SCATTERING OF A POINT GENERATED FIELD BY KIDNEY STONES

A. Charalambopoulos, D.I.Fotiadis and C.V. Massalas

19-2000

Preprint no. 19-00/2000

Department of Computer Science
University of Ioannina
451 10 Ioannina, Greece

Scattering of a Point Generated Field by Kidney Stones

A. Charalambopoulos
Division of Mathematics, Polytechnic School
Aristotle University of Thessaloniki, GR 54 006 Thessaloniki, Greece
E-mail: chara@eng.auth.gr

D.I. Fotiadis
Dept. of Computer Science
University of Ioannina, GR 451 10 Ioannina, Greece
E-mail: fotiadis@cs.uoi.gr

C.V. Massalas
Dept. of Mathematics
University of Ioannina, GR 451 10 Ioannina, Greece
E-mail: cmasalas@cc.uoi.gr

Abstract

In this work, we examine the acoustic scattering problem of spherical waves by a two-layer spheroid simulating the kidney-stone system. Both the theoretical as well as the numerical treatment are presented. The outcome of the analysis is the determination of the scattered field along with its multivariable dependence on the several physical and geometric parameters of the system. A comparison with the simpler case of spherical geometry is realised.

1 Introduction

In a previous communication [1] we have examined, presenting the theoretical formulation, the point source excitation acoustic scattering problem by a multilayer isotropic and homogeneous spheroidal body. Our analysis is based on the possibility to express the several fields entering the scattering process, in terms of the basis of spheroidal wave functions. The solvability of the problem reduces then to the determination of the expansion coefficients of the scattering and stationary waves occuring in the structure.

The solution procedure is to apply the boundary conditions on those expansions, taking place on the discontinuity surfaces. Extended exploitation of orthogonality properties of the involved spheroidal wave functions permits to transform the boundary conditions to an algebraic linear non-homogeneous system satisfied by the unknown expansion coefficients. The system is an infinite one and its specific structure very complicated. However, the general theoretical regime has been built in [1].

In this paper, we adapt this theoretical background to the investigation of a particular case consisting of a two-layer spheroid simulating the system of interest for medicine kidney - stone. The necessity to solve such a problem stems first from the importance to implement the aforementioned analysis to specific problems, affronting simultaneously a lot of peculiarities (convergence of the expansion, stability, etc.) arising when spheroidal geometry is present. In addition, the specific application problem merits special biomedical engineering interest due to the augmented percentage kidney stones appear in medical clients.

Indeed, kidney stones are very common. About 5% of women and 10% of men will have at least one episode by age 70. Kidney stones affect two out of 1,000 people. The common diagnostic tools used is kidney ultrasound, pyelogram, X-rays, etc. The first is of great importance since it is a non-invasive technique and offers a no risk examination. However, a point source generated field can be used for the patient examination and useful results can be obtained for the size, shape and relative position of the kidney stone. Those results can be interpreted by a physician who can suggest a treatment method.

To our knowledge, there is no previous, theoretical or numerical approach to the modelling, use of sound waves for the examination of stones in kidneys. Such an approach can offer valuable information and relate the characteristics of the stone to those of the scattered field. The latter can lead to the inverse scattering problem which needs further investigation. The treatment of kidney stones by shock waves is reviewed in [2].

In this work it is used the approach described in [1] and concerns acoustical scattering from kidney stones, i.e. the problem is simplified to a double layer homogeneous spheroidal body to be solved numerically. The scope is to measure characteristics of the scattered field which can be used for the identification of the morphology and other features of the kidney stones. The major assumption is a geometric one, since for simplicity we consider both the kidney and the stone being spheroids. Another assumption is related to the materials: the stone is supposed to be a rigid body and the surrounding material of kidney homogeneous, elastic and isotropic body.

The present analysis faces extensively the direct scattering problem. Furthermore, we present a multivariable analysis of the measured scattered wave with respect to several parameters of the problem aiming to build an inversion algorithm based on the direct problem structure.

2 The Model

A two - layer isotropic and homogeneous medium occupying region V, centered at the coordinate system origin, has been selected to represent the kidney - stone system under discussion. The geometry of the problem is shown in Fig. 1. The system fitting suitably with the problem geometry in the spheroidal one representing every point in $\mathbf{R}^3$ through the spheroidal coordinates (μ, θ, ϕ) is

