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FOR SURFACE REMODELING

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Abstract

A theoretical analysis of the surface bone remodeling induced by the insertion of a cylindrical implant into the medulla of a long hollow cylinder bone model is presented. The bone is treated as a poroelastic material using Biot's formulation of the theory of consolidation. The theory of small-strain adaptive elasticity proposed by Cowin et al. is employed to develop a new theoretical approach for surface remodeling. Our results predict the movement of the endosteal and periosteal surfaces with remodeling time for various values of porosity, initial radii and thickness of the bone.

1. Introduction

Under the general term bone remodeling, all the processes by which the adult shape, structure and form of a bone is influenced by the mechanical circumstances to which it is subjected, are included. Although these processes occur simultaneously and cannot be separated, the distinction made by Frost [1] between surface and internal remodeling can be adopted to distinguish between the changes in geometry and material properties of the bone tissue as a result of the changes in its loading environment. Under this convention, surface or external remodeling is the process which results in a change of external shape of the bone. This is due to the resorption or deposition of bone material on the endosteal and periosteal surfaces.

The causal relationship between the shape changes of a long bone such as the femur and its loading induced by the forced fit of an endoprosthesis into its medulla was early addressed by Cowin and Van Buskirk [2]. Their approach was based on the small-strain approximation [3] of the thermomechanical continuum theory of adaptive elasticity [4, 5]. Using a simplified geometry of the bone-implant system and treating bone as a porous elastic solid, they predicted the changes in the external shape of the bone model as a result of the insertion of the implant. According to their hypothesis, the rate of the chemical reactions responsible for bone deposition or resorption on the endosteum and periosteum is proportional to the strain in these surfaces.

Theoretical predictions of the surface bone remodeling in the diaphysis of a long bone under a constant superposed compressive load have been made by Cowin et al.[6]. They have shown theoretically that, for different values of the superposed compressive stress and surface remodeling rate coefficients, all types of surface movement are possible. This has been verified by the experimental observations of Woo et al. [7], Uthoft et al.. [8] and Jaworski et al. [9] which indicate that there is a variety of possible movements for the endosteal and periosteal surfaces. Different kinds of variations of the endosteal and periosteal surfaces have

also been illustrated by Cowin *et al* [10]. Misra J.C. *et al.* [11] have studied the effects of cross sectional non-uniformity and the anisotropy of osseous tissues on the remodeling of diaphyseal surfaces of a specimen of long tubular bone.

In this work a new approach of surface remodeling induced by a cylindrical implant is presented. Our objective is to develop a new theoretical model of surface bone remodeling where the role of the fluid part is evident and make predictions about the progress of the surface bone remodeling process, expressed as the change of the endosteum and periosteum radii with time. In the proposed model, a simple geometry of a hollow circular cylinder for the bone and a solid circular cylinder for the endoprosthesis has been employed. Bone is treated as a porous elastic deformable solid in the pores of which a viscous compressible fluid flows, using Biot's formulation of the theory of consolidation [12, 13, 14]. The theory of small-strain adaptive elasticity [3], is appropriately modified in order to incorporate the fluid part according to the new material description. The basic equations of the new theory for surface remodeling are formulated. To predict the movements of the endosteal and periosteal surfaces due to the endoprosthesis force fitted into the medulla a situation is considered where there is a constant axial force applied to the hollow circular cylinder in addition to the stress on the interior surface due to the endoprosthesis. The material coefficients are considered constants while the radii of the hollow cylinder vary with time. The initial value problem of a coupled system of first order differential equations is numerically solved to obtain the evolution of the endosteum and periosteum radii with time. The movement of the endosteal and periosteal surfaces with remodeling time is predicted for various values of porosity, initial radii and bone thickness.

2. Theoretical Background

The basic set of equations in the theory of surface bone remodeling as described by Cowin and Van Buskirk [2], consists of the constitutive equations, the kinematic relations, the stress equations of equilibrium and the constitutive equation for the speed of the remodeling surface. Assuming that the bone is a porous isotropic solid that contains a viscous compressible fluid, the above mentioned set of equations is reformulated by using the theory of consolidation introduced by Biot [12, 13], in cylindrical coordinates. We describe this procedure below.

The stress tensor in a porous material is

$$\overline{T} = T_{ij} + \delta_{ij}T, \tag{1}$$

where δ_{ij} is the Kronecker's symbol and T represents the total normal force applied to the fluid part of the faces of a cube of unit size of the bulk material.

If p is the hydrostatic pressure of the fluid in the pores we may write

$$T = -fp, \tag{2}$$

where f is the porosity defined as

$$f = V_p / V_b, \tag{3}$$

where V_p is the volume of the pores contained in a sample of bulk volume V_b. Thus, f represents the fraction of the volume of the porous material occupied by the pores.

This system of solid and fluid is a general system which has conservation properties. The solid part is considered to have compressibility and shearing rigidity and the fluid is compressible. The deformation of a unit cube is assumed to be completely reversible. By

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deformation is meant here the one determined by the strain tensors in the solid and in the fluid [15].

The kinematic relations for the solid part are

$$E_{rr} = \frac{\partial u_r}{\partial r}, \quad E_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad E_{zz} = \frac{\partial u_z}{\partial z},$$

$$\tag{4}$$

$$E_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right), \quad E_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad E_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right),$$

where u_r, u_θ, u_z are the average displacement components and E_{ij} $i,j = r,\theta,z$, are the strain components of the solid.

