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Abstract

A theoretical analysis for internal bone remodeling induced by a medullary pin is presented. Bone is treated as a poroelastic material using the Biot formulation. Based on the theory of small-strain adaptive elasticity as described by Cowin and Hegedus, a new theoretical approach for internal remodeling is proposed. Our results show that the rate of internal remodeling decreases as the porosity of the bone increases.

Keywords: Bone Modeling; Internal Remodeling; Poroelasticity.

1. Introduction

Bone remodeling is a general term which describes the processes by which bone adapts its histological structure to changes in long duration loading. Following the distinction made by Frost [1] between surface and internal remodeling, internal remodeling refers to the mechanism by which the bulk density of an osseous tissue changes, by means of resorption or reinforcement of existing bone within fixed external boundaries.

The problem of internal bone remodeling induced by a medullary pin has been investigated by Cowin and Van Buskirk [2]. Their approach is based on the small-strain approximation [3] of the thermomechanical continuum theory of adaptive elasticity developed by Cowin and co-workers [4, 5]. Bone is treated as a porous elastic solid representing the matrix structure of bone including the bone cells and a perfusate which represents the extracellular fluid and the blood plasma which flows through the matrix structure. The perfusate is accounted for, only insofar as it transfers mass, momentum, energy or entropy to the bone matrix. They assume that the load-adapting properties of living bone can be modeled by a chemically reacting porous medium in which the rate of reaction is strain-controlled. A modified Hooke's law that incorporates a remodeling term is presented. Their proposed remodeling rate equation uses an approximation with quadratic terms in the internal remodeling magnitude and linear terms in the remodeling rate-strain coefficients. Firoozbakhsh and Cowin [6] have used this equation to predict the devolution of a hypothetical initial inhomogeneity of bone density along the shaft of a long bone. A higher order equation, retaining quadratic terms in strain, has been proposed by the same authors for the remodeling rate [7].

The role of piezoelectricity of osseous tissues in bone remodeling induced as a sequel to intramedullary nailing has been investigated by Misra and Murty [8]. The effect of material damping of osseous tissues on the remodeling of an axially-symmetric specimen of long bone with a force-fitted axially-oriented medullary pin has been studied by Misra and Samanta [9].

The influence of non-isotropy of osseous tissues on the internal remodeling dynamics has been investigated by Misra and co-workers [10].

In our work the problem of internal remodeling induced by a medullary pin is reconsidered. Our objective is to present a new theoretical model of internal bone remodeling where the role of the fluid part is clear and make predictions about the progress of the internal bone remodeling process for a range of porosities. In our proposed model, bone is treated as a porous elastic deformable solid in the pores of which a viscous compressible fluid flows, using Biot's formulation of the theory of consolidation [11, 12, 13]. The theory of small-strain adaptive elasticity [3], is appropriately modified in order to incorporate the fluid part according to the new material description. The basic equations of the modified theory for internal remodeling are formulated. The problem of internal bone remodeling around a medullary pin is solved in two steps as illustrated in Fig. 1. First, the problem of the remodeling of an isotropic hollow circular cylinder of adaptive poroelastic material subjected to an axial load and an internal pressure is solved. Second, the solution of the problem of an isotropic solid elastic cylinder subjected to an external pressure as solved by Cowin and Van Buskirk [2] is given. The two sub-problems are combined to obtain the solution of the forced fit of an isotropic hollow adaptive poroelastic cylinder about an isotropic solid elastic cylinder. The dependence of the elastic coefficients of the theory of consolidation on porosity has been employed [14]. The unknown material functions are approximated for small values of porosity. A constitutive equation incorporating the remodeling process is presented. A strain-controlled remodeling rate equation where both the fluid and solid parts appear is proposed. According to the proposed approximation of the material functions on porosity, numerical solution of the remodeling rate equation with chosen values of the parameters entering the problem leads to a relation between remodeling rate and porosity.

2. Theory

The basic set of equations in the theory of internal bone remodeling as described by Cowin and Van Buskirk [2], consists of the constitutive equations, the kinematic relations, the stress equations of equilibrium and the remodeling rate equation.

Assuming that the bone is a porous isotropic solid that contains a viscous compressible fluid, the above mentioned set of equations is reformulated by using the theory of consolidation introduced by Biot [11, 12], in cylindrical coordinates.

The stress tensor in a porous material is

$$\overline{T} = T_{ij} + \delta_{ij} T, \tag{1}$$

where δ_{ij} is the Kronecker's symbol, and T represents the total normal force applied to the fluid part of the faces of a cube of unit size of the bulk material.

If p is the hydrostatic pressure of the fluid in the pores we may write

$$T = -fp, \tag{2}$$

where f is the porosity defined as

$$f = V_p / V_b, \tag{3}$$

where V_p is the volume of the pores contained in a sample of bulk volume V_b. Thus, f represents the fraction of the volume of the porous material occupied by the pores.

