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1. SUMMARY

We propose a theoretical approach to study the dynamic characteristics of a human long bone during callus formation. The shifting of the eigenfrequency spectrum proves to be a major indicator in the monitoring and diagnosis of the healing process. The bone diaphysis is assumed to be a finite length hollow piezoelectric cylinder of crystal class 6 while the callus area consists of isotropic, elastic material.

2. INTRODUCTION

The monitoring of the bone fracture healing using non-invasive techniques (e.g. wave propagation) has proven of great importance compared to widely used methods such as manual sensing or x-rays. Several researchers have studied wave propagation in long bones, considered it as an infinite piezoelectric cylinder of crystal class 6 with circular or arbitrary cross section [1-5]. A limited number of papers [6-7] address the vibration of bone of finite length. In Ref. [8] the evolution of the dynamic characteristics of isotropic elastic bone during the healing process is studied.

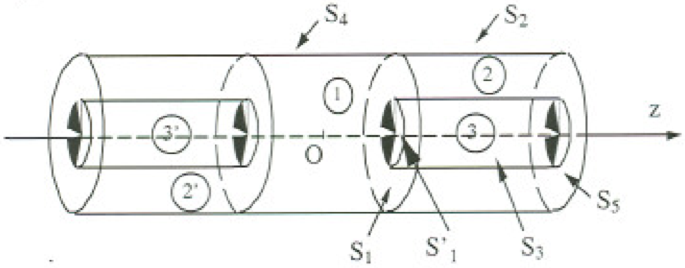
In a previous work [5] we considered wave propagation in a piezoelectric bone of arbitrary cross section. In this work we extend this approach to study the dynamic characteristics of bone during callus formation. The bone diaphysis is modeled as a finite length hollow piezoelectric cylinder of crystal class 6 while the callus area, which approximately follows the same geometry, is made of isotropic elastic material. The analysis can be proven very efficient for the understanding of the relation between the stage of fracture healing and the changes in the eigenfrequency spectrum of the system under consideration.

The description of the problem is based on the three-dimensional theory of elasticity and piezoelectricity (in quasi-static approximation). The solution of the wave equations for the piezoelectric cylinder is derived analytically as it is described in [2]. The solution of the problem for the isotropic callus is presented in terms of the Navier vector eigenfunctions for cylindrical coordinates [7].

The boundary conditions on the plane ends of the cylinder are responsible for the selection of the specific solution (odd or even in z-coordinate) from the general representation. They correspond to those imposed by the external fixator used in the treatment of long bone fracture. The lateral surfaces of the isotropic and piezoelectric parts are assumed to be free of fields (stress and electric potential), while on the contact surfaces of the layers conditions of continuity of the fields are considered. The satisfaction of the boundary conditions on the lateral surfaces leads to discretization of the initially continuous range of the wave numbers for the isotropic and piezoelectric part. The remaining boundary conditions are satisfied by an orthogonalization procedure, which finally leads to the frequency equation.

3. GOVERNING EQUATIONS

The geometry of the system under consideration is shown in the figure below.



Region 1 corresponds to the callus and it is assumed to be filled with isotropic elastic material. Regions 2 and 2' correspond to the cortical bone and they are assumed to be filled with piezoelectric material having crystal class 6 properties, while regions 3 and 3' are assumed to be empty (bone marrow is not included in our model). The cylindrical coordinates are used in the sequel with the z-axis along the axis of the cylinder.

The equation of motion for the isotropic material after suppressing the time-harmonic dependence is

$$c_t^2\nabla^2\mathbf{u}^{(1)}(\mathbf{r})+\left(c_l^2-c_t^2\right)\nabla\left(\nabla\cdot\mathbf{u}^{(1)}(\mathbf{r})\right)+\Omega^2\mathbf{u}^{(1)}(\mathbf{r})=0, \qquad (1)$$

where $\mathbf{u}^{(1)}=(u_x,u_\theta,u_z)$ is the elastic displacement vector field, $\mathbf{r}=(x,\theta,z)$ is the dimensionless position vector, c_t, c_l are the dimensionless velocities of the transverse and the longitudinal waves, respectively, and Ω is the dimensionless frequency.