Similar relations hold for the fluid part with U_r, U_θ, U_z and ε_{ij}, $i,j = r,\theta,z$, denoting the average displacement components and the strains of the fluid, respectively.

The stress-strain equations for an isotropic poroelastic material are given as

$$T_{rr} = 2NE_{rr} + AE + Q\varepsilon,$$

$$T_{\theta\theta} = 2NE_{\theta\theta} + AE + Q\varepsilon,$$

$$T_{zz} = 2NE_{zz} + AE + Q\varepsilon,$$

$$T_{\theta z} = NE_{\theta z},$$

$$T_{zr} = NE_{zr},$$

$$\tag{5}$$

$$T_{r\theta} = NE_{r\theta},$$

$$T = QE + R\varepsilon,$$

where A, N, R, Q are the elastic constants of the material, in accordance with Biot's formulation [12-15] and E and ε are the dilatations of the solid and fluid, that is,

$$E = E_{rr} + E_{\theta\theta} + E_{zz}, \tag{6}$$

and

$$\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}, \tag{7}$$

respectively.

We note that Eq. (7) does not provide the actual strain in the fluid but the divergence of the fluid-displacement field which is derived from the average volume flow through the pores [15].

Inversely, the isotropic strain-stress relations are

$$E_{rr} = \frac{1}{2N}\left\{(1+q)T_{rr} + q(T_{\theta\theta} + T_{zz}) - \frac{Q}{R}(3q+1)T\right\},$$

$$E_{\theta\theta} = \frac{1}{2N}\left\{(1+q)T_{\theta\theta} + q(T_{rr} + T_{zz}) - \frac{Q}{R}(3q+1)T\right\},$$

$$E_{zz} = \frac{1}{2N}\left\{(1+q)T_{\theta\theta} + q(T_{rr} + T_{zz}) - \frac{Q}{R}(3q+1)T\right\}, \tag{8}$$

$$E_{\theta z} = \frac{1}{N}T_{\theta z},$$

$$E_{zr} = \frac{1}{N}T_{zr},$$

$$E_{r\theta} = \frac{1}{N}T_{r\theta},$$

and

$$E = E_{rr} + E_{\theta\theta} + E_{zz} = \frac{1}{2N}\left\{(3q+1)T_{rr} + (3q+1)T_{\theta\theta} + (3q+1)T_{zz} - \frac{3Q}{R}(3q+1)T\right\}, \tag{9}$$

$$\varepsilon = -Qs(T_{rr} + T_{\theta\theta} + T_{zz}) + \left(\frac{1}{R} + \frac{3Q^2}{R}s\right)T, \tag{10}$$

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where

$$q = \frac{Q^2 - AR}{(2N + 3A)R - 3Q^2} \quad \text{and} \quad s = \frac{1}{(2N + 3A) - 3Q^2} = \frac{q}{Q^2 - AR}. \tag{11}$$

The total stress field of the bulk material, in the absence of body forces, satisfies the equilibrium equations

$$\frac{\partial}{\partial r}(T_{rr} + T) + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \frac{\partial T_{rz}}{\partial z} = 0,$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(T_{zz} + T) + \frac{\partial T_{r\theta}}{\partial r} + \frac{2}{r}T_{r\theta} + \frac{\partial T_{rz}}{\partial z} = 0, \tag{12}$$

$$\frac{\partial}{\partial z}(T_{zz} + T) + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zr}}{\partial r} + \frac{T_{rz}}{r} = 0.$$

Darcy's law governing the flow of a fluid in a porous isotropic material, for non-existing body forces, is given as

$$\frac{\partial T}{\partial r} = C\frac{\partial}{\partial t}(U_r - u_r),$$

$$\frac{1}{r}\frac{\partial T}{\partial \theta} = C\frac{\partial}{\partial t}(U_\theta - u_\theta), \tag{13}$$

$$\frac{\partial T}{\partial z} = C\frac{\partial}{\partial t}(U_z - u_z),$$

where C is a constant that depends on the permeability κ, the porosity f of the medium and the viscosity η of the fluid [16, 17], that is

$$C = \frac{\eta f^2}{\kappa}. \tag{14}$$

Eq. (5) can be written in matrix form as

$$
\begin{bmatrix}
T_{rr}+T \\
T_{\theta\theta}+T \\
T_{zz}+T \\
T_{\theta z} \\
T_{zr} \\
T_{r\theta}
\end{bmatrix}
=
$$

(15)

$$
\begin{bmatrix}
2N+A+Q & A+Q & A+Q & 0 & 0 & 0 \\
A+Q & 2N+A+Q & A+Q & 0 & 0 & 0 \\
A+Q & A+Q & 2N+A+Q & 0 & 0 & 0 \\
0 & 0 & 0 & N & 0 & 0 \\
0 & 0 & 0 & 0 & N & 0 \\
0 & 0 & 0 & 0 & 0 & N
\end{bmatrix}
\begin{bmatrix}
E_{rr}+x_1\varepsilon \\
E_{\theta\theta}+x_2\varepsilon \\
E_{zz}+x_3\varepsilon \\
E_{\theta z} \\
E_{zr} \\
E_{r\theta}
\end{bmatrix},
$$

so that the following relationship is satisfied,

$$
\left.
\begin{aligned}
(2N+A+Q)x_1 + (A+Q)x_2 + (A+Q)x_3 &= Q+R \\
(A+Q)x_1 + (2N+A+Q)x_2 + (A+Q)x_3 &= Q+R \\
(A+Q)x_1 + (A+Q)x_2 + (2N+A+Q)x_3 &= Q+R
\end{aligned}
\right\}
\Rightarrow x_1 = x_2 = x_3 = \frac{-(Q+R)}{2N+3A+3Q}. \quad (16)
$$

