & & \ddots & & & \\ & & (b)_{i,j} & & & f_r^l & & \\ & & & & & & \ddots & \\ & & & & & & & f_k^l \end{vmatrix} = \prod_{i=1}^{l} L_i \; det(B'),$$

where

$L_i = a_i$, for $1 \le i \le l$, since the vertices $1, 2, \ldots, l$ are leaves of T, and

$$f_t^l = a_t - b^2 \sum_{\substack{i \in ch(t) \\ 1 \le i \le l}} \frac{1}{L_i}, \quad \text{for } l+1 \le t \le k.$$

We observe that the $(k-l) \times (k-l)$ matrix B' has structure similar to that of the initial matrix B; see Eq. (1). Thus, for the computation of its determinant $\det(B')$, we follow a similar simplification; that is, we start by multiplying each column i, $1 \le i \le s$, of the matrix B' by $-b/f_j^l$ and adding it to the column j if $(b)_{i,j} = b$ $(s < j \le k)$. Then, we obtain:

$$\det(B) = \prod_{i=1}^{l} L_i \prod_{i=l+1}^{s} L_i \begin{vmatrix} f_{s+1}^s & & & & \\ & \ddots & & (b)_{j,i} & \\ & & f_r^s & & \\ & (b)_{i,j} & & \ddots & \\ & & & & f_k^s \end{vmatrix} = \prod_{i=1}^{s} L_i \; det(B''),$$

where

$L_i = f_i^l$, for $l+1 \le i \le s$, and

$$f_t^s = a_t - b^2 \sum_{\substack{i \in ch(t) \\ 1 \le i \le s}} \frac{1}{L_i}, \quad \text{for } s+1 \le t \le k.$$

The matrix B'' has also structure similar to that of the initial matrix B; see Eq. (1). It alters only on the smaller size and on the diagonal values. Thus, continuing in the same fashion we can finally show that

$$\det(B) = \prod_{i=1}^{k} L_i,$$

where L is an st-function of the tree graph T and k is the number of vertices of T.

Thus, based on the formula that gives the number $\tau(G)$ of the spanning trees of the graph $G = K_n - T$ and the fact that $\det(A) = \det(B)$, we obtain the following result.

Theorem 3.1. *Let T be a tree graph on k vertices and let L be an st-function on $V(T)$. The number of spanning trees of the graph $G = K_n - T$ is equal to*

$$\tau(G) = n^{n-2} \prod_{i=1}^{k} L_i,$$

where K_n is the complete graph on n vertices and $n \ge k$.

Remark 3.1. We point out that Theorem 3.1 provides a simple linear-time algorithm for computing the number of spanning trees of the graph $G = K_n - T$, where T is a tree graph on k vertices, $k \le n$.

4 Quasi-threshold Graphs

In this section, we derive a formula for the number of the spanning trees of the graph $K_n - Q$, where Q is a quasi-threshold graph, using the work of the preceding section as motivation.

Let Q be a QT-graph on p vertices and let $V_1, V_2, \ldots, V_k$ be the nodes of its cent-tree $T_c(Q)$ containing $p_1, p_2, \ldots, p_k$ vertices, respectively. We denote d_i the degree of an arbitrary vertex of the node V_i. Recall that all the vertices $u \in V(Q)$ of a node V_i have the same degree. In Figure 3 we show a cent-tree of a QT-graph on 12 vertices. Nodes V_3 and V_{10} contain two vertices, while all the other contain one vertex. The degree of a vertex in node V_3 is 4.

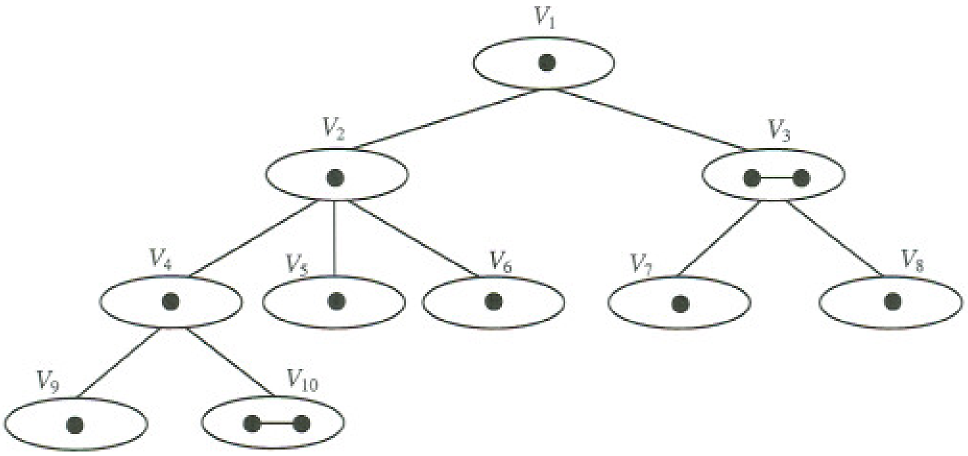

Figure 2: A cent-tree $T_c(Q)$ of a QT-graph on 12 vertices.