The elastic displacement vector field $\mathbf{u}^{(1)}$ can be represented using Navier eigenfunctions [7]. In order to separate the odd and the even eigenstates of the system under consideration (due to the specific geometric symmetry), we introduce the parameter s ($s=1$, for odd solutions and $s=2$, for even solutions). The components of the displacement field can be represented as

$$u_x^{(1)}=\sum_{i=\left[s/2\right]+s}^{\left[s/2\right]+s+1}\sum_{\alpha=1}^{2}\sum_{m=0}^{+\infty}\int_0^{+\infty}\Big\{a^m_{i,\alpha}(\lambda)j^{m,1}\left(x_l^i x\right)+m b^m_{i,\alpha}(\lambda)\frac{J^{m,1}\left(x_t^i x\right)}{x_t^i x}+$$

$$+c^m_{i,\alpha}(\lambda)(-1)^i\lambda^2 j^{m,1}\left(x_t^i x\right)\Big\}\Theta^m_\alpha(\theta)Z_{1,i}(z,\lambda)d\lambda,$$

$$u_\theta^{(1)}=\sum_{i=\left[s/2\right]+s}^{\left[s/2\right]+s+1}\sum_{\alpha=1}^{2}\sum_{m=0}^{+\infty}\int_0^{+\infty}\Big\{m a^m_{i,\alpha}(\lambda)\frac{J^{m,1}\left(x_l^i x\right)}{x_l^i x}+b^m_{i,\alpha}(\lambda)j^{m,1}\left(x_t^i x\right)$$

$$+c_{i,\alpha}^{m}(\lambda)(-1)^{i}\lambda^{2}m\frac{J^{m,1}\left(x_t^i x\right)}{x_t^i x}\Bigg\}\frac{1}{m}\frac{\partial\Theta_{\alpha}^{m}(\theta)}{\partial\theta}Z_{1,i}(z,\lambda)d\lambda\,,$$

$$u_z^{(1)}=\sum_{i=\left[\frac{s}{2}\right]+s}^{\left[\frac{s}{2}\right]+s+1}\sum_{\alpha=1}^{2}\sum_{m=0}^{+\infty}\int_{0}^{+\infty}\left\{a_{i,\alpha}^{m}(\lambda)\frac{J^{m,1}\left(x_l^i x\right)}{x_l^i}+c_{i,\alpha}^{m}(\lambda)x_t^i J^{m,1}\left(x_t^i x\right)\right\}\Theta_{\alpha}^{m}(\theta)\frac{\partial Z_{1,i}(z,\lambda)}{\partial z}d\lambda\,, \tag{2}$$

where $x_{\beta}^{i}=\sqrt{k_{\beta}^{2}+(-1)^{i}\lambda^{2}}$ for i=1,2,3,4, with $k_{\beta}=\dfrac{\Omega}{c_{\beta}}$ for $\beta=l,t$ and λ is the dimensionless wavenumber, $\Theta_{\alpha}^{m}(\theta)=\cos(m\theta)$ for $a=1$ and $\Theta_{\alpha}^{m}(\theta)=\sin(m\theta)$ for $a=2$,

$$Z_{j,i}(z,\lambda)=\begin{cases}\sin\left[\lambda\left(z-\left[\frac{j}{2}\right]L\right)\right]\ for\ i=1, & \cos\left[\lambda\left(z-\left[\frac{j}{2}\right]L\right)\right]\ for\ i=2,\\ \sinh\left[\lambda\left(z-\left[\frac{j}{2}\right]L\right)\right]\ for\ i=3, & \cosh\left[\lambda\left(z-\left[\frac{j}{2}\right]L\right)\right]\ for\ i=4,\end{cases}\quad j=1,2$$

where $2L$ is the length of the cylinder and $[x]$ is the integer part of x, and

$$J^{m,l}(ax)=\begin{cases}J^{m}(ax), & l=1\quad (Bessel\ function\ of\ 1st\ kind)\\ Y^{m}(ax), & l=2\quad (Bessel\ function\ of\ 2nd\ kind)\end{cases}\ \text{for}\ a^{2}>0\,.$$

We note that for $a^2<0$ the Bessel functions are replaced by the modified ones. Also $\dot{J}^{m,l}$ denotes the derivative of $J^{m,l}$ with respect to its argument.