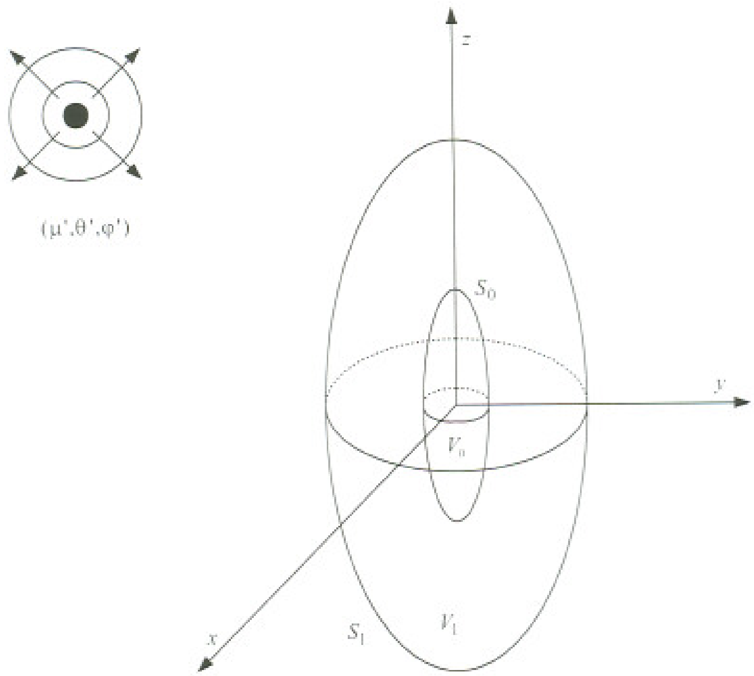

Figure 1: *The System Geometry.*

defined as

$$x = \frac{\alpha}{2}\sinh\mu\sin\theta\cos\phi,$$
$$y = \frac{\alpha}{2}\sinh\mu\sin\theta\sin\phi,$$
$$z = \frac{\alpha}{2}\cosh\mu\cos\theta,$$

with

$$\mu \geq 0,\ \ 0 \leq \theta \leq \pi,\ \ 0 \leq \phi \leq \pi.$$

The two -layer spheroid is reached by an acoustical spherical wave emanating from a point having position vector $\mathbf{r}' = (\mu', \theta', \phi')$. The central spheroidal region, which is surrounded by the surface S_0, corresponds to the kindey stone and is inpenetrable and rigid. The exterior region, surrounded by the surface S_1, corresponds to the kidney tissue. Those two regions (layers) are characterised by their densities ρ_0, ρ_1 and velocities of sound propagation c_0 and c_1, respectively. The surrounding region has density ρ and the velocity of sound is c_{ext}.

For the solution of this specific boundary value problem we follow the general formulation presented in [1], concerning the theoretical setting of the acoustical scattering of spherical waves by multi-layered spheroids.

The interference of the incident field $u^{\text{in}}(\mathbf{r})$ with the structure leads to the emanation of the scattered field in the surrounding space as well as the creation of acoustic field in the kidney tissue not penetrating the interior rigid body. The incident and the secondary fields obey to Helmholtz equation in their definition domains and are connected through boundary and impedence conditions on the discontinuity surfaces.

For the solution of the specific boundary value problem, we follow the general formulation presented in [1], concerning the theoretical setting of the acoustical scattering of spherical waves by multi-layered spheroids.

The aforementioned acoustic fields are expanded in a complete set of scalar Helmholtz equation solutions, known as spheroidal wave functions [1]. Those can be computed with methods described in [3].

More precisely, we have the representations

$$\begin{aligned} u^{(1)}(\mathbf{r}) &= \sum_{mn} \frac{1}{\Lambda_{mn}(c^{(1)})} \{\alpha^1_{mn}\cos(m\phi)S_{mn}(\cos\theta;c^{(1)})je_{mn}(\cosh\mu;c^{(1)}) \\ &+ \beta^1_{mn}\sin(m\phi)S_{mn}(\cos\theta;c^{(1)})je_{mn}(\cosh\mu;c^{(1)}) \\ &+ \gamma^1_{mn}\cos(m\phi)S_{mn}(\cos\theta;c^{(1)})ye_{mn}(\cosh\mu;c^{(1)}) \\ &+ \delta^1_{mn}\sin(m\phi)S_{mn}(\cos\theta;c^{(1)})ye_{mn}(\cosh\mu;c^{(1)})\}, \qquad \mathbf{r}\in V_1, \end{aligned} \tag{1}$$

$$\begin{aligned} u^{\mathrm{sc}}(\mathbf{r}) &= \sum_{n,m} \frac{1}{\Lambda_{mn}(c)} \{\alpha^{ext}_{mn}\cos(m\phi)S_{mn}(\cos\theta;c)he_{mn}(\cosh\mu;c) \\ &+ \beta^{ext}_{mn}\sin(m\phi)S_{mn}(\cos\theta;c)he_{mn}(\cosh\mu;c)\}, \qquad \mathbf{r}\in V. \end{aligned} \tag{2}$$

$$u^{\mathrm{in}}(\mathbf{r}) = 2ik\sum_{mn}\frac{1}{\Lambda_{mn}(c)}S_{mn}(\cos\theta;c)S_{mn}(\cos\theta';c)$$

$$\cos(m(\phi-\phi'))\begin{cases} je_{mn}(\cosh\mu';c)he_{mn}(\cosh\mu;c) & \mu>\mu' \\ je_{mn}(\cosh\mu;c)he_{mn}(\cosh\mu';c) & \mu<\mu' \end{cases}. \tag{3}$$

(4)

In the above relations, we meet the angular wave functions S_{mn} and the spheroidal radial functions $je_{mn}, ye_{mn}, he_{mn} = je_{mn} + iye_{mn}$. Their definition is given in [1], but the general concept is that these functions are represented as infinite expansions of the corresponding spherical functions. Notice here that the special selection of radial functions incorporates the outgoing propagating behavior of the scattered field.

