HUMAN HEAD INTERACTION WITH MOBILE PHONES: THE SPHEROIDAL HEAD MODEL

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1. SUMMARY

In the present work an attempt is made to compute the amount and distribution of the absorbed power inside the human head system. We consider the case that the human head system is simulated by a multi-layered confocal spheroidal structure while electromagnetic excitation is realised through a localised point source emanating spherical wave. The solution of the problem constitutes the Green's function of the general problem and can be exploited through a straightforward manner.

2. INTRODUCTION

The investigation of the mechanism describing the interaction of electromagnetic waves with specific parts of the human body constitutes a scientific area of great interest. Several researchers have addressed the biological effects of the electromagnetic radiation to human tissues and organs. Particularly, the necessity to study the interference of electromagnetic waves with the human head has been stimulated, now days, drastically by the use of handheld transceivers for mobile communications. Rest of the work focuses on the computation of distribution of the absorbed power inside the head system to reconsider and ameliorate the characteristics of electromagnetic emission.

The interference of electromagnetic waves with the human head, in the framework of localized source exposure, constitutes a very complicated scattering problem. Complexity is first due to the fact that the human head system disposes complex geometrical and physiological properties. On the other hand, the consideration of antenna emission goes out of the usual plane wave excitation regime, rendering the scattering problem more difficult. Several papers provide with numerical solutions of the problem. Morgan [1] applied finite element techniques to determine the distribution of the specific absorption rate in the human head. In Ref. [2-6], finite difference time domain methods have been applied.

However, the development of analytical techniques handling the problem seems necessary and indispensable not only as benchmarks for numerical solutions but in order to form a structured and hierarchical knowledge about the specific area. Analytical techniques are expected to deal with simpler models simulating the realistic one. We mention here the case of the concentric layered spherical structure model considered by Shapiro et al. [7] as well as the case of the layered eccentric spheres model for the head presented in [10].

In this paper, the human head system is simulated by a multilayer confocal spheroidal structure while the electromagnetic excitation is realized through a localized point source emanating spherical waves. Actually, the solution of the problem constitutes the Green's function of the general problem and can be exploited through a straightforward manner. The consideration of the spheroidal model is realistic enough, given that only the human skull, is proven to form a slightly perturbed spheroidal shell with axes ratio almost 0.75.

The model, developed in this paper, permits an arbitrary number of layers. The investigation of the scattering problem under discussion is based on the suitable use of Navier vector eigenfunctions, which constitute a function basis for the electromagnetic field. More precisely, the incident and scattered fields are expressed as finite expansions of the vector spheroidal wave functions. These fields are forced to satisfy the boundary and impedance conditions on the discontinuity surfaces, imposed by Maxwell's equations. In order to obtain algebraic systems concerning the expansion coefficients, we need to handle the boundary conditions for the basis eigenfunctions. This is accomplished, instead of using the classical T-matrix theory through a theoretical approach. This approach assumes the optimal treatment of vector spheroidal wave functions as it minimizes the analytical and numerical burden by suitable differential and integral transformations. The non-homogeneous algebraic system obtained through the above analysis is solved numerically after suitable truncation and stability analysis. There exists a plethora of parameters entering the problem, as the position and orientation of the point source, the relevant position of the source and the scatterer, as well as the relevant geometrical features of the spheroidal structure itself.

3. THE MODEL

3.1 Spheroidal Geometry

The geometrical system fitting with the investigated structure is the spheroidal one sharing the property to describe with simplicity all configurations lacking symmetry in only one direction. The prolate spheroidal coordinates are connected with the Cartesian ones through the relations

$$\begin{aligned} x &= \alpha \sinh\mu \sin\theta \cos\phi, \\ y &= \alpha \sinh\mu \sin\theta \sin\phi, \\ z &= \alpha \cosh\mu \cos\theta, \end{aligned} \tag{1}$$

where $\mu \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi$.

In the above relation α stands for the semi distance of the foci of the coaxial spheroidal structure. The unit spheroidal vectors, which play a crucial role in our analysis, are:

$$\hat{\mu} = \frac{\hat{\mathbf{x}}\cosh\mu\sin\theta\sin\phi + \hat{\mathbf{y}}\cosh\mu\sin\theta\sin\phi + \hat{\mathbf{z}}\sinh\mu\cos\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}} ,$$
$$\hat{\theta} = \frac{\hat{\mathbf{x}}\sinh\mu\cos\theta\sin\phi + \hat{\mathbf{y}}\sinh\mu\cos\theta\sin\phi - \hat{\mathbf{z}}\cosh\mu\sin\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}} , \quad (2)$$
$$\hat{\phi} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi .$$

The position vector can then be expressed as

$$\mathbf{r} = \hat{\mu}\frac{\alpha\sinh\mu\cosh\mu}{\sqrt{\cosh^2\mu - \cos^2\theta}} - \hat{\theta}\frac{\alpha\sin\theta\cos\theta}{\sqrt{\cosh^2\mu - \cos^2\theta}} , \quad (3)$$

while the grad operator, appearing in the forthcoming differential calculus obtains the following form:

$$\nabla = \hat{\mu}\frac{1}{\alpha\sqrt{\cosh^2\mu - \cos^2\theta}}\frac{\partial}{\partial\mu} + \hat{\theta}\frac{1}{\alpha\sqrt{\cosh^2\mu - \cos^2\theta}}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{\alpha\sinh\mu\sin\theta}\frac{\partial}{\partial\phi} . \quad (4)$$

The scale factor $\sqrt{\cosh^2\mu - \cos^2\theta}$ has made already its appearance and is going to constitute the major complicator factor of our analysis, as it implicates in a non-separable way the spheroidal coordinates μ and θ.