The stress tensor for an isotropic elastic material is given by the constitutive relation:

$$\mathbf{T}^{(1)}=\mathbf{c}^{(1)}:\nabla_s\mathbf{u}^{(1)}\,, \tag{3}$$

where $\nabla_s\mathbf{u}$ denotes the symmetric gradient of $\mathbf{u}$ and the components of the stiffness tensor $\mathbf{c}^{(1)}$ are given as

$$c_{ijkl}=\lambda\delta_{ji}\delta_{kl}+\mu\left(\delta_{jk}\delta_{il}+\delta_{jl}\delta_{ik}\right). \tag{4}$$

Using relations (2)-(4) we can calculate the components of the stress tensor $\mathbf{T}^{(1)}$.

The equations describing the behavior of a piezoelectric material are the equation of motion and the Gauss equation given as

$$div\mathbf{T}^{(2)}=\rho_2\frac{\partial^2\mathbf{u}^{(2)}}{\partial t^2}\,,\qquad \nabla\cdot\mathbf{D}=0\,, \tag{5}$$

where $\mathbf{u}^{(2)}$ is the displacement field for the piezoelectric material, ρ_2 is the mass density, $\mathbf{T}^{(2)}$ and $\mathbf{D}$ are the stress tensor and the electric displacement field, respectively, given by the constitutive relations:

$$\mathbf{T}^{(2)}=\mathbf{c}^{(2)}:\nabla_s\mathbf{u}^{(2)}+\mathbf{e}\cdot\nabla V\,,\qquad \mathbf{D}=\mathbf{e}:\nabla_s\mathbf{u}^{(2)}-\boldsymbol{\varepsilon}\cdot\nabla V\,. \tag{6}$$

In the above relations, V denotes the electrostatic potential, $\mathbf{c}$ the stiffness tensor, $\mathbf{e}$ the piezoelectric stress tensor and $\boldsymbol{\varepsilon}$ the dielectric tensor (for a piezoelectric material of crystal class 6 are given in Ref. [2]).

Using the methodology proposed in Ref.[2], we can find a wave-type solution in the form

$$u_x^{(2)} = \sum_{i=[s/2]+s}^{[s/2]+s+1} \sum_{j=1}^{4} \sum_{l=1}^{2} \sum_{m=0}^{+\infty} \int_0^{+\infty} \left\{ \left[\alpha_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p1} k_{i,j} \dot{J}^{m,l}(k_{i,j}x) + \beta_{i,j}^{m,l}(\lambda) m \delta_{i,j}^{p2} \frac{J^{m,l}(k_{i,j}x)}{x} \right] \cos(m\theta) + \right.$$
$$\left. + \left[-\alpha_{i,j}^{m,l}(\lambda) m \delta_{i,j}^{p2} \frac{J^{m,l}(k_{i,j}x)}{x} + \beta_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p1} k_{i,j} \dot{J}^{m,l}(k_{i,j}x) \right] \sin(m\theta) \right\} Z_{2,i}(z,\lambda) d\lambda e^{-i\Omega t}$$

$$u_\theta^{(2)} = \sum_{i=[s/2]+s}^{[s/2]+s+1} \sum_{j=1}^{4} \sum_{l=1}^{2} \sum_{m=0}^{+\infty} \int_0^{+\infty} \left\{ \left[-\alpha_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p2} k_{i,j} \dot{J}^{m,l}(k_{i,j}x) + \beta_{i,j}^{m,l}(\lambda) m \delta_{i,j}^{p1} \frac{J^{m,l}(k_{i,j}x)}{x} \right] \cos(m\theta) + \right.$$
$$\left. - \left[\alpha_{i,j}^{m,l}(\lambda) m \delta_{i,j}^{p1} \frac{J^{m,l}(k_{i,j}x)}{x} + \beta_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p2} k_{i,j} \dot{J}^{m,l}(k_{i,j}x) \right] \sin(m\theta) \right\} Z_{2,i}(z,\lambda) d\lambda e^{-i\Omega t}$$