In addition, the parameters $c^{(1)}, c$ are given as $c^{(1)} = \frac{1}{2}k_1\alpha$ and $c = \frac{1}{2}k\alpha$ (k, k_1 are the wave numbers) and constitute a measure of the relation between the geometric characteristic dimension of the system and the wavelengths. Λ_{mn} stand for the normalization constants of the angular functions S_{mn}.

The above expansions share the important privillege that the unknown character of the acoustic fields is transfered to the determination of the expansion coefficients orientating the problem to the linear algebra regime.

The solution process consists in forcing the expansions (1), (2), and (4) to satisfy the boundary conditions and exploit the orthogonality properties of the involved functions to obtain proper algebraic relations between the expansion coefficients.

We present briefly here the main steps of the analysis. The quantitative form of the boundary conditions is

$$\frac{\partial u^{(1)}}{\partial n} = 0, \qquad \mathbf{r} \in S_0. \tag{5}$$

$$\left.\begin{array}{r} \rho(u^{\rm sc} + u^{\rm in}) = \rho_1 u^{(1)} \\ \frac{\partial}{\partial n}(u^{\rm sc} + u^{\rm in}) = \frac{\partial}{\partial n} u^{(1)} \end{array}\right\}, \mathbf{r} \in S_1.$$

Equation (5) and orthogonality of $S_{mn}(\cos\theta; c)$ lead to the relations

$$\alpha^1_{mn} je'_{mn}(\cosh\mu_0; c^{(1)}) + \gamma^1_{mn} ye'_{mn}(\cosh\mu_0; c^{(1)}) = 0, \qquad m \geq 0, \qquad n = m, m+1, ..., \tag{6}$$

$$\beta^1_{mn} je'_{mn}(\cosh\mu_0; c^{(1)}) + \delta^1_{mn} ye'_{mn}(\cosh\mu_0; c^{(1)}) = 0, \qquad m \geq 1, \qquad n = m, m+1, ... \tag{7}$$

Unfortunately, Equations (2) involve angular function of different parameters $c, c^{(1)}$, which do not share mutual orthogonality. However, we can project on $S_{mn}(\cos\theta; c)$ to obtain algebraic equations but we cannot avoid infinite summation in the final expressions (in constrast to the separable Equations (6-7)).

More precisely, we obtain

$$\begin{aligned} &\rho[\alpha^{ext}_{mn} he_{mn}(\cosh\mu_1; c) \\ &+2ik\epsilon_m S_{mn}(\cos\theta'; c)\cos(m\phi' je_{mn}(\cosh\mu_1; c) he_{mn}(\cosh\mu'; c)] = \\ &\rho_1 \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(1)})}\{\alpha^1_{mn'} je_{mn'}(\cosh\mu_1; c^{(1)}) + \\ &\gamma^1_{mn'} ye_{mn'}(\cosh\mu_1; c^{(1)})\} < S^1_{mn'}, S_{mn} >, \end{aligned} \tag{8}$$

$$\begin{aligned} &\rho[\beta^{ext}_{mn} he_{mn}(\cosh\mu_1; c) \\ &+2ik\epsilon_m S_{mn}(\cos\theta'; c)\sin(m\phi') je_{mn}(\cosh\mu_1; c) he_{mn}(\cosh\mu'; c)] = \\ &\rho_1 \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(1)})}\{\beta^1_{mn'} je_{mn'}(\cosh\mu_1; c^{(1)}) + \\ &\delta^1_{mn'} ye_{mn'}(\cosh\mu_1; c^{(1)})\} < S^1_{mn'}, S_{mn} >, \end{aligned} \tag{9}$$

$$\begin{aligned} &\alpha^{ext}_{mn} he'_{mn}(\cosh\mu_1; c) \\ &+2i\epsilon_m k S_{mn}(\cos\theta'; c)\cos(m\phi') je'_{mn}(\cosh\mu_1; c) he_{mn}(\cosh\mu'; c) = \\ &\sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(1)})}\{\alpha^1_{mn'} je'_{mn}(\cosh\mu_1; c^{(1)}) \\ &+\gamma^1_{mn'} ye'_{mn'}(\cosh\mu_1; c^{(1)})\} < S^1_{mn'}, S_{mn} >, \end{aligned} \tag{10}$$

$$\beta^{ext}_{mn} he'_{mn}(\cosh\mu_1; c)$$

$$+2i\epsilon_m k S_{mn}(\cos\theta';c)\sin(m\phi')je'_{mn}(\cosh\mu_1;c)he_{mn}(\cosh\mu';c) = \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(1)})}\{\beta^1_{mn'}je'_{mn}(\cosh\mu_1;c^{(1)}) + \delta^1_{mn'}ye'_{mn'}(\cosh\mu_1;c^{(1)})\} < S^1_{mn'}, S_{mn} >, \tag{11}$$

where brackets $< S'_{mn'}, S_{mn} >$ indicate the "inner products"

$$< S'_{mn'}, S_{mn} >= \int_{-1}^{1} S_{mn'}(\eta;c^{(1)})S_{mn}(\eta;c)d\eta. \tag{12}$$