3.2 The Scalar Hemholtz Equation

The investigated scattering problem involves time-harmonic propagating waves. Although the electromagnetic field is a vector function, its study can be based on the investigation of the corresponding scalar waves, as it will be apparent in the sequel. After suppressing the harmonic time-dependence, every scalar wave satisfies the Helmholtz equation, given by

$$\nabla^2\psi + k^2\psi = 0 , \quad (5)$$

where k is the wave number for the propagating process. Expressing ∇^2 in spheroidal coordinates and applying separation of variable techniques by requiring

$$\psi = R(\xi)S(\eta)\Phi(\phi),\ \xi = \cosh\mu,\ \eta = \cos\theta,$$

we obtain that

$$\Phi(\phi) \in \{\cos(m\phi),\ \sin(m\phi),\ m = 0,1,2,\ldots\},$$
$$S_{mn}(\eta;c) = \sum_{k=0,1}^{\infty}{}' d_k^{mn}(c)P_{m+k}^m(\eta) = \begin{cases} \sum_{k=0}^{\infty} d_{2k}^{mn}P_{m+2k}^m(\eta),\ n = m, m+2,\ldots \\ \sum_{k=0}^{\infty} d_{2k+1}^{mn}P_{m+2k+1}^m(\eta),\ n = m+1, m+3,\ldots \end{cases}, \quad (6)$$
$$R_{mn}^{(p)}(\xi;c) = \frac{(m+n)!}{(n-m)!}\left(1-\frac{1}{\xi^2}\right)^{\frac{m}{2}} \sum_{k=0,1}^{\infty}{}' i^{k+m-n}\frac{(2m+k)!}{k!}d_{2k}^{mn}(c)Z_{m+k}^{(p)}(c\xi),$$

where P_n^m stands for the Legendre functions, while $Z_n^{(p)}$ runs over four alternatives of spherical Bessel functions, as follows

$$\begin{aligned} Z_n^{(1)} &= j_n(z), \\ Z_n^{(2)} &= y_n(z), \\ Z_n^{(3)} &= h_n^{(1)}(z) = j_n(z) + iy_n(z), \\ Z_n^{(4)} &= h_n^{(2)}(z) = j_n(z) - iy_n(z), \end{aligned} \tag{7}$$

where we encounter the spherical Hankel functions $h_n^{(1)}(z), h_n^{(2)}(z)$ of the first and the second kind, respectively.

4. FORMULATION OF THE PROBLEM

Let us consider a multi-layered spheroidal structure, every layer of which constitutes a linear, homogeneous, isotropic and non-conductive electromagnetic propagation medium. Every component of the human head (e.g. skull, brain, cerebrospinal fluid, etc.) is completely characterized by its electrical permittivity ε and its magnetic permeability μ, as far as the electromagnetic properties are concerned. The interfaces of this structure constitute coordinate spheroidal surfaces of the same spheroidal coordinate system whose center coincides with the structure center and which is totally determined once the distance 2α of the focii is given. This structure is surrounded by an infinite homogenous and non-conductive medium. The aforementioned configuration is shown in Fig. 1. The spheroidal body is located in the near field region of an antenna emanating spherical electromagnetic waves in the surrounding space. Actually the electric current density of the source is given by

$$\mathbf{J} = I\, l\, \delta(\mathbf{r} - \mathbf{r}')\hat{\gamma}, \tag{8}$$

where $\mathbf{r}' = (\alpha \sinh \mu' \sin\theta' \cos\phi',\ \alpha \sinh \mu' \sin\theta' \sin\phi',\ \alpha \cosh \mu' \cos\theta')$ is the position of the source, I the current density, l its length and $\hat{\gamma}$ stands for the unit vector denoting its orientation. Notice that the pilot character of point source (8), since it constitutes the Dirac stimulation and every other excitation is handled via superposition principle. The electromagnetic field, $\mathbf{E}^{inc}$, $\mathbf{H}^{inc}$ of the point source are considered as the incident field for the spheroidal structure and their interference leads to the emanation of the scattered electromagnetic field $\mathbf{E}^{sc}$, $\mathbf{H}^{sc}$ propagating in the exterior space, as well as the creation of a plethora of stationary waves oscillating in the several components of the structure. Each one of these interior waves is labeled by the corresponding region symbol while in the exterior region the total field is the superposition of the incident and the scattered field as

$$\begin{aligned} \mathbf{E}_{L+1} &= \mathbf{E}^{inc} + \mathbf{E}^{sc}, \\ \mathbf{H}_{L+1} &= \mathbf{H}^{inc} + \mathbf{H}^{sc}. \end{aligned} \tag{9}$$

In every subregion, the electromagnetic field satisfies Maxwell's equations, which under the assumption of time-harmonic dependence, with frequency ω, $e^{-j\omega t}$,obtain the form

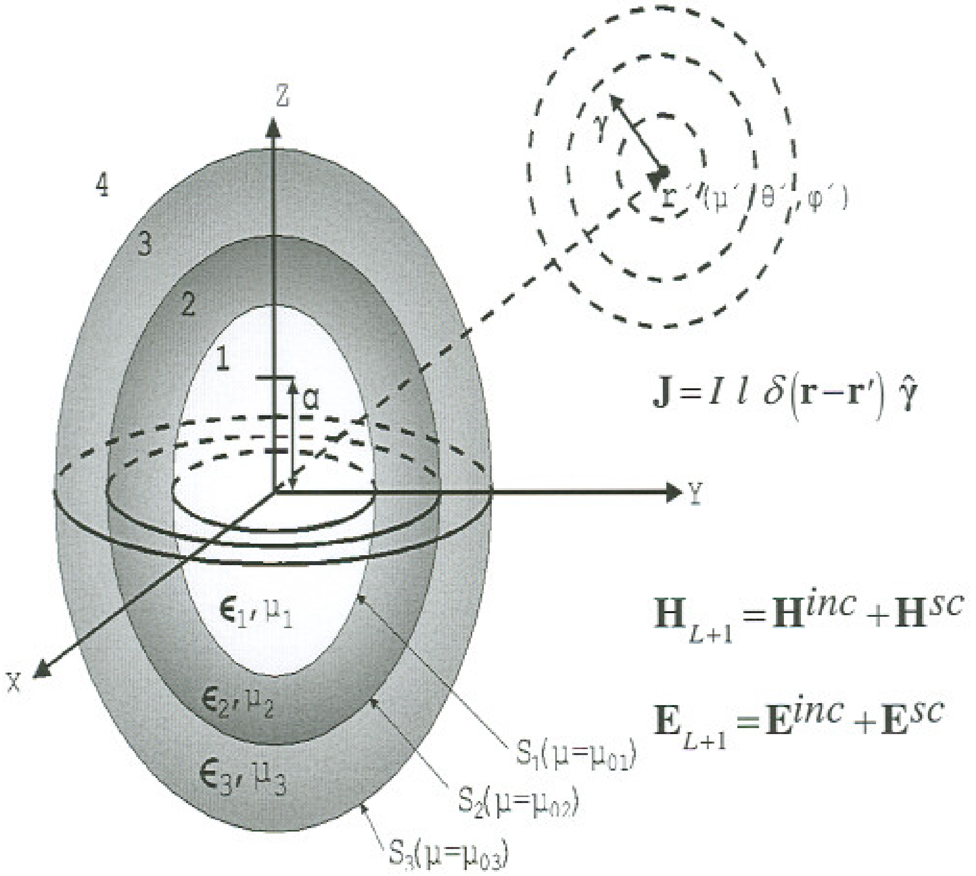

Figure 1: The Problem Geometry

$$\begin{aligned} \nabla\times\mathbf{H} &= \mathbf{J} - j\omega\varepsilon\mathbf{E} \\ \nabla\times\mathbf{E} &= j\omega\mu\mathbf{H} \\ \nabla\cdot\mathbf{H} &= \mathbf{0} \\ \nabla\cdot\mathbf{E} &= \mathbf{0}. \end{aligned} \tag{10}$$