Thus, the stress-strain relations for the isotropic poroelastic bone can be written as

$$
\left(T_{ij}+\delta_{ij}T\right) = C_{ijkm}\left(E_{km} - \delta_{km}\frac{Q+R}{2N+3A+3Q}\varepsilon\right). \quad (17)
$$

In analogy to Cowin and Van Buskirk's theory of surface remodeling [2] and using Eq. (17) the constitutive equation for the speed of the remodeling surface can be written as

$$
U = C_{ij}(\mathbf{n},P)\left[E_{ij}(P) - \delta_{ij}\frac{Q+R}{2N+3A+3Q}\varepsilon(P) - E_{ij}^0(P) - \delta_{ij}\frac{Q+R}{2N+3A+3Q}\varepsilon^0(P)\right], \quad (18)
$$

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where $E_{ij}^0(P) - \delta_{ij} \dfrac{Q+R}{2N+3A+3Q} \varepsilon^0(P)$ is a reference value of strain where no remodeling occurs and $C_{ij}(\mathbf{n}, P)$ are surface remodeling rate coefficients which, in general, depend on the point P and the normal $\mathbf{n}$ to the surface at P.

In cylindrical coordinates, this expression can be written as

$$U = C_{rr}\left(E_{rr} - \frac{Q+R}{2N+3A+3Q}\varepsilon \right) + C_{\theta\theta}\left(E_{\theta\theta} - \frac{Q+R}{2N+3A+3Q}\varepsilon \right)$$

$$+ C_{zz}\left(E_{zz} - \frac{Q+R}{2N+3A+3Q}\varepsilon \right) + C_{r\theta}E_{r\theta} + C_{rz}E_{rz} + C_{\theta z}E_{\theta z} - C^0,$$

(19)

where

$$C^0 = C_{rr}\left(E_{rr}^{\ 0} - \frac{Q+R}{2N+3A+3Q}\varepsilon^0 \right) + C_{\theta\theta}\left(E_{\theta\theta}^{\ 0} - \frac{Q+R}{2N+3A+3Q}\varepsilon^0 \right)$$

$$+ C_{zz}\left(E_{zz}^{\ 0} - \frac{Q+R}{2N+3A+3Q}\varepsilon^0 \right) + C_{r\theta}E_{r\theta}^{\ 0} + C_{rz}E_{rz}^{\ 0} + C_{\theta z}E_{\theta z}^{\ 0}.$$

(20)

In terms of stress, the constitutive equation for the speed of the remodeling surface can be written as

$$U = B_{rr}T_{rr} + B_{\theta\theta}T_{\theta\theta} + B_{zz}T_{zz} + B_{r\theta}T_{r\theta} + B_{rz}T_{rz} + B_{\theta z}T_{\theta z} + BT - C^0,$$

(21)

where

$$B_{rr} = \left[\frac{1}{2N}(1+q) + \frac{Q+R}{2N+3A+3Q}Qs \right]C_{rr} + \left[\frac{1}{2N}q + \frac{Q+R}{2N+3A+3Q}Qs \right](C_{\theta\theta} + C_{zz}),$$

$$B_{\theta\theta} = \left[\frac{1}{2N}(1+q) + \frac{Q+R}{2N+3A+3Q}Qs \right]C_{\theta\theta} + \left[\frac{1}{2N}q + \frac{Q+R}{2N+3A+3Q}Qs \right](C_{rr} + C_{zz}),$$

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$$B_{zz} = \left[\frac{1}{2N}(1+q) + \frac{Q+R}{2N+3A+3Q}Qs \right]C_{zz} + \left[\frac{1}{2N}q + \frac{Q+R}{2N+3A+3Q}Qs \right](C_{rr}+C_{\theta\theta}),$$

$$B_{r\theta} = \frac{1}{N}C_{r\theta}, \qquad\qquad (22)$$

$$B_{rz} = \frac{1}{N}C_{rz},$$

$$B_{\theta z} = \frac{1}{N}C_{\theta z},$$

and

$$B = -\left[\frac{1}{2N}\frac{Q}{R}(3q+1) + \frac{Q+R}{2N+3A+3Q}\left(\frac{1}{R} + \frac{3Q^2}{R}s \right) \right](C_{rr}+C_{\theta\theta}+C_{zz}).$$

We assume that the term BT must equivalently contribute to the terms $B_{rr}T_{rr}$, $B_{\theta\theta}T_{\theta\theta}$ and $B_{zz}T_{zz}$, so that the constitutive equation for the speed of the remodeling surface can be written in terms of stress as follows

$$U = B_{rr}(T_{rr}+T) + B_{\theta\theta}(T_{\theta\theta}+T) + B_{zz}(T_{zz}+T) + B_{r\theta}T_{r\theta} + B_{rz}T_{rz} + B_{\theta z}T_{\theta z} - C^0. \qquad (23)$$

It holds

$$B_{rr}+B_{\theta\theta}+B_{zz} = \left[\frac{1}{2N}(3q+1) + 3\frac{Q+R}{2N+3A+3Q}Qs \right](C_{rr}+C_{\theta\theta}+C_{zz}). \qquad (24)$$