We define

$$\mathbf{\Delta}_1^{m,n} = \begin{bmatrix} \rho_1 je_{mn}(\cosh\mu_1;c^{(1)}) & \rho_1 ye_{mn}(\cosh\mu_1;c^{(1)}) \\ je'_{mn}(\cosh\mu_1;c^{(1)}) & ye'_{mn}(\cosh\mu_1;c^{(1)}) \end{bmatrix}$$

$$\mathcal{B}_1^{m,n,n'} = \frac{< S^1_{mn'}, S_{mn} >}{\Lambda_{mn'}(c^{(1)})}\mathbf{\Delta}_1^{m,n'} \tag{13}$$

$$\mathbf{y}_{m,n} = [\rho he_{mn}(\cosh\mu_1;c), he'_{mn}(\cosh\mu_1,c)]^T \tag{14}$$

$$\omega_{m,n} = [je'_{mn}(\cosh\mu_0;c^{(1)}), ye'_{mn}(\cosh\mu_0;c^{(c1)})] \tag{15}$$

$$\mathbf{B}^m_{n,n} = \begin{bmatrix} \omega_{m,n} & 0 \\ -\mathcal{B}_1^{m,n,n} & \mathbf{y}_{m,n} \end{bmatrix}$$

$$\mathbf{B}^m_{n,n'} = \begin{bmatrix} \mathbf{0} & 0 \\ -\mathcal{B}_1^{m,n,n'} & \mathbf{0}^T \end{bmatrix}$$

for $n \neq n'$ and $\mathbf{0} = (0,0)$

The final non-homogeneous system of equations is

$$\mathbf{D}^{(m)}\mathbf{x}^{(m)} = \mathbf{b}^{(m)} \tag{16}$$

where

$$\mathbf{D}^{(m)} = \begin{bmatrix} \mathbf{B}^m_{m,m} & \mathbf{B}^m_{m,m+1} & \mathbf{B}^m_{m,m+2} & \cdots & \cdots \\ \mathbf{B}^m_{m+1,m} & \mathbf{B}^m_{m+1,m+1} & \mathbf{B}^m_{m+1,m+2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{B}^m_{m+\nu,m} & \mathbf{B}^m_{m+\nu,m+1} & \cdots & \mathbf{B}^m_{m+\nu,m+\nu} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

$$\mathbf{x}^{(m)} = [\mathbf{x}^{(m)}_m, \mathbf{x}^{(m)}_{m+1}, ..., \mathbf{x}^{(m)}_n, ...]^T, \tag{17}$$

$$\mathbf{x}^{(m)}_n = [\alpha^1_{mn}, \gamma^1_{mn}, \alpha^{ext}_{mn}] \tag{18}$$

or

$$\mathbf{x}^{(m)}_n = [\beta^1_{mn}, \delta^1_{mn}, \beta^{\text{ext}}_{mn}] \tag{19}$$

and

$$\mathbf{b}^{(m)} = [\mathbf{b}^{(m)}_m, \mathbf{b}^{(m)}_{m+1}, ..., \mathbf{b}^{(m)}_n, ...] \tag{20}$$

where

$$\mathbf{b}^{(m)}_n = [0, -\rho 2ik\epsilon_m S_{mn}(\cos\theta';c)\cos(m\phi')je_{mn}(\cosh\mu_1;c)he_{mn}(\cosh\mu';c), -2ik\epsilon_m S_{mn}(\cos\theta';c)\cos(m\phi')je'_{mn}(\cosh\mu_1;c)he_{mn}(\cosh\mu';c)] \tag{21}$$

(in the case of $\beta^1_{mn}, \delta^1_{mn}, \beta^{\text{ext}}_{mn}$, the function $\cos(m\phi')$ is replaced by $\sin(m\phi')$).

3 Numerical Solution - Results

The non-homogeneous system (16) is solved numerically. For given m the system (16) is truncated to a finite system of dimension $3 \times N$

$$\mathbf{D}_N^{(m)} \mathbf{x}_N^{(m)} = \mathbf{b}_N^{(m)} \tag{22}$$

such that $||u_N^{\mathrm{sc}} - u_{N+1}^{\mathrm{sc}}|| \leq 10^{-6}$. This is performed by an iterative procedure. We observe that the value of N is strongly dependent on the problem characteristics.

The elements of $\mathbf{D}_N^{(m)}$ depend on the computation of spheroidal angular and radial wave functions and their derivatives of the first, second and third kind [3]. The presence of the latter introduces complex terms and the system (21) is translated to

$$\mathbf{D}_{N_r}^{(m)} \mathbf{x}_{N_r}^{(m)} + i\mathbf{D}_{N_i}^{(m)} \mathbf{x}_{N_i}^{(m)} = \mathbf{b}_{N_r}^{(m)} + i\mathbf{b}_{N_i}^{(m)}, \tag{23}$$

where the subscripts r and i denote real and imaginary parts, respectively.

The system (22) can be easily transformed to the inflated equivalent

$$\begin{bmatrix} \mathbf{D}_{N_r}^{(m)} & -\mathbf{D}_{N_i}^{(m)} \\ \mathbf{D}_{N_i}^{(m)} & \mathbf{D}_{N_r}^{(m)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{N_r}^{(m)} \\ \mathbf{x}_{N_i}^{(m)} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{N_r}^{(m)} \\ \mathbf{b}_{N_i}^{(m)} \end{bmatrix}$$

of order $2 \times (3 \times N)$.