Actually, for every point except source location point, the non-homogenous term $\mathbf{J}$ vanishes as we have assumed non-conductive media. It is well known [15] that the first order differential system (10) can provide with second order equations concerning every field separately. More precisely the electromagnetic field satisfies the vector Helmholtz equation

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = \mathbf{0},\ \mathbf{r} \neq \mathbf{r}' \tag{11a}$$

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = \mathbf{0}, \tag{11b}$$

where $k = \omega\sqrt{\mu\varepsilon}$ stands for the wave number in the specific surrounding medium. In addition, the electromagnetic field satisfies suitable boundary conditions on the discontinuity surfaces of the structure. Restricting ourselves to the magnetic field only, the boundary conditions are

$$\begin{aligned} &\hat{\mathbf{n}}_{S_i} \times \left(\mathbf{H}_{i+1} - \mathbf{H}_i\right) = \mathbf{0}, \\ &\mu_{i+1}\hat{\mathbf{n}}_{S_i} \cdot \mathbf{H}_{i+1} = \mu_i \hat{\mathbf{n}}_{S_i} \cdot \mathbf{H}_i,\ \mathbf{r} \in S_i,\ i = 1,2,\dots,L, \end{aligned} \tag{12}$$

which in accordance with (11a) – for every sub region, the asymptotic radiation conditions and the superposition (9), establish a well – posed boundary value problem. Our aim is the selection of this problem by determining the scattered field as well as the trapped stationary waves inside the spheroidal layers, in terms of the several alternatives concerning physical properties, relevant geometrical characteristics and point source location.

5. SOLUTION OF THE PROBLEM IN TERMS OF SOLENOIDAL SPHEROIDAL EIGENVECTORS

The electromagnetic field satisfies, as we mentioned in the previous section, the vector Helmholtz equation. As explained in [13] every solution $\mathbf{F}(\mathbf{r})$ of the vector Helmholtz equation is written as the superposition of an irrotational and a solenoidal field as follows

$$\mathbf{F}(\mathbf{r}) = \nabla\Phi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r}). \tag{13}$$

However, the solenoidal character of the electromagnetic field excludes potential $\Phi(\mathbf{r})$ from the representation (13) and further analysis [13] guarantees that a basis space for the solution of vector Helmholtz equation consists of the vector functions

$$\begin{aligned} &\mathbf{M} = \nabla \times (\mathbf{P}\Psi) \\ &\mathbf{N} = \frac{1}{k}\nabla \times \left(\nabla \times (\mathbf{P}\Psi)\right) = \frac{1}{k}\nabla \times \mathbf{M} \end{aligned}, \tag{14}$$

where $\mathbf{P}$ alternatively may be $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}},$ or $\mathbf{r}$ and runs over all solutions ψ (see Eq. (7) of the scalar Helmholtz equation). The completeness of the spheroidal eigenvectors (14), assures that the magnetic fields of the several regions have the following representation

$$\begin{aligned} &\mathbf{H}_1 = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{t=e,o}\left(\alpha^{1}_{mn,t}\mathbf{M}^{1,(1)}_{mn,t} + \beta^{1}_{mn,t}\mathbf{N}^{1,(1)}_{mn,t}\right), \\ &\mathbf{H}_i = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{t=e,o}\sum_{j=1}^{2}\left(\alpha^{i,j}_{mn,t}\mathbf{M}^{i,(j)}_{mn,t} + \beta^{i,j}_{mn,t}\mathbf{N}^{i,(j)}_{mn,t}\right),\ i = 2,3,\dots,L, \\ &\mathbf{H}^{inc} = Il\nabla G\left(\mathbf{r},\mathbf{r}'\right) \times \hat{\gamma}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\mathbf{H}^{sc} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{t=e,o}\left(\alpha^{sc}_{mn,t}\mathbf{M}^{L+1,(3)}_{mn,t} + \beta^{sc}_{mn,t}\mathbf{N}^{L+1,(3)}_{mn,t}\right), \\ &\mathbf{H}_{L+1} = \mathbf{H}^{inc} + \mathbf{H}^{sc} = Il\nabla G\left(\mathbf{r},\mathbf{r}'\right) \times \hat{\gamma} + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{t=e,o}\left(\alpha^{sc}_{mn,t}\mathbf{M}^{L+1,(3)}_{mn,t} + \beta^{sc}_{mn,t}\mathbf{N}^{L+1,(3)}_{mn,t}\right), \end{aligned}$$

where, index t characterizes the even or odd azimuthal dependence, index i indicates the specific region, indices m, n correspond to the separation of variables parameters of the scalar Helmholtz equation while index j defines the selected radial dependence as indicated in Eqs. (6)-(7). Notice that in region 1, the regularity of the magnetic field near the origin demands j=1, while the asymptotic outgoing behavior of the scattered field requires that j=3 in the surrounding space. What remains to be commented, between the above expression, is the incident field presented in the third of Eqs. (15) in a primitive form. More precisely, $G(\mathbf{r},\mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$ is the free-space Green's function of the scalar Helmholtz operator and the third of expressions (15) is the outcome of the standard analysis [14], producing the magnetic field from the vector potential through the relations

$$\begin{aligned}&\mu_0\mathbf{H}=\nabla\times\mathbf{A}\\&\mathbf{A}(\mathbf{r})=\mu_0\iiint\mathbf{J}(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}')d\mathbf{r}'.\end{aligned}\tag{16}$$