$$u_z^{(2)} = \sum_{i=[s/2]+s}^{[s/2]+s+1} \sum_{j=1}^{4} \sum_{l=1}^{2} \sum_{m=0}^{+\infty} \int_0^{+\infty} \left\{ \left[\alpha_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p3} J^{m,l}(k_{i,j}x) \right] \cos(m\theta) + \right.$$
$$\left. + \left[\beta_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p3} J^{m,l}(k_{i,j}x) \right] \sin(m\theta) \right\} \frac{1}{\lambda} \frac{\partial Z_{2,i}(z,\lambda)}{\partial z} d\lambda e^{-i\Omega t}$$

$$V = \sum_{i=[s/2]+s}^{[s/2]+s+1} \sum_{j=1}^{4} \sum_{l=1}^{2} \sum_{m=0}^{+\infty} \int_0^{+\infty} \left\{ \left[\alpha_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p4} J^{m,l}(k_{i,j}x) \right] \cos(m\theta) + \right.$$
$$\left. + \left[\beta_{i,j}^{m,l}(\lambda)\delta_{i,j}^{p4} J^{m,l}(k_{i,j}x) \right] \sin(m\theta) \right\} \frac{1}{\lambda} \frac{\partial Z_{2,i}(z,\lambda)}{\partial z} d\lambda e^{-i\Omega t}, \qquad (7)$$

where the coefficients $\delta_{i,j}^{pq}, k_{i,j}$ $(i, j, p, q = 1,2,3,4)$ depend on λ, Ω and the material constants. Substituting Eqs. (7) into the constitutive relations (6) we can calculate the components of the stress tensor $\mathbf{T}^{(2)}$ and the electric displacement vector $\mathbf{D}$.

4. BOUNDARY CONDITIONS

The unknown coefficients entering the solution of the field equations are determined by the boundary conditions. We assume that the lateral surfaces S_2, S_3 and S_4 are stress-free and that S_2 and S_3 are coated with electrodes which are shorted. Also it is assumed that continuity conditions apply on S_1, while the plane surface S_1' is stress free. Finally, for the surfaces S_5 and S_5' we can impose two sets of boundary conditions:

(i) $u_x^{(2)} = u_\theta^{(2)} = T_{zz}^{(2)} = 0$,

(ii) $T_{zx}^{(2)} = T_{z\theta}^{(2)} = u_z^{(2)} = 0$.

It can be proved that the (i) set of boundary conditions implies that the parameter $i=1,2$ in the expressions of the solution (7) (odd solution), while the set of boundary conditions (ii) implies $i=3,4$ (even solution).

On the surface S_4 the following conditions are hold:

$$T_{xx}^{(1)} = T_{zx}^{(1)} = T_{x\theta}^{(1)} = 0. \tag{8}$$

Introducing the solution (7) in the above conditions, we are leading for every pair m,i to a system of algebraic equations

$$\mathbf{D}_{1,i}^{m}(\Omega,\lambda)\mathbf{x}_{1,i}^{m}(\lambda) = \mathbf{0}, \tag{9}$$

where

$$\mathbf{x}_{1,i}^{m}(\lambda) = \left\lfloor a_{i,\alpha}^{m}, b_{i,\alpha}^{m}, c_{i,\alpha}^{m} \right\rfloor, \quad m = 0,1,2,\ldots,\ i = \left[\frac{s}{2}\right]+s, \left[\frac{s}{2}\right]+s+1,\ \alpha = 1,2$$

In order that the system (9) to have non-trivial solutions the following equation must be satisfied

$$\det\left\{\mathbf{D}_{1,i}^{m}(\Omega,\lambda)\right\} = 0 \tag{10}$$

This equation discretizes the continuous range of λ in the expressions (2) to be transformed to sums over the possible values of λ's which form the sequence

$$\lambda_{1,i}^{m,1}(\Omega), \lambda_{1,i}^{m,2}(\Omega), \ldots, \lambda_{1,i}^{m,n}(\Omega), \ldots$$

which depends explicitly on Ω.