Equation (23) is satisfied only if

$$B = B_{rr}+B_{\theta\theta}+B_{zz}, \qquad\qquad (25)$$

which, using Eq. (22), is equivalent to the constraint

$$R(3A+4N) = 3Q^2. \qquad\qquad (26)$$

3.   Formulation of the implant problem

In this section we apply the theoretical results obtained in the previous section to predict the movements of the endosteal and periosteal surfaces in the diaphyseal region of a long bone due to a cylindrical implant force-fitted into the medulla. We assume that the diaphyseal region of a long bone is a hollow circular cylinder of isotropic poroelastic material (bone) of initial endosteal and periosteal radii a_0 and b_0, respectively (Fig. 1). The cylindrical implant is assumed to be an isotropic linear elastic material with Lamé's constants λ and μ and initial radius $a_0 + \dfrac{\rho}{2}$. The internal and external radii of the hollow cylinder as functions of time t are denoted by $a(t)$ and $b(t)$, respectively. They correspond to the radii of the endosteal and periosteal surfaces at time t.

We consider the situation where there is a constant axial force P applied to the hollow cylinder in addition to the stress on the interior surface due to the implant (Fig. 2). The solution to this problem has been given by Papathanasopoulou *et al.* [18]. The displacement of the hollow cylinder in the circumferential or hoop direction is zero. The components of the displacement vector for the solid and fluid part in the radial and axial directions are given [18] as

$$u_r = (Q+R)(B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{\frac{-m^2 t}{k}} - A_1(t)r - A_2(t)\frac{1}{r},$$

$$(27)$$

$$U_r = -(2N + A + Q)(B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{\frac{-m^2 t}{k}} - A_1(t)r - A_2(t)\frac{1}{r},$$

and

$$u_z = -D_1(t)z,$$

$$(28)$$

$$U_z = -D_2(t)z,$$

where $A_1(t), A_2(t), D_1(t), D_2(t)$ are given in Appendix I, B_6, B_7 are constants, r is the radial distance from the long axis of the cylinder and m is a parameter that must satisfy the equation [18],

$$I_1(ma)K_1(mb) - I_1(mb)K_1(ma) = 0,\qquad(29)$$

where $I_1(ma)$, $I_1(mb)$ and $K_1(ma)$, $K_1(mb)$ are modified Bessel functions of the first and second kind, respectively, of order one.

The parameter k is given [18] as

$$k = \frac{C(2N + 2Q + R + A)}{Q^2 - 2NR - AR}\qquad(30)$$

where C is the constant in Darcy's law.

In the case under discussion the material coefficients are constant while the radii of the hollow cylinder vary with time.

The surface remodeling equation (23) will determine the changes in a and b with time. For the endosteal surface, Eq. (23) can be written in the following form

$$U_e = B_{rr_e}(T_{rr} + T) + B_{zz_e}(T_{zz} + T) - C_e^0,\qquad(31)$$

while for the periosteal surface we can write that

$$U_p = B_{rr_p}(T_{rr} + T) + B_{zz_p}(T_{zz} + T) - C_p^0.\qquad(32)$$

The speeds normal to the endosteal and periosteal surfaces can be written as time derivatives of $a(t)$ and $b(t)$, respectively,

$$U_e = -\frac{da}{dt}, \qquad U_p = \frac{db}{dt}. \tag{33}$$

When Eqs (4), (5), (27), (28) and (33) are substituted into Eqs (31) and (32) two ordinary differential equations for $a(t)$ and $b(t)$ are obtained,

$$-\frac{da}{dt} = B_{rr_e} \begin{pmatrix} -2N(Q+R)m^2 e^{\frac{-m^2 t}{k}} \left(B_6 \frac{1}{a} I_1(ma) + B_7 \frac{1}{a} K_1(ma) \right) - \\ -2N(N+A+2Q+R)A_1(t) + 2NA_2(t)\frac{1}{a^2} - \\ -(A+2Q+R)D_1(t) - (Q+R)\Theta \end{pmatrix}$$
$$+ \frac{-B_{zz_e}P}{\pi(b^2 - a^2)} - B_{zz_e}T_0 \quad , \tag{34}$$

and

$$\frac{db}{dt} = B_{rr_p} \begin{pmatrix} -2N(Q+R)m^2 e^{\frac{-m^2 t}{k}} \left(B_6 \frac{1}{b} I_1(mb) + B_7 \frac{1}{b} K_1(mb) \right) - \\ -2N(N+A+2Q+R)A_1(t) + 2NA_2(t)\frac{1}{b^2} - \\ -(A+2Q+R)D_1(t) - (Q+R)\Theta \end{pmatrix}$$
$$+ \frac{-B_{zz_p}P}{\pi(b^2 - a^2)} - B_{zz_p}T_0 \quad , \tag{35}$$

where the stress in the axial direction of the hollow cylinder is given by [18]

$$T_{zz} + T = \frac{-P}{\pi(b^2 - a^2)}. \tag{36}$$

To solve the above differential system the values of the known parameters must be substituted into Eqs (34) and (35). This can be done by inserting $A_1(t), A_2(t), D_1(t)$ given in Appendix I into Eqs (34) and (35) and taking [18]

$$\Theta = \frac{3A + 2N + 6Q + 3R}{Q + R}, \quad B_7 = 1 \text{ and } B_6 = -B_7 \frac{K_1(ma)}{I_1(ma)}. \tag{37}$$

The internal pressure $p(t)$ is given as

$$p(t) = \frac{1}{\Lambda_3}\left(\frac{\rho}{2} - \Lambda_1 P(t) - \Lambda_2\right), \tag{38}$$

where Λ_1, Λ_2 and Λ_3 are given in Appendix II and $P(t)$ is the axial force.