In fact, the combination of Eqs. (8)-(16) leads to the third of expressions (15). In order to express the incident field in spheroidal coordinates, all we need is the expansion of $G(\mathbf{r},\mathbf{r}')$ in terms of the spheroidal functions [14]

$$\begin{aligned}G(\mathbf{r},\mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}&=\frac{ik}{2\pi}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{\varepsilon_m}{\Lambda_{mn}}S_{mn}(\cos\theta;c)S_{mn}(\cos\theta';c)\\&\cos(m(\phi-\phi'))\begin{Bmatrix}R_{mn}^{(1)}(\cosh\mu';c)R_{mn}^{(3)}(\cosh\mu;c)\ \text{if}\ \mu>\mu'\\R_{mn}^{(1)}(\cosh\mu;c)R_{mn}^{(3)}(\cosh\mu';c)\ \text{if}\ \mu<\mu'\end{Bmatrix},\end{aligned}\tag{17}$$

where

$$\varepsilon_m=\begin{Bmatrix}1 & m=0\\2 & m\neq0\end{Bmatrix},$$

and Λ_{mn} are the normalization coefficients of the spheroidal functions S_{mn}, i.e.

$$\Lambda_{mn}=\int_{-1}^{1}|S_{mn}(x;c)|^2dx=\sum_{k=0}^{\infty}{}'\left|d_k^{mn}\right|^2\left(\frac{2}{2k+2m+1}\right)\frac{(k+2m)!}{k!}.\tag{18}$$

The representations (15) transfer the unknown character of the magnetic fields to their expansion coefficients. These coefficients will be determined once the expansions (15) are substituted in the boundary conditions (12). Actually these boundary conditions must first projected to the unit spheroidal coordinate vectors to obtain scalar equations. These new equations constitute functional equations as they are defined in the space $0\le\theta\le\pi$, $0\le\phi\le2\pi$. To obtain purely algebraic equations governing the field expansion coefficients, we have to "project" functionally these scalar boundary conditions on a complete set of functions in the space of square integrable functions on the unit sphere. We obtain then, instead of Eqs. (12), the infinite algebraic equations

$$
\begin{aligned}
&\oiint_{S_i} \mu_{i+1}\hat{\boldsymbol{\mu}}_{S_i}\cdot \mathbf{H}_{i+1}S_1\Phi_1 w_\mu dS = \oiint_{S_i} \mu_i\hat{\boldsymbol{\mu}}_{S_i}\cdot \mathbf{H}_i S_1\Phi_1 w_\mu dS,\\
&\oiint_{S_i} \hat{\boldsymbol{\theta}}_{S_i}\cdot \mathbf{H}_{i+1}S_1\Phi_1 w_\mu dS = \oiint_{S_i} \hat{\boldsymbol{\theta}}_{S_i}\cdot \mathbf{H}_i S_1\Phi_1 w_\theta dS,\\
&\oiint_{S_i} \hat{\boldsymbol{\phi}}_{S_i}\cdot \mathbf{H}_{i+1}S_1\Phi_1 w_\mu dS = \oiint_{S_i} \hat{\boldsymbol{\phi}}_{S_i}\cdot \mathbf{H}_i S_1\Phi_1 w_\phi dS,
\end{aligned}
\tag{19}
$$

where

$$
\Phi_1 = \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix},\ S_1 = P_\nu^m(\cos\theta),\ i = 1,2,...,L,\ m = 0,1,2,...,\ \nu = m, m+1, m+2,...
$$

In the previous equations, w_μ, w_θ, w_ϕ stand for weight functions, which are proved very helpful for compensating the scale factor appearing in the denominator of the integrals. These weight functions need not to be the same and their functionality as well as the methodology of handling the appeared surface integrals are described extensively in [13]. The replacement of expansions (15) in the boundary conditions (19) reveals that all we need to handle these surface integrals is the treatment of the corresponding "inner-products" referring to vector eigenvectors. More precisely the following integrals emerge and suffice to be determined

$$
\oiint_S \hat{\mathbf{p}}\cdot \mathbf{M}S_1\Phi_1 w_p dS,\ \oiint_S \hat{\mathbf{p}}\cdot \mathbf{N}S_1\Phi_1 w_p dS,\ \hat{\mathbf{p}} = \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}.
$$

As we mentioned above, their determination has been implemented elsewhere[13]. The outcome of this analysis leads to the following results

$$
\oiint_S \hat{\boldsymbol{\mu}}\cdot \mathbf{M}_{mn} S_1\Phi_1 w_\mu dS = -R_{mn}\alpha^2\ {}_\mu\mathbf{M}_{mn\nu}^{\Phi\Phi_1'},
$$

$$
{}_\mu\mathbf{M}_{mn\nu}^{\Phi\Phi_1'} = \int_0^{2\pi}\Phi\Phi_1' d\phi\left[\mathbf{I}_\nu^{5,mn} - \mathbf{I}_\nu^{7,mn} + \cosh^4\mu\left(\mathbf{I}_\nu^{1,mn} - \mathbf{I}_\nu^{3,mn}\right) - 2\cosh^2\mu\left(\mathbf{I}_\nu^{3,mn} - \mathbf{I}_\nu^{5,mn}\right)\right],
$$

$$
\oiint_S \hat{\boldsymbol{\mu}}\cdot \mathbf{N}_{mn} S_1\Phi_1 w_\mu dS = \frac{\alpha}{k}R_{mn}\cosh\mu\ {}_\mu N_{mn\nu}^{\Phi'\Phi_1'} + \frac{\alpha}{k}R_{mn}\sinh^2\mu\cosh\mu\ {}_\mu N_{mn\nu}^{\Phi\Phi_1,1} + \frac{\alpha}{k}R_{mn}'\sinh^2\mu\ {}_\mu N_{mn\nu}^{\Phi\Phi_1,2},
$$

$$
{}_\mu N_{mn\nu}^{\Phi'\Phi_1'} = \int_0^{2\pi}\Phi'\Phi_1' d\phi\left(\mathbf{I}_\nu^{4,mn} + \cosh^4\mu\mathbf{I}_\nu^{0,mn} - 2\cosh^2\mu\mathbf{I}_\nu^{2,mn}\right)
$$

$$
{}_\mu N_{mn\nu}^{\Phi\Phi_1,1} = \int_0^{2\pi}\Phi\Phi_1 d\phi\left(\cosh^2\mu\mathbf{R}_\nu^{0,mn} - \mathbf{R}_\nu^{2,mn} - 2\cosh^2\mu\mathbf{Q}_\nu^{1,mn} - 4\mathbf{Q}_\nu^{1,mn} + 6\mathbf{Q}_\nu^{3,mn}\right)
$$