We next form the submatrix B of the complement spanning-tree matrix A for the graph $K_n - Q$ based on the structure of the cent-tree $T_c(Q)$, as well as on the st-labeling of $T_c(Q)$.

Let $l_1, l_2, \ldots, l_k$ be the st-labels of the nodes $V_{\pi(1)}, V_{\pi(2)}, \ldots, V_{\pi(k)}$, respectively, of the cent-tree $T_c(Q)$. Then, we label the vertices of the graph Q from $n - p + 1$ to n as follows: First, we label the vertices in $V_{\pi(1)}$ from $(n-p)+1$ to $(n-p)+p_1$; next, we label the vertices in $V_{\pi(2)}$ from $(n-p)+p_1+1$ to $(n-p)+p_1+p_2$; finally, we label the vertices in $V_{\pi(k)}$. For example, in the QT-graph with cent-tree $T_c(Q)$ that of Figure 2, we have $\pi = (9, 10, 5, 6, 7, 8, 4, 3, 2, 1)$.

Thus, based on the above labelling of the vertices of the QT-graph Q, we can easily construct the matrix B of the graph $K_n - Q$; it is an $p \times p$ matrix and has the following form:

$$B = \begin{bmatrix} M_1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & M_l & & & & & & & & \\ & & & M_{l+1} & & & [b]_{j,i} & & & & \\ & & & & \ddots & & & & & & \\ & & & & & M_s & & & & & \\ & & & & & & M_{s+1} & & & & \\ & & & [b]_{i,j} & & & & \ddots & & & \\ & & & & & & & & M_r & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & M_k \end{bmatrix}, \tag{2}$$

where M_i is an $p_i \times p_i$ submatrix of the form

$$M_i = \begin{bmatrix} a_i & b & \cdots & b \\ b & a_i & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a_i \end{bmatrix}$$

and the entry $[b]_{i,j}$ of the off-diagonal position (i, j) corresponds to an $p_i \times p_j$ submatrix with all its elements $b's$ if node V_j is a descendant of node V_i in $T_c(Q)$ and $0's$ otherwise, $1 \leq i,\ j \leq k$. Recall that $a_i = 1 - d_i b$, where d_i is the degree of an arbitrary vertex in node V_i of $T_c(Q)$, and $b = 1/n$.

For example, the matrix B of the graph $K_n - Q$, where Q is the QT-graph with cent-tree $T_c(Q)$ that of Figure 2, has the following form:

$$B = \begin{bmatrix} a_9 & & & & & & & b & & & b & b \\ & a_{10} & b & & & & & b & & & b & b \\ & b & a_{10} & & & & & b & & & b & b \\ & & & a_5 & & & & & & & b & b \\ & & & & a_6 & & & & & & b & b \\ & & & & & a_7 & & & b & b & & b \\ & & & & & & a_8 & & b & b & & b \\ b & b & b & & & & & a_4 & & & b & b \\ & & & & & b & b & & a_3 & b & & b \\ & & & & & b & b & & b & a_3 & & b \\ b & b & b & b & b & & & b & & & a_2 & b \\ b & b & b & b & b & b & b & b & b & b & b & a_1 \end{bmatrix}.$$

In order to compute the determinant of the matrix B we will first simplify the determinants of the matrices M_i, $1 \leq i \leq k$. To this end, we multiply the row of the matrix B which corresponds to the first row of the matrix M_i by -1 and add it to rows of B which correspond to rows $2, 3, \ldots, p_i$ of M_i. Then, the determinant of matrix M_i becomes:

$$\det(M_i) = \begin{vmatrix} a_i & b & \cdots & b \\ -(a_i - b) & a_i - b & & \\ \vdots & & \ddots & \\ -(a_i - b) & & & a_i - b \end{vmatrix}.$$