4. Numerical Solution and Results

In what follows the expressions of A, N, Q, R given by Biot and Willis [15] are employed, that is

$$Q = \frac{f\left(1 - f - \dfrac{\delta}{\kappa}\right)}{\gamma + \delta - \dfrac{\delta^2}{\kappa}}, \qquad R = \frac{f^2}{\gamma + \delta - \dfrac{\delta^2}{\kappa}}, \qquad A = \frac{\dfrac{\gamma}{\kappa} + f^2 + (1 - 2f)\left(1 - \dfrac{\delta}{\kappa}\right)}{\gamma + \delta - \dfrac{\delta^2}{\kappa}} - \frac{2}{3}\mu,$$

and $\quad N = \mu_b,$

(39)

where f is the porosity, κ the coefficient of jacketed compressibility, δ the coefficient of unjacketed compressibility, γ the coefficient of fluid content and μ_b the shear modulus of bone.

Taking $f = 0.02$ and assuming that $\mu_b = 5.5 GPa$, $\kappa = 5.5(GPa)^{-1}$, $\delta = 0.02(GPa)^{-1}$, and $\gamma = 4.68(GPa)^{-1}$ we obtain a numerical expression for N, A, Q and R.

The material properties of the implant are taken as $\lambda_p = 120 GPa$ and $\mu_p = 80 GPa$ [16].

The internal radius of the bone at $t = 0$ is taken as $a_0 = 10 mm$ and the external one as $b_0 = 15 mm$. The ratio of ρ to a_0 is assumed to be 0.005. The axial force is taken as a constant and equal to $P = 1631N$. The internal pressure p can be calculated from Eq. (38).

Using Eq. (29) and the expressions for B_6 and B_7, Eqs. (34) and (35) can be simplified to

$$-\frac{da}{dt} = B_{rr_e}\left(\begin{array}{l} -2N(N + A + 2Q + R)A_1(t) + 2NA_2(t)\dfrac{1}{a^2} - \\ -(A + 2Q + R)D_1(t) - (Q + R)\Theta \end{array} \right) + \frac{-B_{zz_e}P}{\pi(b^2 - a^2)} - B_{zz_e}T_0,$$

(40)

and

$$\frac{db}{dt} = B_{rr_p} \left(\begin{array}{c} -2N(N+A+2Q+R)A_1(t) + 2NA_2(t)\frac{1}{b^2} - \\ -(A+2Q+R)D_1(t) - (Q+R)\Theta \end{array} \right) + \frac{-B_{zz_p}P}{\pi(b^2-a^2)} - B_{zz_p}T_0 \; , \quad (41)$$

respectively.

The remodeling coefficients $B_{rr_e}, B_{rr_p}, B_{zz_e}, B_{zz_p}$ are calculated from Eq. (22) by using the values of the remodeling coefficients C_{ij_p} and C_{ij_e}, where $i,j=r,\theta,z$, given by Misra *et al.* [8], as

$$B_{rr_e} = -5.78697 \times 10^{-9} \, ms^{-1}GPa^{-1},$$

$$B_{zz_e} = 2.60312 \times 10^{-8} \, ms^{-1}GPa^{-1}, \quad (42)$$

$$B_{rr_e} = -5.78697 \times 10^{-9} \, ms^{-1}GPa^{-1},$$

and

$$B_{rr_e} = -5.78697 \times 10^{-9} \, ms^{-1}GPa^{-1} \; .$$

The values of the coefficients C_e^0, C_p^0 are calculated from the relations

$$C_e^0 = B_{zz_e}T_0,$$
$$\quad (43)$$
$$C_p^0 = B_{zz_p}T_0,$$

where T_0 is the initial reference value of stress before remodeling starts which is assumed to be $T_0 = -0.5MPa$ [10].

The system of Eqs. (41) and (42) with initial conditions $a(t=0) = a_0 = 0.01m$ and $b(t=0) = b_0 = 0.015m$ is then numerically solved by using the Runge-Kutta method.

Figs. 3 and 4 show the variation of the endosteal and periosteal radius with time, respectively, for constant initial endosteum radius and gradually increasing initial periosteum radius, for a constant value of porosity $f = 0.02$. It can be seen that as the time of remodeling evolves both the endosteum and periosteum radius increases.

Figs. 5 and 6 show the variation of the endosteal and periosteal radius with time, respectively, for constant initial bone thickness and gradually increasing initial periosteum and periosteum radius, for a constant value of porosity $f = 0.02$. The bone thickness increases with time but the rate of increase is slower when the initial radii are larger as is shown in Fig. 7. Increase of the porosity f causes no considerable change of the endosteum and periosteum radii in the physiological (for bone) porosity range 0.02 to 0.03.

5. Discussion

A theoretical analysis of surface remodeling induced by the forced fit of a medullary pin in a hollow cylindrical poroelastic bone model has been presented. A common situation when this occurs is when an endoprosthesis is fitted into the medulla of a femoral bone. In the proposed model, the bone is modelled as a hollow poroelastic isotropic cylinder, consisting of a solid elastic bone matrix and interstitial fluid flowing through the interconnected pores inside the matrix. The prosthesis is modelled as a solid elastic isotropic cylindrical rod which is forced into the cylindrical cavity of the bone. The formulation of the problem is based on the three-dimensional theory of consolidation for poroelastic media introduced by Biot [12, 13, 14]. A constitutive relation for the poroelastic bone, which incorporates the surface bone remodeling process is proposed. The contribution of the fluid term to the remodeling process is clearly indicated. Using Biot and Willis [15] expression for the elastic constants of a poroelastic medium and the Cowin et al. [2, 6] expression for the speed of the remodeling surfaces, a new surface remodeling rate equation is proposed.