This system of solid and fluid is a general system which has conservation properties. The solid part is considered to have compressibility and shearing rigidity, and the fluid is compressible. The deformation of a unit cube is assumed to be completely reversible. By deformation is meant here the one determined by the strain tensors in the solid and in the fluid [14].

The kinematic relations for the solid part are

$$E_{rr} = \frac{\partial u_r}{\partial r}, \quad E_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad E_{zz} = \frac{\partial u_z}{\partial z},$$

$$E_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right), \quad E_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad E_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right), \tag{4}$$

where u_r, u_θ, u_z are the average displacement components and E_{ij} $i, j = r, \theta, z$, are the strain components of the solid.

Analogous relations hold for the fluid part with U_r, U_θ, U_z and ε_{ij}, $i, j = r, \theta, z$, denoting the average displacement components and the strains of the fluid, respectively.

The constitutive equations for an isotropic poroelastic material are given as

$$T_{rr} = 2NE_{rr} + AE + Q\varepsilon,$$

$$T_{\theta\theta} = 2NE_{\theta\theta} + AE + Q\varepsilon,$$

$$T_{zz} = 2NE_{zz} + AE + Q\varepsilon,$$

$$T_{\theta z} = NE_{\theta z},$$

$$T_{zr} = NE_{zr}, \tag{5}$$

$$T_{r\theta} = NE_{r\theta},$$

$$T = QE + R\varepsilon,$$

where A, N, R, Q are the elastic constants of the material, in accordance with the Biot formulation [11-14] and E and ε are the dilatations of the solid and fluid, that is,

$$E = E_{rr} + E_{\theta\theta} + E_{zz}, \tag{6}$$

and

$$\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}, \tag{7}$$

respectively.

We note that Eq. (7) does not provide the actual strain in the fluid but the divergence of the fluid-displacement field which is derived from the average volume flow through the pores [14].

The total stress field of the bulk material, in the absence of body forces, satisfies the equilibrium equations

$$\frac{\partial}{\partial r}(T_{rr} + T) + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + \frac{\partial T_{rz}}{\partial z} = 0,$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(T_{zz} + T) + \frac{\partial T_{r\theta}}{\partial r} + \frac{2}{r}T_{r\theta} + \frac{\partial T_{rz}}{\partial z} = 0, \tag{8}$$

$$\frac{\partial}{\partial z}(T_{zz} + T) + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zr}}{\partial r} + \frac{T_{rz}}{r} = 0.$$

Darcy's law governing the flow of a fluid in a porous isotropic material, for non-existing body forces, is given as

$$\frac{\partial T}{\partial r} = C\frac{\partial}{\partial t}(U_r - u_r),$$

$$\frac{1}{r}\frac{\partial T}{\partial \theta} = C\frac{\partial}{\partial t}(U_\theta - u_\theta), \tag{9}$$

$$\frac{\partial T}{\partial z} = C\frac{\partial}{\partial t}(U_z - u_z),$$

where C is a constant that depends on the permeability κ, the porosity f of the medium and the viscosity η of the fluid [15, 16], that is,

$$C = \frac{\eta f^2}{\kappa}. \tag{10}$$

In the theory of small-strain adaptive elasticity [3], the stress-strain relationship for bone that incorporates internal remodeling is given as

$$T_{ij} = (\xi_0 + e)C_{ijkm}(e)E_{km}, \tag{11}$$

where e is a change in the volume fraction ξ of bone matrix material from its reference value ξ_0 as a result of internal remodeling, $e = \xi - \xi_0$.

Equation (5) can be written in matrix form as

$$
\begin{bmatrix} T_{rr}+T \\ T_{\theta\theta}+T \\ T_{zz}+T \\ T_{\theta z} \\ T_{zr} \\ T_{r\theta} \end{bmatrix} =
$$

(12)

$$
\begin{bmatrix} 2N+A+Q & A+Q & A+Q & 0 & 0 & 0 \\ A+Q & 2N+A+Q & A+Q & 0 & 0 & 0 \\ A+Q & A+Q & 2N+A+Q & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 & 0 \\ 0 & 0 & 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 & 0 & N \end{bmatrix} \begin{bmatrix} E_{rr}+x_1\varepsilon \\ E_{\theta\theta}+x_2\varepsilon \\ E_{zz}+x_3\varepsilon \\ E_{\theta z} \\ E_{zr} \\ E_{r\theta} \end{bmatrix},
$$

so that the following relationship is satisfied,

$$
\left.\begin{aligned} (2N+A+Q)x_1+(A+Q)x_2+(A+Q)x_3=Q+R \\ (A+Q)x_1+(2N+A+Q)x_2+(A+Q)x_3=Q+R \\ (A+Q)x_1+(A+Q)x_2+(2N+A+Q)x_3=Q+R \end{aligned}\right\} \Rightarrow x_1=x_2=x_3=\frac{-(Q+R)}{2N+3A+3Q}. \quad (13)
$$