$$
{}_\mu N_{mn\nu}^{\Phi\Phi_1,2} = \int_0^{2\pi}\Phi\Phi_1 d\phi\left[2\left(2+\cosh^2\mu\right)\left(\mathbf{I}_\nu^{2,mn} - \mathbf{I}_\nu^{4,mn}\right) + 6\left(\mathbf{I}_\nu^{6,mn} - \mathbf{I}_\nu^{4,mn}\right) + \mathbf{L}_\nu^{3,mn} - \mathbf{L}_\nu^{5,mn} - \cosh^2\mu\left(\mathbf{L}_\nu^{1,mn} - \mathbf{L}_\nu^{3,mn}\right)\right],
$$

$$\oiint_S \hat{\theta} \cdot \mathbf{M}_{mn} S_1 \Phi_1 w_\theta dS = R_{mn} \alpha^2 \sinh \mu \cosh \mu \, {}_\theta M_{mn\nu}^{\Phi'\Phi_1},$$

$${}_\theta M_{mn\nu}^{\Phi'\Phi_1} = \int_0^{2\pi} \Phi' \Phi_1 d\phi \left(\mathbf{I}_\nu^{4,mn} + \cosh^4 \mu \mathbf{I}_\nu^{0,mn} - 2\cosh^2 \mu \mathbf{I}_\nu^{2,mn} \right),$$

$$\oiint_S \hat{\theta} \cdot \mathbf{N}_{mn} S_1 \Phi_1 w_\theta dS = \alpha \sinh \mu \left(-\alpha^2 k R_{mn} \, {}_\theta N_{mn\nu}^{\Phi\Phi_1,1} - \frac{R_{mn}}{k} \, {}_\theta N_{mn\nu}^{\Phi\Phi_1,2} + \frac{R'_{mn}}{k} \sinh^2 \mu \cosh \mu \, {}_\theta N_{mn\nu}^{\Phi\Phi_1,3} + \frac{R_{mn}}{k} \, {}_\theta N_{mn\nu}^{\Phi\Phi_1,4} \right),$$

$${}_\theta N_{mn\nu}^{\Phi\Phi_1,1} = \int_0^{2\pi} \Phi \Phi_1 d\phi \left[\cosh^4 \mu \left(\mathbf{I}_\nu^{1,mn} - \mathbf{I}_\nu^{3,mn} \right) + \mathbf{I}_\nu^{5,mn} - \mathbf{I}_\nu^{7,mn} - 2\cosh^2 \mu \left(\mathbf{I}_\nu^{3,mn} - \mathbf{I}_\nu^{5,mn} \right) \right]$$

$${}_\theta N_{mn\nu}^{\Phi\Phi_1,2} = \int_0^{2\pi} \Phi \Phi_1 d\phi \left(\mathbf{Q}_\nu^{4,mn} - 2\cosh^2 \mu \mathbf{Q}_\nu^{2,mn} + \cosh^4 \mu \mathbf{Q}_\nu^{0,mn} \right)$$

$${}_\theta N_{mn\nu}^{\Phi\Phi_1,3} = \int_0^{2\pi} \Phi \Phi_1 d\phi \left(\cosh^2 \mu \mathbf{L}_\nu^{0,mn} - \mathbf{L}_\nu^{2,mn} - 2\cosh^2 \mu \mathbf{I}_\nu^{1,mn} - 4\mathbf{I}_\nu^{1,mn} + 6\mathbf{I}_\nu^{3,mn} \right)$$

$${}_\theta N_{mn\nu}^{\Phi\Phi_1,4} = \int_0^{2\pi} \Phi \Phi_1 d\phi \left(\cosh^2 \mu \mathbf{R}_\nu^{1,mn} - \mathbf{R}_\nu^{3,mn} - 2\cosh^2 \mu \mathbf{Q}_\nu^{2,mn} - 4\mathbf{Q}_\nu^{2,mn} + 6\mathbf{Q}_\nu^{4,mn} \right),$$

$$\oiint_S \hat{\phi} \cdot \mathbf{M}_{mn} S_1 \Phi_1 w_\phi dS = \alpha^2 \sinh^2 \mu \left(R'_{mn} \, {}_\phi M_{mn\nu}^{\Phi\Phi_1,1} + R_{mn} \cosh \mu \, {}_\phi M_{mn\nu}^{\Phi\Phi_1,2} \right),$$

$${}_\phi M_{mn\nu}^{\Phi\Phi_1,1} = \int_0^{2\pi} \Phi \Phi_1 d\phi \left(\mathbf{I}_\nu^{1,mn} - \mathbf{I}_\nu^{3,mn} \right)$$

$${}_\phi M_{mn\nu}^{\Phi\Phi_1,2} = \int_0^{2\pi} \Phi \Phi_1 d\phi \mathbf{Q}_\nu^{0,mn},$$

$$\oiint_S \hat{\phi} \cdot \mathbf{N}_{mn} S_1 \Phi_1 w_\phi dS = \frac{\alpha}{k} \left(R_{mn} \, {}_\phi N_{mn\nu}^{\Phi'\Phi_1} - R_{mn} \, {}_\phi N_{mn\nu}^{\Phi\Phi'_1,1} - R'_{mn} \sinh^2 \mu \cosh \mu \, {}_\phi N_{mn\nu}^{\Phi\Phi'_1,2} \right),$$

$${}_\phi N_{mn\nu}^{\Phi'\Phi_1} = \int_0^{2\pi} \Phi' \Phi_1 d\phi \left(\cosh^2 \mu \mathbf{I}_\nu^{0,mn} - \mathbf{I}_\nu^{2,mn} \right)$$

$${}_\phi N_{mn\nu}^{\Phi\Phi'_1,1} = \int_0^{2\pi} \Phi \Phi'_1 d\phi \mathbf{I}_\nu^{0,mn}$$

$${}_\phi N_{mn\nu}^{\Phi\Phi'_1,2} = \int_0^{2\pi} \Phi \Phi'_1 d\phi \mathbf{Q}_\nu^{1,mn}, \tag{20}$$

where the quantities in bold are defined in the Appendix.