The values of the surface remodeling coefficients B_{ij} which have been employed have been derived by making use of the values of the coefficients C_{ij} given in Misra et al. [11]. The surface remodeling of the endosteal and periosteal surfaces is expressed as a temporal variation of the radii of the corresponding cylindrical surfaces.

According to our results, both the periosteal and endosteal radii increase as the remodeling evolves, which corresponds to bone deposited on the periosteum and absorbed from the endosteum, as a result of the applied interior radial pressure from the medullary pin. This is also predicted by other models presented [6, 11] and experimental studies [7, 8, 9, 10]. This means that the insertion of the medullary pin will gradually cause bone loosening and finally failure of the implant. Bone deposition on the periosteum is stronger than bone resorption on the endosteum, which leads to net increase of bone thickness with remodeling.

Increasing the initial thickness of the bone matrix material lying between the endosteum and periosteum surfaces for a constant periosteum radius results in more prominent remodeling on the endosteum and periosteum. In this case, the percentage increase of the bone thickness is larger. Keeping a constant initial thickness of the bone and gradually increasing the periosteum and endosteum radius results in less prominent remodeling on the endosteum and periosteum and the net bone thickness totally decreases. Finally, a variation of porosity in the range 0.020 to 0.030 (normal for bone) causes no observable change in the depiction of surface remodeling.

To our knowledge, the present work constitutes the first attempt to introduce the poroelasticity theory in the formulation of the problem of surface remodeling induced by a medullary pin. The use of Biot's formulation for the material description of bone offers a direct visualisation of the contribution of the fluid term. The employment of values for the surface remodeling rate coefficients derived using our model assumptions from corresponding experimental values given in the literature, yields results whose order of magnitude lies within the predicted experimental range for the endosteum and periosteum radius rate of change. This is a clear indication that the suggested solution in our model allows a realistic estimate of the relative rate of surface remodeling in a bone model.

The work presented here could be extended in the future to include a more realistic symmetry for bone such as transverse isotropy or orthotropy [19]. In addition, the cavity of the hollow poroelastic cylinder could be assumed to be filled with fluid which is free to flow in and out of the pores of the bone matrix.

Appendix I

$$A_1(t) = \frac{(A+2Q+R)P(t) + a^2\pi(A+2N+2Q+R)p(t) + 2N\pi(b^2-a^2)(Q+R)\Theta}{2(a^2-b^2)N\pi(3A+2N+6Q+3R)},$$

$$A_2(t) = -\frac{a^2b^2p(t)}{2(b^2-a^2)N},$$

$$D_1(t) = \frac{-(A+2Q+R+N)P(t) - a^2\pi(A+2Q+R)p(t) + N\pi(b^2-a^2)(Q+R)\Theta}{(a^2-b^2)(3A+2N+6Q+3R)}.$$

Appendix II

$$\Lambda_1 = \frac{-a(A+2Q+R)}{(a^2-b^2)N\pi(3A+2N+6Q+3R)},$$

$$\Lambda_2 = \frac{2a(Q+R)\Theta}{(3A+2N+6Q+3R)},$$

$$\Lambda_3 = \left(\frac{ab^2}{(b^2-a^2)N} - \frac{a^3(N+A+2Q+R)}{(a^2-b^2)N(3A+2N+6Q+3R)} + \frac{(\lambda+2\mu)a}{2\mu(3\lambda+2\mu)} \right).$$

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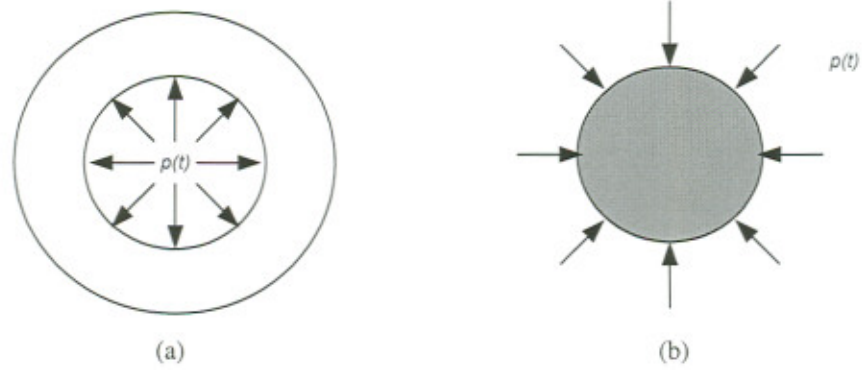
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(a)   (b)

Figure 1: (a) a hollow isotropic poroelastic cylinder of initial internal radius a_0 and initial external radius b_0 subjected to an internal radial pressure $p(t)$.