Now we multiply the column of B which corresponds to the last column p_i of M_i by -1 and add it to the columns of B which correspond to columns $1, 2, \ldots, p_i - 1$ of M_i. Consequently, we add the columns of the matrix B which correspond to the columns $2, 3, \ldots, p_i - 1$ of M_i to the column of B which corresponds to the first column of M_i. Then, we obtain:

$$\det(M_i) = \begin{vmatrix} a_i - b & & & b \\ & a_i - b & & \\ & & \ddots & \\ -p_i(a_i - b) & -(a_i - b) & & a_i - b \end{vmatrix}.$$

In order to simplify the determinant of matrix M_i we multiply the column of B which corresponds to the first column of M_i by $-b/(a_i - b)$ and add it to the column of B which correspond to (i) the last column of the matrix M_i, and (ii) the last column of the matrix M_j if node V_j is a descendant of node V_i in $T_c(Q)$, where $i + 1 \leq j \leq k$. Then, we obtain:

$$
\det(M_i) = \begin{vmatrix} a_i - b & & & \\ & a_i - b & & \\ & & \ddots & \\ -p_i(a_i-b) & -(a_i-b) & & a_i - b + p_i b \end{vmatrix} = (a_i - b)^{p_i - 1}(a_i - (1-p_i)b).
$$

It now suffices to substitute the above value in the determinant of matrix B. We point out that after simplifying the determinant of matrices M_i only the diagonal and the last row of each matrix M_i have non-zero's entries; the diagonal has non-zero's entries since $d_i < n-1$. Thus, we have:

$$
\det(B) = \prod_{i=1}^{k} p_i (a_i - b)^{p_i - 1} \det(D), \tag{3}
$$

where

$$
D = \begin{bmatrix} \sigma_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \sigma_l & & & & & & & \\ & & & \sigma_{l+1} & & & (b)_{j,i} & & & \\ & & & & \ddots & & & & & \\ & & & & & \sigma_s & & & & \\ & & & & & & \sigma_{s+1} & & & \\ & & & (b)_{i,j} & & & & \ddots & & \\ & & & & & & & & \sigma_r & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & \sigma_k \end{bmatrix} \tag{4}
$$

is an $k \times k$ matrix with diagonal elements $\sigma_i = \frac{a_i - (1-p_i)b}{p_i}$, $1 \le i \le k$, and the entry $(b)_{i,j}$ of the off-diagonal position (i,j) is b if node V_j is a descendant of node V_i in $T_c(Q)$ and 0 otherwise.

We observe that if we set $p_i = 1$ in matrix D, $1 \le i \le k$, then D is equal to the submatrix B of the graph $K_n - Q$, where Q is a graph of a special type; it is a QT-graph on k vertices possessing the property that each node of its cent-tree $T_c(Q)$ contains a single vertex; see Figure 3. The matrix D of the QT-graph of Figure 3 is the following:

$$
D = \begin{bmatrix} a_2 & & & & & & b & & b & b \\ & a_1 & & & & & b & & b & b \\ & & a_6 & & & & & & b & b \\ & & & a_5 & & & & & b & b \\ & & & & a_4 & & & b & & b \\ & & & & & a_3 & & b & & b \\ b & b & & & & & a_7 & & b & b \\ & & & & b & b & & a_8 & & b \\ b & b & b & b & & & b & & a_9 & b \\ b & b & b & b & b & b & b & b & b & a_{10} \end{bmatrix}.
$$

It is easy to see that, if we form the submatrix B of the complement spanning-tree matrix A of the graph $K_n - Q$, where Q is the QT-graph of Figure 3, using an appropriate vertex labeling, that is, $l_2 = n-9$, $l_1 = n-8$, $\ldots$, $l_{10} = n$, then we obtain $D = B$.