Let us define the following expressions which will help us to form the final system of the equations

$$\mu_{mn\nu}^{i,j,\alpha}(\mu_i) = m\pi R_{mn}^j(\mu_i) \alpha^2 \left[\mathbf{I}_\nu^{5,mn} - \mathbf{I}_\nu^{7,mn} + \cosh^4 \mu_i \left(\mathbf{I}_\nu^{1,mn} - \mathbf{I}_\nu^{3,mn} \right) - 2\cosh^2 \mu_i \left(\mathbf{I}_\nu^{3,mn} - \mathbf{I}_\nu^{5,mn} \right) \right],$$

$$\mu_{mn\nu}^{i,j,\beta}(\mu_i) = m^2 \pi \frac{\alpha}{k_i} R_{mn}^j(\mu_i) \cosh \mu_i \left(\mathbf{I}_\nu^{4,mn} + \cosh^4 \mu_i \mathbf{I}_\nu^{0,mn} - 2\cosh^2 \mu_i \mathbf{I}_\nu^{2,mn} \right) +$$

$$+\pi \frac{\alpha}{k_i} R_{mn}^j(\mu_i) \sinh^2 \mu_i \cosh \mu_i \left(\cosh^2 \mu_i \mathbf{R}_\nu^{0,mn} - \mathbf{R}_\nu^{2,mn} - 2\cosh^2 \mu_i \mathbf{Q}_\nu^{1,mn} - 4\mathbf{Q}_\nu^{1,mn} + 6\mathbf{Q}_\nu^{3,mn} \right) +$$

$$+\pi \frac{\alpha}{k_i} R'^j_{mn}(\mu_i) \sinh^2 \mu_i \left[2\left(2 + \cosh^2 \mu_i\right)\left(\mathbf{I}_\nu^{2,mn} - \mathbf{I}_\nu^{4,mn} \right) + 6\left(\mathbf{I}_\nu^{6,mn} - \mathbf{I}_\nu^{4,mn} \right) + \mathbf{L}_\nu^{3,mn} - \mathbf{L}_\nu^{5,mn} - \cosh^2 \mu_i \left(\mathbf{L}_\nu^{1,mn} - \mathbf{L}_\nu^{3,mn} \right) \right],$$

$$\theta_{mnv}^{i,j,\alpha}(\mu_i) = m\pi R_{mn}^{j}(\mu_i)\alpha^2 \sinh\mu_i \cosh\mu_i \left(\mathbf{I}_{v}^{4,mn} + \cosh^4\mu_i \mathbf{I}_{v}^{0,mn} - 2\cosh^2\mu_i \mathbf{I}_{v}^{2,mn}\right),$$

$$\theta_{mnv}^{i,j,\beta}(\mu_i) = \pi\alpha \sinh\mu_i \left\{ \begin{array}{l} -\alpha^2 k_i R_{mn}^{j}(\mu_i)\left[\cosh^4\mu_i\left(\mathbf{I}_{v}^{1,mn} - \mathbf{I}_{v}^{3,mn}\right) + \mathbf{I}_{v}^{5,mn} - \mathbf{I}_{v}^{7,mn} - 2\cosh^2\mu_i\left(\mathbf{I}_{v}^{3,mn} - \mathbf{I}_{v}^{5,mn}\right)\right] - \\ -\dfrac{R_{mn}^{j}(\mu_i)}{k_i}\left(\mathbf{Q}_{v}^{4,mn} - 2\cosh^2\mu_i \mathbf{Q}_{v}^{2,mn} + \cosh^4\mu_i \mathbf{Q}_{v}^{0,mn}\right) + \\ +\dfrac{R'^{j}_{mn}}{k_i}\sinh^2\mu_i \cosh\mu_i\left(\cosh^2\mu_i \mathbf{L}_{v}^{0,mn} - \mathbf{L}_{v}^{2,mn} - 2\cosh^2\mu_i \mathbf{I}_{v}^{1,mn} - 4\mathbf{I}_{v}^{1,mn} + 6\mathbf{I}_{v}^{3,mn}\right) + \\ \dfrac{R_{mn}^{j}(\mu_i)}{k_i}\left(\cosh^2\mu_i \mathbf{R}_{v}^{1,mn} - \mathbf{R}_{v}^{3,mn} - 2\cosh^2\mu_i \mathbf{Q}_{v}^{2,mn} - 4\mathbf{Q}_{v}^{2,mn} + 6\mathbf{Q}_{v}^{4,mn}\right) \end{array} \right\},$$

$$\phi_{mnv}^{i,j,\alpha}(\mu_i) = \pi\alpha^2 \sinh^2\mu_i \left[R'^{j}_{mn}(\mu_i)\left(\mathbf{I}_{v}^{1,mn} - \mathbf{I}_{v}^{3,mn}\right) + R_{mn}^{j}(\mu_i)\cosh\mu_i \mathbf{Q}_{v}^{0,mn}\right],$$

$$\phi_{mnv}^{i,j,\beta}(\mu_i) = m\pi\frac{\alpha}{k_i}\left[R_{mn}^{j}(\mu_i)\left(\cosh^2\mu_i \mathbf{I}_{v}^{0,mn} - \mathbf{I}_{v}^{2,mn}\right) + R_{mn}^{j}(\mu_i)\mathbf{Q}_{v}^{1,mn} + R'^{j}_{mn}(\mu_i)\sinh^2\mu_i \cosh\mu_i \mathbf{I}_{v}^{0,mn}\right]. \quad (21)$$

Let us remind that for every azimuthal number m, the free index v in the above systems can take the values $v=m$, $m+1$, $m+2$, … . For every pair (m, v) we remark that the system consists of L groups, each one of them is formed by six equations. However, there exists some inner structure of these subsystems, which can be decoded in order to provide with a systematic and efficient algorithm for the solution of the reduced truncated systems. More precisely, the unknowns can be grouped as

$$\begin{gathered} \mathbf{c}_{(1),mn}^{1} = \left[\alpha_{o,mn}^{1}, \beta_{e,mn}^{1}\right]^T, \ \mathbf{c}_{(2),mn}^{1} = \left[-\alpha_{e,mn}^{1}, \beta_{o,mn}^{1}\right]^T, \\ \mathbf{c}_{(1),mn}^{i,j} = \left[\alpha_{o,mn}^{i,j}, \beta_{e,mn}^{i,j}\right]^T, \ \mathbf{c}_{(2),mn}^{i,j} = \left[-\alpha_{e,mn}^{i,j}, \beta_{o,mn}^{i,j}\right]^T, \\ 2 \le i \le L, \ j = 1,2, \\ \mathbf{c}_{(1),mn}^{sc} = \left[\alpha_{o,mn}^{sc}, \beta_{e,mn}^{sc}\right]^T, \ \mathbf{c}_{(2),mn}^{sc} = \left[-\alpha_{e,mn}^{sc}, \beta_{o,mn}^{sc}\right]^T, \end{gathered} \quad (22)$$

since these pairs satisfy individual subsystems as they are involved in only three equations of every subgroup. Indeed, let us define the coefficient matrices