      (b) an elastic isotropic solid cylinder of radius $a_0 + \dfrac{\rho}{2}$ subjected to an external radial pressure $p(t)$.
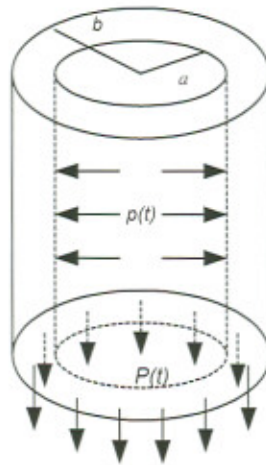

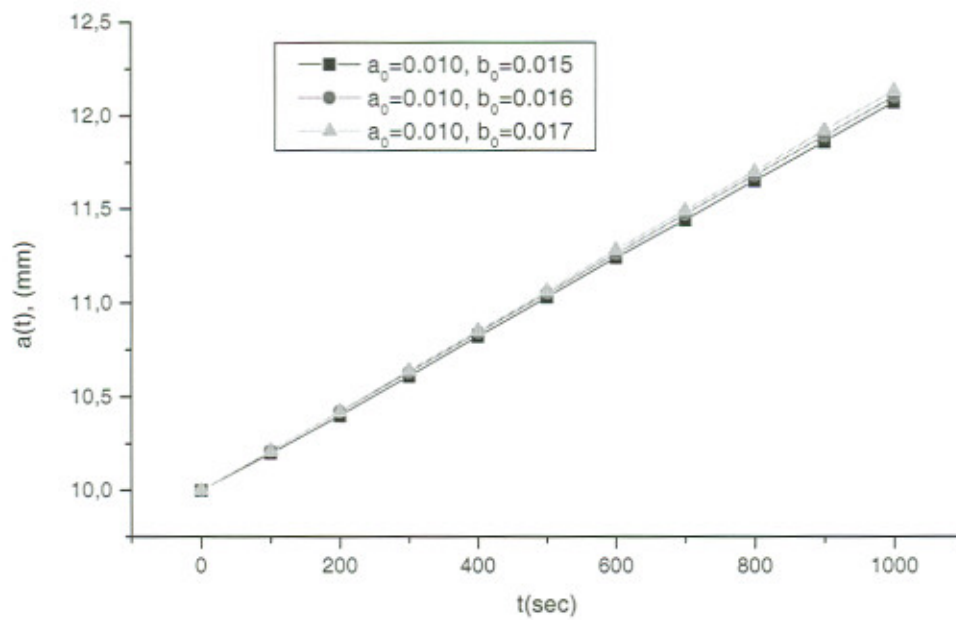**Figure 2:** The poroelastic hollow cylinder is subjected to an axial load $P(t)$ and a radial internal pressure $p(t)$.

Figure 3: Variation of the endosteum radius with time for constant initial periosteum radius and gradually increasing bone thickness.
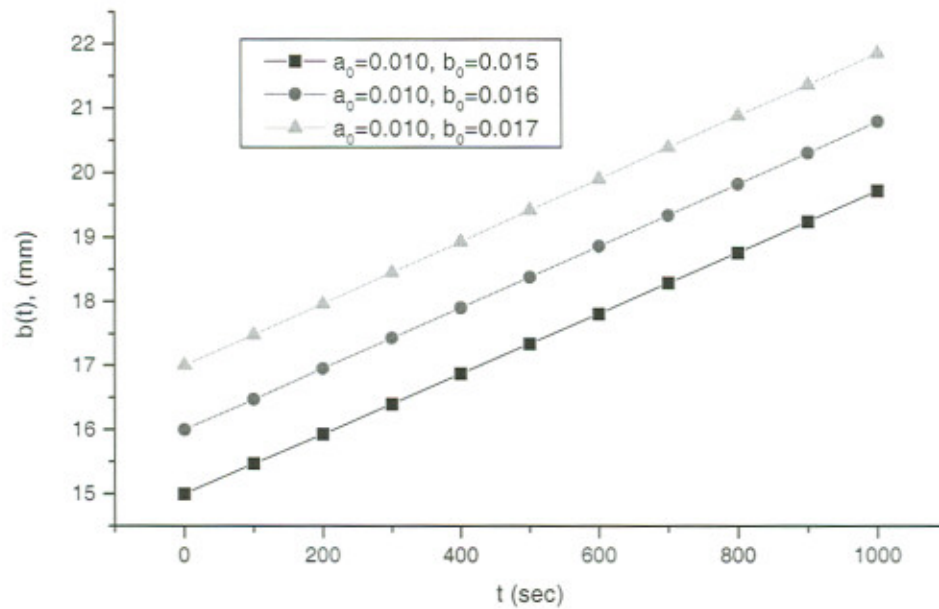
Figure 4: Variation of the periosteum radius with time for constant initial periosteum radius and gradually increasing bone thickness.
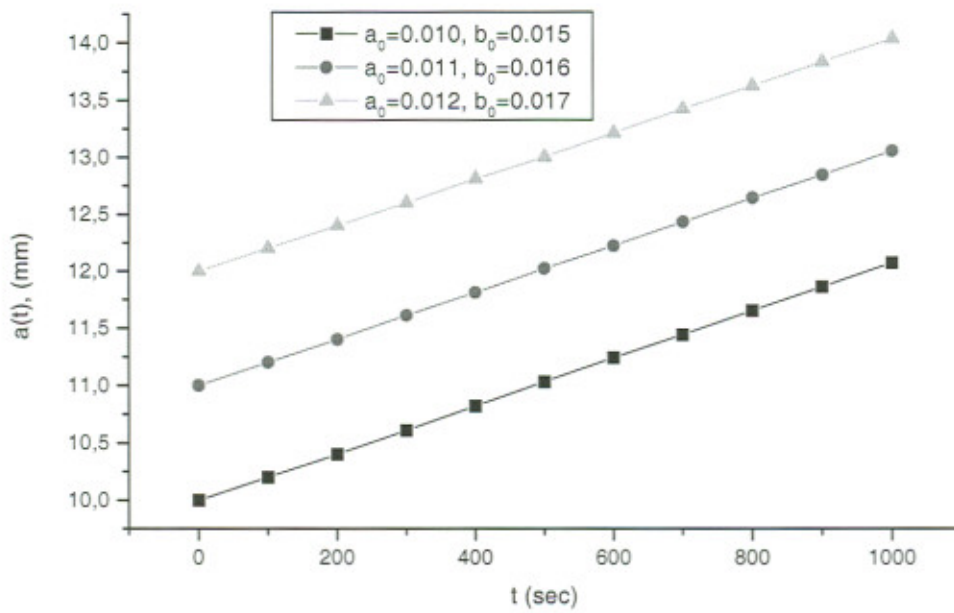
Figure 5: Variation of the endosteum radius with time for constant initial bone thickness and gradually increasing endosteum and periosteum radius.
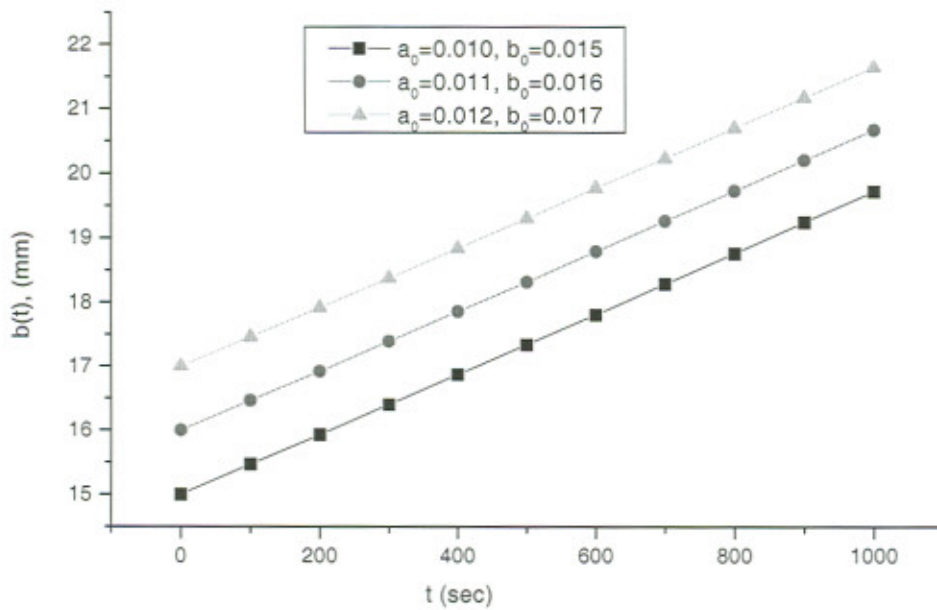