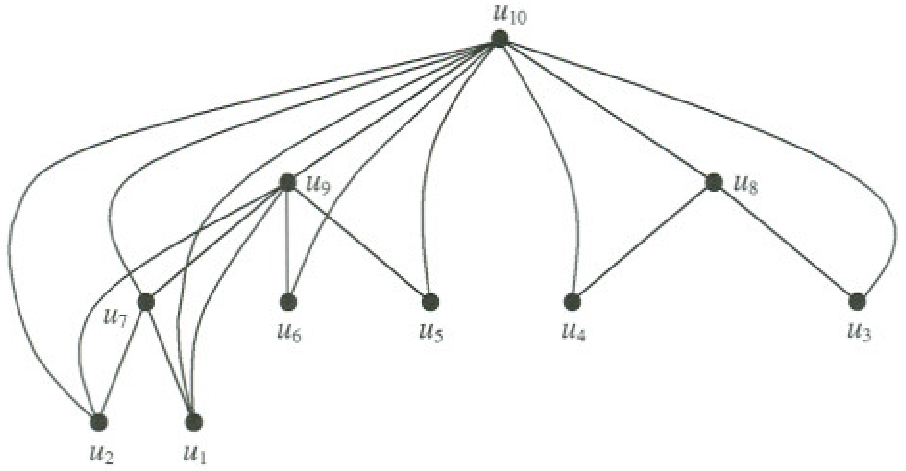

Figure 3: A QT-graph Q on 10 vertices. Every node V_i of the cent-tree $T_c(Q)$ contains exactly one vertex.

The idea now is to transform the $k \times k$ matrix D into a form similar to that of the $k \times k$ matrix B of a tree graph T on k vertices; see Eq. (1) in Section 3.

To this end, we first apply the following operations to each row $i = 1, 2, \ldots, k$ of the matrix D:

- We find the minimum index j such that $i < j \leq k$ and $D[i, j] \neq 0$, and then
- we multiply the jth column by -1 and add it to the lth column, if $D[i, l] = D[i, j]$ and $j + 1 \leq l \leq k$.

Next, we apply similar operations to each column $j = 1, 2, \ldots, k$ of the matrix D:

- We find the minimum index i such that $1 \leq j < i$ and $D[i, j] \neq 0$, and then
- we multiply the ith row by -1 and add it to to the lth row, if $D[l, j] = D[i, j]$ and $i + 1 \leq l \leq k$.

Thus, we obtain:

$$\det(D) = \begin{vmatrix} a'_1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & a'_l & & & (b'_j)_{j,i} & & & & & \\ & & & a'_{l+1} & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & a'_s & & & & & \\ & & (b'_j)_{i,j} & & & & a'_{s+1} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & a'_r & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & a'_k \end{vmatrix},$$

where

$$a'_i = \begin{cases} \sigma_i & \text{if } V_i \text{ is a leaf of } T_c(Q), \\ \sigma_i + \sum\limits_{\substack{j \in ch(i) \\ i+1 \leq j \leq k}} (\sigma_j - 2b) & \text{otherwise,} \end{cases} \tag{5}$$

and

$$b'_i = \begin{cases} b & \text{if } V_i \text{ is a leaf of } T_c(Q), \\ b - \sigma_i & \text{otherwise.} \end{cases} \tag{6}$$

.

Recall that $\sigma_i = \frac{a_i-(1-p_i)b}{p_i}$; in the case where each node of the cent tree $T_c(Q)$ contains a single vertex, we have $\sigma_i = a_i$ (in this case $p_i = 1$, for every $i = 1, 2, \ldots, k$).

It is easy to see that the structure of the resulting $k \times k$ matrix D is similar to that of the the $k \times k$ matrix B of a tree graph; see Eq. (1) in Section 3. Thus, for the computation of the determinant $\det(D)$, we can use similar techniques.

We next define the following function ϕ on the nodes on the cent-tree of a QT-graph Q:

$$\phi(i) = \begin{cases} a'_i & \text{if } i \in V_i \text{ and } V_i \text{ is a leaf of } T_c(Q), \\ a'_i - \sum\limits_{j \in ch(i)} \dfrac{(b'_j)^2}{\phi(j)} & \text{otherwise,} \end{cases}$$

where a'_i and b'_i are defined in Eq. (5) and Eq. (6), respectively. We call the function ϕ *cent-function* of the graph Q or, equivalently, *cent-function* of the cent-tree $T_c(Q)$; hereafter, we use ϕ_i to denote $\phi(i)$, $1 \leq i \leq k$.

Following the same elimination schemes as that for the computation of the determinant of the matrix B in Section 3, we obtain that

$$\det(D) = \prod_{i=1}^{k} \phi_i. \tag{7}$$

Now we are in a position to prove the following theorem.

Theorem 4.1. *Let Q be a quasi-threshold graph on p vertices and let $V_1, V_2, \ldots, V_k$ be the nodes of the cent-tree of Q. Let ϕ be the cent-function of the graph Q. Then, the number of spanning trees of the graph $G = K_n - Q$ is equal to*

$$\tau(G) = n^{n+k-p-2} \prod_{i=1}^{k} p_i (n - d_i - 1)^{p_i - 1} \phi_i,$$

where p_i is the number of vertices of the node V_i and d_i is the degree of an arbitrary vertex in node V_i, $1 \leq i \leq k$.