$$\begin{gathered} D_{mnv}^{1} = \begin{bmatrix} \mu_1 \mu_{mnv}^{1,\alpha}(\mu_{01}) & \mu_1 \mu_{mnv}^{1,\beta}(\mu_{01}) \\ \theta_{mnv}^{1,\alpha}(\mu_{01}) & \theta_{mnv}^{1,\beta}(\mu_{01}) \\ \phi_{mnv}^{1,\alpha}(\mu_{01}) & -\phi_{mnv}^{1,\beta}(\mu_{01}) \end{bmatrix}, \\ D_{mnv}^{i,j,l} = \begin{bmatrix} \mu_i \mu_{mnv}^{i,j,\alpha}(\mu_{0l}) & \mu_i \mu_{mnv}^{i,j,\beta}(\mu_{0l}) \\ \theta_{mnv}^{i,j,\alpha}(\mu_{0l}) & \theta_{mnv}^{i,j,\beta}(\mu_{0l}) \\ \phi_{mnv}^{i,j,\alpha}(\mu_{0l}) & -\phi_{mnv}^{i,j,\beta}(\mu_{0l}) \end{bmatrix}. \end{gathered} \quad (23)$$

Next, for every specific triple (m, n, v) we form the kernel coefficient matrices referring each of them to $3L$ equations. More precisely, we define the matrix A_{mnv} as

$$A_{mnv} = \begin{bmatrix} D_{mnv}^{1} & -D_{mnv}^{2,1,1} & -D_{mnv}^{2,2,1} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & D_{mnv}^{2,1,2} & D_{mnv}^{2,2,2} & -D_{mnv}^{3,1,2} & -D_{mnv}^{3,2,2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{mnv}^{3,1,3} & D_{mnv}^{3,2,3} & -D_{mnv}^{4,1,3} & -D_{mnv}^{4,2,3} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & D_{mnv}^{L,1,L} & D_{mnv}^{L,2,L} & -D_{mnv}^{L+1,3,L} \end{bmatrix}. \quad (24)$$

Furthermore, the unknown expansion coefficients for every pair (m, n) are grouped as

$$\mathbf{c}_{(1),mn} = \left[\mathbf{c}^{1\ T}_{(1),mn}, \mathbf{c}^{2,1\ T}_{(1),mn}, \mathbf{c}^{2,2\ T}_{(1),mn}, \ldots, \mathbf{c}^{L,1\ T}_{(1),mn}, \mathbf{c}^{L,2\ T}_{(1),mn}, \mathbf{c}^{sc\ T}_{(1),mn} \right]^T,$$
$$\mathbf{c}_{(2),mn} = \left[\mathbf{c}^{1\ T}_{(2),mn}, \mathbf{c}^{2,1\ T}_{(2),mn}, \mathbf{c}^{2,2\ T}_{(2),mn}, \ldots, \mathbf{c}^{L,1\ T}_{(2),mn}, \mathbf{c}^{L,2\ T}_{(2),mn}, \mathbf{c}^{sc\ T}_{(1),mn} \right]^T. \tag{25}$$

It is now an easy task to verify that the boundary conditions obtain a matrix form and they are written as

$$\sum_{n=m}^{\infty} \mathbf{A}_{mnv} \mathbf{c}_{(1),mn} = \mathbf{d}_{(1),mv}$$
$$\sum_{n=m}^{\infty} \mathbf{A}_{mnv} \mathbf{c}_{(2),mn} = \mathbf{d}_{(2),mv} \tag{26}$$
$$m = 0,1,2,\ldots,\quad v = m, m+1, m+2, \ldots$$

where

$$\mathbf{d}_{(1),mv} = \left[0,\ 0,\ \ldots,\ 0,\ H^{inc,\mu}_{e,mv},\ H^{inc,\theta}_{e,mv},\ H^{inc,\phi}_{o,mv}\right]^T,$$
$$\mathbf{d}_{(2),mv} = \left[0,\ 0,\ \ldots,\ 0,\ H^{inc,\mu}_{o,mv},\ H^{inc,\theta}_{o,mv},\ -H^{inc,\phi}_{e,mv}\right]^T, \tag{27}$$

are the non-homogeneous terms. The matrix $\mathbf{A}_{mnv}$ has dimension $3L\times 4L$ and all column matrices have dimension $4L\times 1$. This fact orientates the specific manner according to which the truncation of the systems will be realized. However, the forthcoming suggestion obeys just to the rule not to divide apart the $6L$ subgroups and is not restrictive for the final numerical scheme. More precisely, we define the $12kL\times 12kL$ square matrices

$$\mathbf{B}_k^{(m)} = \begin{bmatrix} \mathbf{A}_{m,m,m} & \mathbf{A}_{m,m+1,m} & \mathbf{A}_{m,m+2,m} & \cdots & \mathbf{A}_{m,m+3k-1,m} \\ \mathbf{A}_{m,m,m+1} & \mathbf{A}_{m,m+1,m+1} & \mathbf{A}_{m,m+2,m+1} & \cdots & \mathbf{A}_{m,m+3k-1,m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{m,m,m+4k-1} & \mathbf{A}_{m,m+1,m+4k-1} & \mathbf{A}_{m,m+2,m+4k-1} & \cdots & \mathbf{A}_{m,m+3k-1,m+4k-1} \end{bmatrix}, \tag{28}$$

where m=0,1,2… and k=1,2,3,… .We define also the truncated columns

$$\hat{\mathbf{c}}^{(m)}_{(i),k} = \left[\mathbf{c}^T{}_{(i),m,m},\ \ \mathbf{c}^T{}_{(i),m,m+1},\ \ \ldots,\ \ \mathbf{c}^T{}_{(i),m,m+3k-1}\right]^T,\ i = 1,2 \tag{29}$$

containing the unknown coefficients with $n=m$, $m+1$, … , $m+3k-1$. The non-homogenous terms of the same order are

$$\hat{\mathbf{d}}^{(m)}_{(i),k} = \left[\mathbf{d}^T{}_{(i),m,m},\ \ \mathbf{d}^T{}_{(i),m,m+1},\ \ \ldots,\ \ \mathbf{d}^T{}_{(i),m,m+3k-1}\right]^T,\ i = 1,2. \tag{30}$$

Then, the truncated systems under investigation, for specific azimuthal dependence, are the following

$$\mathbf{B}_k^{(m)} \hat{\mathbf{c}}^{(m)}_{(1),k} = \hat{\mathbf{d}}^{(m)}_{(1),k},$$
$$\mathbf{B}_k^{(m)} \hat{\mathbf{c}}^{(m)}_{(2),k} = \hat{\mathbf{d}}^{(m)}_{(2),k}, \tag{31}$$

for k=1,2,3, … .