Figure 6: Variation of the periosteum radius with time for constant initial bone thickness and gradually increasing endosteum and periosteum radius.
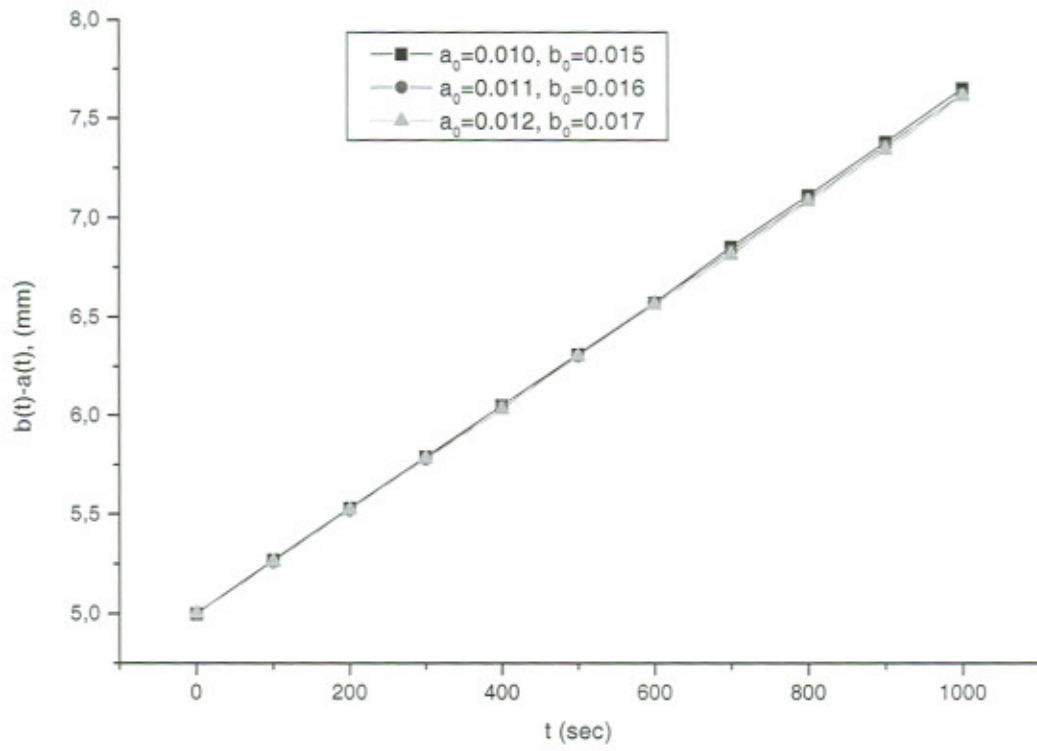
Figure 7: Variation of bone thickness with time for constant initial bone thickness and gradually increasing endosteum and periosteum radius.