The above solution is repeated for $m = 0, 1, 2, \ldots$ and the field in the kidney tissue and the scattered field are computed from (1) and (5), respectively.

The quantities $< S^1_{m,n'}, S_{m,n} >$ and $\Lambda_{mn}(c^{(1)})$ are computed as

$$\Lambda_{mn}(c^{(1)}) = \sum_k{}' |d_k^{m,n}(c^{(1)})|^2 \frac{2}{2k+2m+1} \frac{(k+2m)!}{k!}, \tag{24}$$

and

$$< S^1_{mn'}, S_{mn} >= \sum_k{}' d_k^{mn'}(c^{(1)}) d_k^{mn}(c) \frac{2}{2k+2m+1} \frac{(k+2m)!}{k!}, \tag{25}$$

where $d_k^{mn'}(c^{(1)}), d_k^{mn}(c)$ are the expansion coefficients which are computed following [4].

3.1 The Limiting Case of a Sphere

The investigation of the corresponding two-layer spherical structure is very helpful for the understanding of our problem. To alleviate the analysis, without loss of generality, we assume the point source to be located on z - axis at distance r_0 from the spherical system origin, which coincides with the centres of the two spheres of radii α, β with $\beta > \alpha$. The fields under investigation obtain the much simpler form

$$u^{\mathrm{in}} = \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} = ik \sum_{n=0}^{+\infty} (2n+1) j_n(kr) h_n^{(1)}(kr_0) P_n(\cos\theta), \tag{26}$$

$$u^{(1)}(\mathbf{r}) = \sum_n [\alpha_n j_n(k_1 r) + \beta_n y_n(k_1 r)] P_n(\cos\theta), \tag{27}$$

and

$$u^{\text{sc}}(\mathbf{r}) = \sum_n \gamma_n h_n^{(1)}(kr) P_n(\cos\theta), \tag{28}$$

where P_n are the Legendre polynomials, while j_n, y_n stand for the spherical Bessel functions and $h_n^{(1)}$ for the spherical Hankel function of the first kind. The boundary condition on S_0 is

$$\frac{\partial u_1}{\partial n} = 0, \tag{29}$$

leading to

$$\alpha_n j'_n(k_1\alpha) + \beta_n y'_n(k_1\alpha) = 0 \tag{30}$$

while on S_1

$$\frac{\partial u^{(1)}}{\partial n} = \frac{\partial u^{\text{in}}}{\partial n} + \frac{\partial u^{\text{sc}}}{\partial n}, \tag{31}$$

gives

$$k_1\alpha_n j'_n(k_1\beta) + k_1\beta_n y'_n(k_1\beta) = k\gamma_n h'^{(1)}_n(k^2\beta) + ki(2n+1)h_n^{(1)}(kr_0)j'_n(k\beta), \tag{32}$$

and

$$\rho_1 u^{(1)} = \rho(u^{\text{in}} + u^{\text{sc}}), \tag{33}$$

provides

$$\begin{aligned} \rho_1[\alpha_n j_n(k_1\beta) + \beta_n y_n(k_1\beta)] &= \rho\gamma_n h_n^{(1)}(k\beta) \\ &+ \rho ik(2n+1) j_n(k\beta) h_n^{(1)}(kr_0) \end{aligned} \tag{34}$$

The boundary conditions can be written in matrix form as

$$\begin{bmatrix} j'_n(k_1\alpha) & y'_n(k_1\alpha) & 0 \\ k_1 j'_n(k_1\beta) & k_1 y'_n(k_1\beta) & -kh'^{(1)}_n(k\beta) \\ \rho_1 j_n(k_1\beta) & \rho_1 y_n(k_1\beta) & -\rho h_n^{(1)}(k\beta) \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ k^2 i(2n+1) h_n^{(1)}(kr_0) j'_n(k\beta) \\ \rho ki(2n+1) j_n(k\beta) h_n^{(1)}(kr_0) \end{bmatrix}.$$

The above system is solved (see Ref. [4]) for every n and particularly the scattered field is obtained using Eq. (28).

The following properties for the kidney material and the surrounding medium are used in our computations

$$\rho_1 = 1,022Kg/m^3, c_1 = 1,533m/sec.$$

$$\rho_0 = 1,000Kg/m^3, c_{\text{ext}} = 1,493m/sec.$$

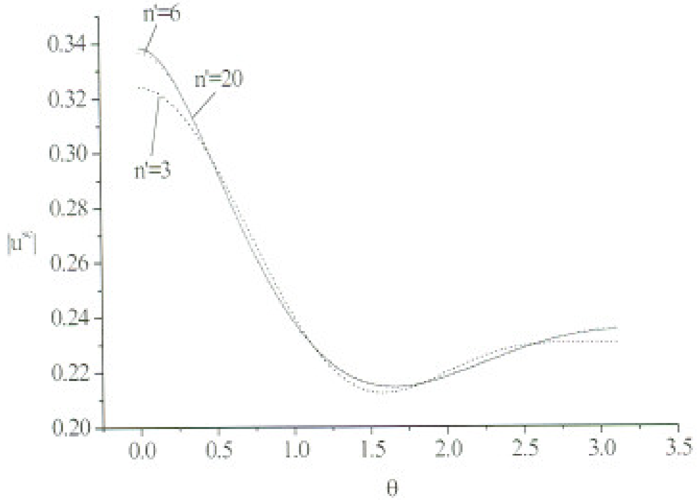

Figure 2: $|u^{sc}|$ *as a function of* θ *for* $n' = 3, n' = 6$ *and* $n' = 20$.