5. CONCLUSIONS

In our work we presented a theoretical treatment of the electromagnetic wave interaction emanating from a point source with a multilayer spheroidal body. This configuration corresponds to an indicative system for the mobile phone antenna placed close to the human head. The analysis does not follow classical T-matrix theory and is based on the expansion of incident and scattered electromagnetic waves in terms of solenoidal spheroidal eigenvectors. This results to a linear algebraic system of equation, which can be solved numerically if it can be truncated appropriately. Our approach can be easily expanded to include conductivity and it is independent of the various parameters (geometrical and physiological) entering the system.

6. APPENDIX

The quantities $\mathbf{I}_{\nu}^{\mu,mn}, \mathbf{L}_{\nu}^{\mu,mn}, \mathbf{Q}_{\nu}^{\mu,mn}, \mathbf{R}_{\nu}^{\mu,mn}$ are defined as

$$\mathbf{I}_{\nu}^{\mu,mn} = \int_{-1}^{1} x^{\mu} P_{\nu}^{m}(x) S_{mn}(x)\, dx = \sum_{i=0}^{\mu} d_{\nu+\mu-2i-m}^{mn}(c) B_{\nu,\nu+\mu-2i}^{\mu,m} A_{\nu+\mu-2i}^{m},$$

if and only if $n-\nu-\mu \equiv 0 (\mathrm{mod}\, 2)$ else $\mathbf{I}_{\nu}^{\mu,mn} = 0$,

$$\mathbf{L}_{\nu}^{\mu,mn} = \int_{-1}^{1} x^{\mu} \left(1-x^2\right) \frac{dP_{\nu}^{m}(x)}{dx} S_{mn}(x)\, dx = \sum_{i=0}^{\mu+1} d_{\nu-\mu-1+2i-m}^{mn}(c) C_{\nu,\nu-\mu-1+2i}^{\mu+1,m} A_{\nu-\mu-1+2i}^{m},$$

if and only if $n-\nu+\mu+1 \equiv 0 (\mathrm{mod}\, 2)$ else $\mathbf{L}_{\nu}^{\mu,mn} = 0$,

$$\mathbf{Q}_{\nu}^{\mu,mn} = \int_{-1}^{1} x^{\mu} P_{\nu}^{m}(x) \left(1-x^2\right) \frac{dS_{mn}(x)}{dx}\, dx = \sum_{i=0}^{\mu} \left[d_{\nu+\mu+1-2i-m}^{mn}(c) C_{\nu+\mu-2i+1,\nu+\mu-2i}^{1,m} + d_{\nu+\mu+1-2i-m}^{mn}(c) C_{\nu+\mu-2i-1,\nu+\mu-2i}^{1,m} \right] B_{\nu,\nu+\mu-2i}^{\mu,m} A_{\nu-\mu+2i}^{m},$$

if and only if $n-\nu-\mu-1 \equiv 0 (\mathrm{mod}\, 2)$ else $\mathbf{Q}_{\nu}^{\mu,mn} = 0$,

$$\mathbf{R}_{\nu}^{\mu,mn} = \int_{-1}^{1} x^{\mu} \left(1-x^2\right) \frac{dP_{\nu}^{m}(x)}{dx} \left(1-x^2\right) \frac{dS_{mn}(x)}{dx}\, dx = \sum_{i=0}^{\mu} \left[d_{\nu-\mu+2i-m}^{mn}(c) C_{\nu-\mu+2i,\nu-\mu+2i-1}^{1,m} + d_{\nu-\mu-2+2i-m}^{mn}(c) C_{\nu-\mu+2i-2,\nu-\mu+2i-1}^{1,m} \right] C_{\nu,\nu-1-\mu+2i}^{\mu+1,m} A_{\nu-\mu-1+2i}^{m},$$

if and only if $n-\nu+\mu \equiv 0 (\mathrm{mod}\, 2)$ else $\mathbf{R}_{\nu}^{\mu,mn} = 0$.

In the above expressions the coefficients $B_{\nu,n}^{k,m}, C_{\nu,n}^{k,m}$ as

$B_{\nu,\nu}^{0,m} = 1$ while,

$$B_{\nu,\nu+k-2i}^{k,m} = u(k-1-i) B_{\nu,\nu+k-1-2i}^{k-1,m} \frac{\nu+k-2i-m}{2(\nu+k-1-2i)+1} + u(i-1) B_{\nu,\nu+k+1-2i}^{k-1,m} \frac{\nu+k+1-2i+m}{2(\nu+k+1-2i)+1},$$

they expand the product $x^k P_{\nu}^{m}(x)$ as follows

$$x^k P_{\nu}^{m}(x) = \sum_{i=0}^{k} P_{\nu+k-2i}^{m} B_{\nu,\nu+k-2i}^{k,m},$$

$$C_{\nu,\nu-1}^{0,m} = \frac{(\nu+1)(\nu+m)}{(2\nu+1)}, C_{\nu,\nu+1}^{0,m} = -\nu \frac{(\nu+1-m)}{(2\nu+1)} \text{ while,}$$

$$C_{\nu,\nu-k-1+2i}^{k,m} = u(i-1) C_{\nu,\nu-k-2+2i}^{k-1,m} \frac{\nu-k-1+2i-m}{2(\nu-k-2+2i)+1} + u(k-i) C_{\nu,\nu-k+2i}^{k-1,m} \frac{\nu-k+2i+m}{2(\nu-k+2i)+1},$$

and they expand the product $x^k \left(1-x^2\right) \frac{dP_{\nu}^{m}(x)}{dx}$ as follows

$$x^k \left(1-x^2\right) \frac{dP_{\nu}^{m}(x)}{dx} = \sum_{i=0}^{k+1} P_{\nu-k-1+2i}^{m} C_{\nu,\nu-k-1+2i}^{k+1,m}.$$

The coefficients A_{ν}^{m} are defined from the relation

$$\int_{-1}^{1} P_n^m(x) P_\nu^m(x)\, dx = \frac{2}{2\nu+1}\frac{(\nu+m)!}{(\nu-m)!}\delta_{n\nu} = A_\nu^m \delta_{n\nu}$$

where $\delta_{n\nu}$ is the well known Dirac function

$$\delta_{n\nu} = \begin{Bmatrix} 1, & n=\nu \\ 0, & n \neq \nu \end{Bmatrix}$$

and

$$u(i) = \begin{Bmatrix} 1, & i \geq 0 \\ 0, & i < 0 \end{Bmatrix}.$$

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