On the lateral surfaces S_2, S_3 the following boundary conditions are hold:

$$T_{xx}^{(2)} = T_{zx}^{(2)} = T_{x\theta}^{(2)} = 0 = V \tag{11}$$

When the appropriate solution for the stresses and electrostatic potential are substituted into these boundary conditions we obtain for each pair m,i the system

$$\mathbf{D}_{2,i}^{m}(\Omega,\lambda)\mathbf{x}_{2,i}^{m}(\lambda) = \mathbf{0} \tag{12}$$

where

$$\mathbf{x}_{2,i}^{m}(\lambda) = \left\lfloor \alpha_{i,1}^{m,1}, \ldots, \alpha_{i,4}^{m,1}, \alpha_{i,1}^{m,2}, \ldots, \alpha_{i,4}^{m,2}, \beta_{i,1}^{m,1}, \ldots, \beta_{i,4}^{m,1}, \beta_{i,1}^{m,2}, \ldots, \beta_{i,4}^{m,2} \right\rfloor,$$

for $m = 0,1,2,\ldots,\ i = \left[\frac{s}{2}\right]+s, \left[\frac{s}{2}\right]+s+1,$

In order that the system (12) to have non-trivial solutions the following equation must be satisfied

$$\det\left\{\mathbf{D}_{2,i}^{m}(\Omega,\lambda)\right\} = 0. \tag{13}$$

This equation gives rise to a sequence

$$\lambda_{2,i}^{m,1}(\Omega), \lambda_{2,i}^{m,2}(\Omega), \ldots, \lambda_{2,i}^{m,n}(\Omega), \ldots$$

of the possible values of λ for every value of Ω.

Since the boundary conditions examined until now, have discretized the parameter λ, the corresponding coefficients in the expressions (2) and (7) depend on n giving $a_{i,\alpha}^{m}(\lambda) = a_{i,\alpha}^{m,n}, b_{i,\alpha}^{m}(\lambda) = b_{i,\alpha}^{m,n}$ etc.

The remaining boundary conditions are the following:

$$u_x^{(1)} = u_x^{(2)}, \qquad u_\theta^{(1)} = u_\theta^{(2)}, \qquad u_z^{(1)} = u_z^{(2)}, \qquad V = 0, \qquad \text{on } S_1$$

$$T_{zx}^{(1)} = T_{zx}^{(2)}, \quad T_{z\theta}^{(1)} = T_{z\theta}^{(2)}, \quad T_{zz}^{(1)} = T_{zz}^{(2)}, \quad D_z = 0, \quad \text{on } S_1$$
$$T_{zz}^{(1)} = T_{zx}^{(1)} = T_{z\theta}^{(1)} = 0 \quad \text{on } S_1{}'. \tag{14}$$

These boundary conditions are satisfied by making the solution orthogonal to a complete set of functions $J_0(\xi_k x)$, where $\xi_k, k = 1,2,\ldots$ stand for the roots of equation $J_0(\xi_k R) = 0$ [8, 9]. After projecting equations (14) onto the basis $J_0(\xi_k x)$ we obtain, for each m, an infinite system of algebraic equations

$$\mathbf{D}^m(\Omega)\mathbf{x}^m = \mathbf{0}, \tag{15}$$

where

$$\mathbf{x}^m = \left[\mathbf{x}_i^{m,1}, \mathbf{x}_i^{m,2}, \ldots, \mathbf{x}_i^{m,n} \ldots\right], \ \mathbf{x}_i^{m,n} = \left[\alpha_{i,j}^{m,1,n}, \alpha_{i,j}^{m,2,n}, \beta_{i,j}^{m,1,n}, \beta_{i,j}^{m,2,n}, a_{i,\alpha}^{m,n}, b_{i,\alpha}^{m,n}, c_{i,\alpha}^{m,n}\right],$$

and $m = 0,1,2,\ldots$, $i = \left[\frac{s}{2}\right] + s, \left[\frac{s}{2}\right] + s + 1$.

Truncating suitably the system (15) we obtain a sequence of linear algebraic homogeneous systems of increasing dimension N:

$$\mathbf{D}_N^m(\Omega)\mathbf{x}_N^m = \mathbf{0}, \tag{16}$$

which lead to the equations

$$\det\left\{\mathbf{D}_N^m(\Omega)\right\} = 0, \quad N = 1,2,3,\ldots \tag{17}$$

The truncated system (17) is solved numerically to provide with the shown frequency Ω [8].

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