The summation in Equation (28) depends on the value of n', which ensures convergence of $|u^{\rm sc}|$. This computation has been done by using an iterative procedure and the results obtained are shown in Fig. 2. The value of n' for convergence is strongly dependent on k, k_1, α and β. In the case under discussion, for fixed values of k, k_1, α and β, the computational procedure was repeated until $||u^{\rm sc}|_{n'} - |u^{\rm sc}|_{n'+1}|| \approx O(10^{-4})$.

Figs. 3 and 4 show the dependence of $|u^{\rm sc}|$ and $|u^{\rm in}|$ on θ and the distance from the center of the sphere for $k\alpha = 0.9240$ and $k_1\alpha = 0.2244$ which correspond to surrounding air and kidney tissue with sound speed $c_1 = 1533\, m/sec$. It is shown that the scattered field decreases as the distance from the center of the sphere (r/α) increases and the incident field increases with decreasing distance.

Fig. 5 indicates the dependence of the scattered field on the size (α) of the kidney stone for water surrounding $c = 340 m/sec$ and kindey tissue with $c_1 = 1533 m/sec$ which corresponds to $k\alpha = 0.263$ and $k_1\alpha = 0.256$. It is shown that the scattered field increases as the size of the spherical kidney stone increases. The scattered field disappears as the radius of the sphere becomes very small. This observation indicates that the use of acoustic waves can be used for the determination of the kidney stone size.

3.2 The Spheroidal Kidney

The results for the spheroidal geometry are shown in Figs. 6 - 9. The relations given in Appendix A are used to represent the obtained quantities in spherical geometry. The shape dependence of the scattered field is shown in Fig. 6 as

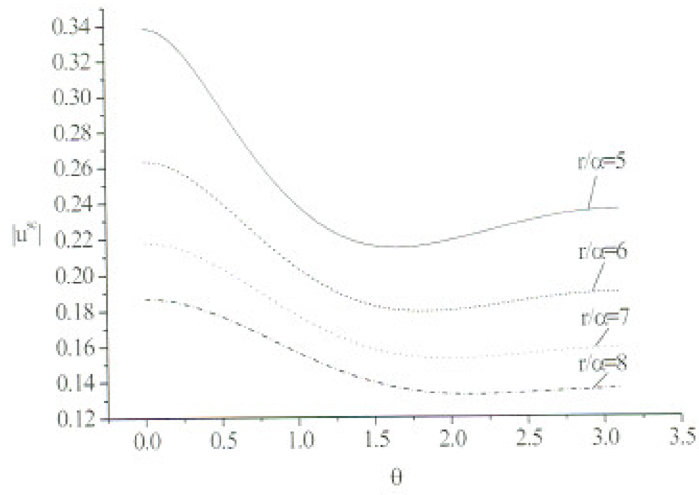

Figure 3: $|u^{sc}|$ *as a function of* θ *and distance from the center of the sphere* r/α.

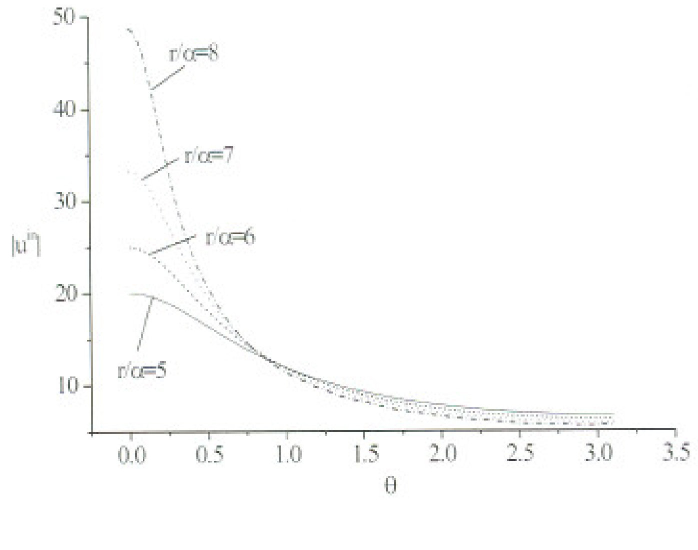

Figure 4: $|u^{in}|$ *as a function of* θ *and distance from the center of the sphere* r/α.