Proof. As mentioned in Section 3, the complement spanning-tree matrix A of a graph $K_n - Q$ can be represented by

$$A = \begin{bmatrix} I_{n-p} & \\ & B \end{bmatrix},$$

where the submatrix B concerns those vertices of the graph $K_n - Q$ that have degree less than $n-1$; these vertices induce the graph $\overline{Q}$. Since $a_i = 1 - d_i b$ and $b = 1/n$, from Eq. (3) we have

$$\det(B) = n^{k-p} \prod_{i=1}^{k} p_i(n - d_i - 1)^{p_i - 1} \det(D).$$

From the above equality and Eq. (7), we obtain:

$$\det(B) = n^{k-p} \prod_{i=1}^{k} p_i(n - d_i - 1)^{p_i - 1} \phi_i.$$

Consequently, the number of spanning trees $\tau(G)$ of the graph G is equal to $n^{n-2} \det(A)$. Thus, since $\det(A) = \det(B)$, the theorem follows. □

Remark 4.1. As mentioned in Introduction, the class of quasi-threshold graphs contains the class of c-split graphs (complete split graphs); recall that a graph is defined to be a c-split if there is a partition of its vertex set into a stable set S and a complete set K and every vertex in S sees all the vertices in K) [8].

Thus, the cent-tree of a c-split H consists of $|S| + 1$ nodes $V_1, V_2, \dots, V_{|S|+1}$ such that $V_1 = K$ and the nodes $V_2, V_3, \dots, V_{|S|+1}$ are children of the root V_1; each child contains exactly one vertex $u \in S$.

Let H be a c-split graph on p vertices and let $V(H) = K + S$ be the partition of its vertex set. Then, by Theorem 4.1, we obtain that the number of spanning trees of the graph $G = K_n - H$ is given by the following close formula:

$$\tau(G) = n^{n-p-1}(n - |K|)^{|S|-1}(n-p)^{|K|},$$

where $p = |K| + |S|$ and $p \leq n$.

5 Concluding Remarks

In this paper we derived formulas regarding the number of spanning trees of the classes of graphs of the form $K_n - H$, where H is (i) a tree graph, and (ii) a quasi-threshold graph. We took advantage of the structural properties of these graphs and used the Complement Spanning-Tree Matrix (CSTM) theorem as a tool for deriving the proposed formulas.

The results of this paper generalized previously known results; path graphs, star graphs, ice-graphs are all special cases of tree graphs, and extend the family of graphs of the form $K_n - H$ having formulas regarding the number of spanning trees; the classes of tree graphs, quasi-threshold graphs, threshold graphs and c-split graphs are now members of this family; for the class of c-split graphs, see Remark 4.1.

It is well-known that the classes of threshold and quasi-threshold graphs are perfect graphs. Thus, it is reasonable to ask whether the CSTM theorem can be efficiently used for deriving formulas, regarding the number of spanning trees, for other classes of perfect graphs such as the class of cographs [14]. We note that cographs contain the class of quasi-threshold graphs and they also have a unique tree representation. We pose it as an open problem.

A more general class of perfect graphs is that of permutation graphs; an undirected graph G is a permutation graph if there exists a permutation π on $N_n = \{1, 2, \ldots, n\}$ such that G is isomorphic to the inversion graph $G[\pi]$: $V(G[\pi]) = N_n$ and $(i, j) \in E(G[\pi])$ iff $(i-j)(\pi_i^{-1} - \pi_j^{-1}) < 0$, where π_i^{-1} is the index of the element i in π [8, 18]. It has been shown that a permutation graph $G[\pi]$ can be transform into a directed acyclic graph and, then, into a rooted tree by exploiting the inversion relation on the elements of π [18]. Based on the results of this paper, it is interesting to investigate whether the class of graphs of the form $K_n - G[\pi]$ belong to the family of graphs that have formulas regarding the number of spanning trees.

In closing, we point out that calculating the determinant of the complement spanning-tree matrix seems to be a promising approach for computing the number of spanning trees of families of graphs of the form $K_n - H$, where H posses an inherent symmetry (see [1, 5, 6, 20, 26]). We note that there have been also developed other methods for determining the number of spanning trees and deriving formulas in more general classes of graphs; see, for example [3, 4, 10, 12, 27, 28].

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