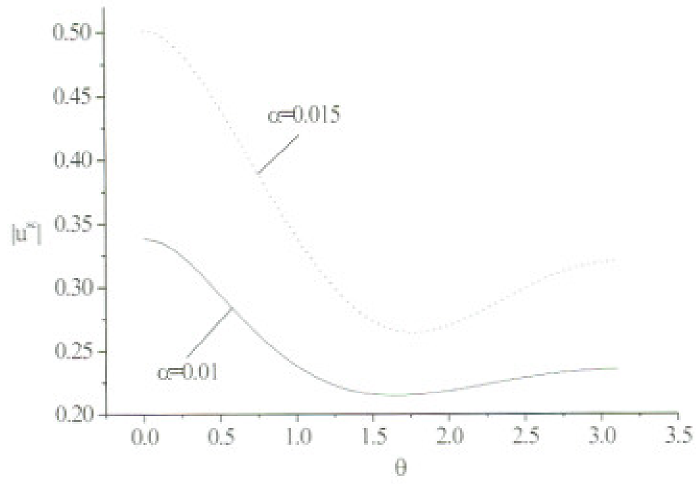

Figure 5: $|u^{sc}|$ *as a function of* θ *and size of the stone* α.

the spheroid deviates from the spherical geometry ($\alpha_0/\beta_0 = 1$). This shows that the scattered wave can provide with information on the spheroidal ratio of radii. The dependence of the scattered wave, measured on the minimum sphere containing the kidney, on the distance of the point source is shown in Fig. 7. The position of the diagnostic equipment should be as close as possible to the kidney outside surface and if it possible in contact with it. Usually a lubricant is used to obtain the optimum contact. Our results show that if a medium with smaller sound speed exists the obtained scattered field (and the received signal) decreases significantly (Fig. 9).

Another important parameter which enters the problem is the size of the spheroidal stone which affects the scattered field. An increase in the scattered field is obtained with increasing size of the kidney stone (Fig. 8).

4 Concluding Remarks

In this work we examined the acoustic scattering problem of spherical waves by the kidney-stone system. Our analysis is based on the expression of the fields entering the problem in terms of the basis of spheroidal wave function. The determination of the expansion coefficients is an easy process which minimises the computational effort required.

The multivariable parametrization we performed indicates that the process can be proven a useful tool in clinical practice in order to predict non-invasively the size and the shape of kidney stones. However, some practical difficulties might enter since the sound transmitter should be in touch with the human body and in the resonance region framework, the distance between the source and the scatterer is not large at all.

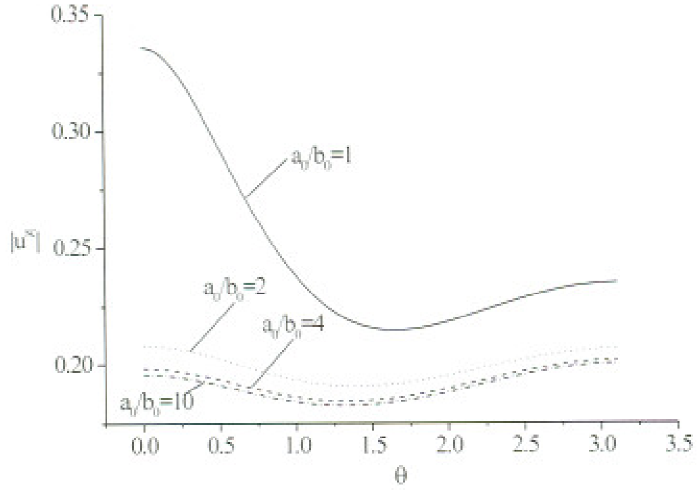

Figure 6: $|u^{sc}|$ *as a function of* θ *and ratio of kidney radii* (α_0/β_0)

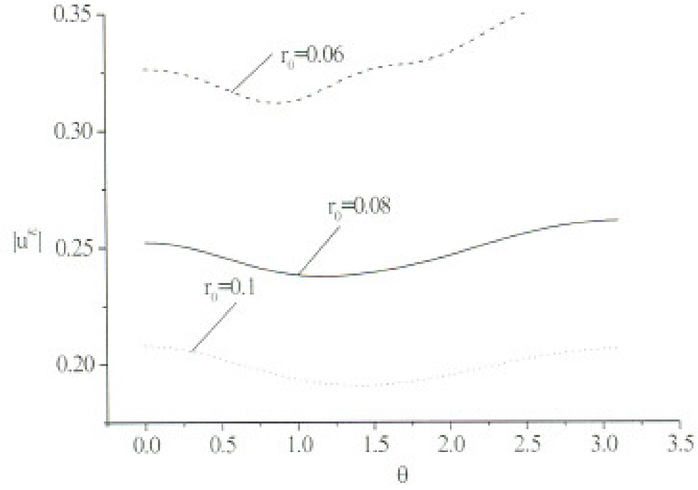

Figure 7: $|u^{sc}|$ *as a function of* θ *and distance of the point source* (r_0).

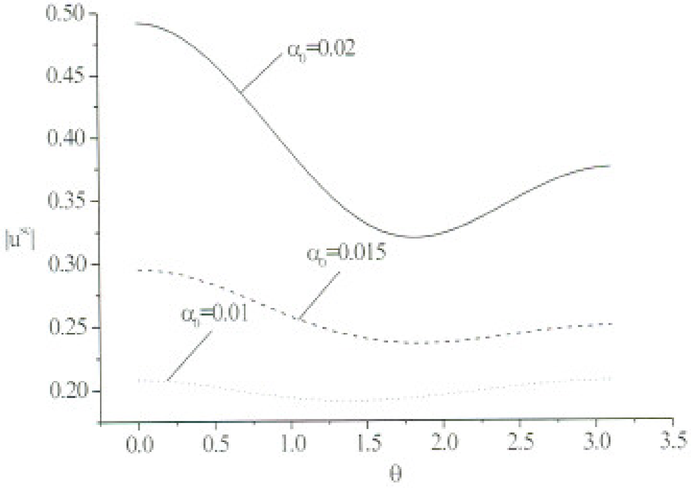

Figure 8: $|u^{sc}|$ *as a function of* θ *and size of the kidney* (α_0).

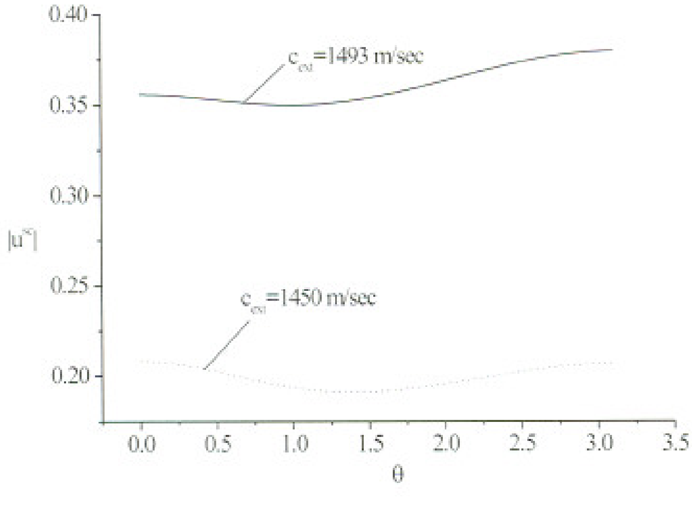

Figure 9: $|u^{sc}|$ *as a function of* θ *and speed of sound in the external medium* (c_{ext}).

The geometric approximation we treated is very close to the realistic but other geometries can be investigated too. The results of the direct problem we solve can be used in a database which along with pattern recognition techniques can be used to build an inversion algorithm to provide with accurate diagnostic decisions.

5 Appendix A

The computations for the spheroidal kidney correspond to the coordinate θ which is transformed for better understanding to the coordinate $\theta_{\text{spherical}}$. In this case the scattered and incident field are computed as

$$u^{\text{sc}}(r = r_1, \theta_{\text{spherical}}) = \sum_{n' \geq 0} \frac{1}{\Lambda_{0n'}(c)} \alpha_{0n'}^{\text{ext}} S_{0n'}(\cos\theta; c) he_{0n'}(\cosh\mu'; c),$$

and

$$u^{\text{in}}(r = r_1, \theta_{\text{spherical}}) =$$

$$2ik \sum_{n' \geq 0} \frac{1}{\Lambda_{0n'}(c)} \alpha_{0n'}^{\text{ext}} S_{0n'}(1; c) S_{0n'}(\cos\theta; c) je_{0n'}(\cosh\mu'; c) he_{0n'}(\cosh\mu; c), \quad \mu > \mu',$$

$$u^{\text{in}}(r = r_1, \theta_{\text{spherical}}) =$$

$$2ik \sum_{n' \geq 0} \frac{1}{\Lambda_{0n'}(c)} \alpha_{0n'}^{\text{ext}} S_{0n'}(1; c) S_{0n'}(\cos\theta; c) je_{0n'}(\cosh\mu; c) he_{0n'}(\cosh\mu'; c), \quad \mu < \mu',$$

respectively, where

$$\cos\theta = \frac{\sqrt{2}\cos\theta_{\text{spherical}}}{[(r_1^2 + \frac{\alpha^2}{4}) + \sqrt{(r_1^2 + \frac{\alpha^2}{4})^2 - \alpha^2 r^2 \cos^2\theta_{\text{spherical}}}]^{\frac{1}{2}}},$$

and

$$\cosh\mu = [1 + \frac{4}{\alpha^2} r_1^2 - \frac{2r_1^2 \cos^2\theta_{\text{spherical}}}{r_1^2 + \frac{\alpha^2}{4} + \sqrt{(r_1^2 + \frac{\alpha^2}{4})^2 - \alpha^2 r_1^2 \cos^2\theta_{\text{spherical}}}}]^{\frac{1}{2}}.$$

References

[1] A. Charalambopoulos, G. Dassios, D.I. Fotiadis and C.V. Massalas, *Scattering of a Point Generated Field by a Multilayered Spheroid*, *Acta Mechanica* (accepted) (2000).

[2] L. Howle, D.G. Schraffer, M. Shearer and P. Zhong, *Lithotripsy: The Treatment of Kidney Stones with Shock Waves*, SIAM Review 40(2), 356-371 (1998).

[3] S. Zhang and J. Jin, *Computation of Special Functions*, John Wiley and Sons, New York (1996).

[4] W.H. Press, S.A. Teukolsky, W.T. Vettering and B.P. Flannery, *Numerical Recipes in FORTRAN, The Art of Scientific Computing*, 2nd Edition, Cambridge University Press, Cambridge (1992).