Thus, the stress-strain relations for the isotropic poroelastic bone can be written as

$$
\left(T_{ij}+\delta_{ij}T\right)=C_{ijkm}\left(E_{km}-\delta_{km}\frac{Q+R}{2N+3A+3Q}\varepsilon\right). \quad (14)
$$

In order to incorporate the adaptive nature of bone, we assume that the constitutive relations describing the isotropic poroelastic adaptive bone can be written, in analogy to Eq. (11) as

$$
\left(T_{ij}+\delta_{ij}T\right)=C_{ijkm}(e)\left(\xi E_{km}-(1-\xi)\delta_{km}\frac{Q+R}{2N+3A+3Q}\varepsilon\right). \quad (15)
$$

The bulk volume of the poroelastic medium has been assumed to remain constant throughout the remodeling process, which is denoted by the sum of coefficients ξ and $1-\xi$ being equal to unity. We further assume that the remodeling rate equation, in accordance with the formulation used in the theory of small-strain adaptive elasticity [3], for an isotropic material, is given as

$$\dot{e} = A_1(e) + A_2(e)\left(E - \frac{3(Q+R)}{2N+3A+3Q}\varepsilon\right), \tag{16}$$

where $A_1(e)$ and $A_2(e)$ are material coefficients dependent upon the volume fraction e. Finally, a relation between the porosity f as defined in Eq. (3) and the change in the volume fraction of the bone matrix material e, can be derived. By writing

$$f = 1-\xi_0, \tag{17}$$

where f is the porosity before internal remodeling takes place, and

$$f' = 1-\xi, \tag{18}$$

where f' is the porosity after internal remodeling has occurred, the change in porosity due to the internal remodeling process can be expressed as

$$\Delta f = f' - f = \xi_0 - \xi = -e. \tag{19}$$

Thus, the set of equations describing the proposed theory of internal bone remodeling for a porous isotropic solid bone containing a viscous compressible fluid, consists of Eqs. (4), (5), (8), (9), (15) and (16).

3. The hollow cylinder problem

We now consider a problem in which a hollow circular cylinder of poroelastic bone (Fig. 2) is subjected to a quasi-static axial load $-P(t)$ and an internal radial pressure $p(t)$. The boundary conditions at the inner and outer surfaces of the cylinder are

$$\text{at} \quad r = a: \quad T_{rr} + T = -p(t), \quad T_{\theta\theta} = T_{r\theta} = T_{rz} = T_{\theta z} = 0, \tag{20}$$

and

$$\text{at} \quad r = b: \quad T_{rr} + T = -P_0, \quad T_{r\theta} = T_{rz} = T_{\theta z} = 0, \tag{21}$$

where a and b denote the inner and outer radii of the cylinder and P_0 is a constant pressure outside the cylinder. The boundary conditions (20) and (21) reflect the radial internal pressure exerted on the cylindrical surface of the medullary canal by the forced fit of the medullary pin.

We assume that the fluid is not allowed to flow out of the bone matrix, i.e., P_0 must always be greater than the hydrostatic pressure anywhere inside the poroelastic cylinder. For simplicity, it can be assumed that $P_0 = 0$.

The boundary condition at a transverse cross-section S of the hollow cylinder can be written as [3]

$$\int_S (T_{zz} + T)\,dS = -P(t), \tag{22}$$

where

$$dS = r\,dr\,d\theta\,, \tag{23}$$

is the unit cross-sectional area of the hollow cylinder.

The system of Eqs. (4), (5), (8), (9), (15), (16) and the boundary conditions (20), (21), (22) constitute a well-posed mathematical problem. We assume a solution for the displacements which satisfies the above equations. The proposed solution is

$$\begin{aligned} &u_r = u_r(r,t), && u_\theta = 0, && u_z = -D_1(t)z \\ &U_r = U_r(r,t), && U_\theta = 0, && U_z = -D_2(t)z\ . \end{aligned} \tag{24}$$

Then, the constitutive equations become

$$\begin{aligned} T_{rr} &= (2N + A)\frac{\partial u_r}{\partial r} + A\left(\frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + Q\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z}\right), \\ T_{\theta\theta} &= (2N + A)\frac{u_r}{r} + A\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right) + Q\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z}\right), \\ T_{zz} &= (2N + A)\frac{\partial u_z}{\partial z} + A\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + Q\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z}\right), \\ T &= Q\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + R\left(\frac{\partial U_r}{\partial r} + \frac{U_r}{r} + \frac{\partial U_z}{\partial z}\right), \\ T_{\theta z} &= T_{zr} = T_{r\theta} = 0\,. \end{aligned} \tag{25}$$

The equilibrium equations yield

$$\frac{\partial}{\partial r}(T_{rr}+T)+\frac{1}{r}T_{rr}-\frac{1}{r}T_{\theta\theta}=0,$$
$$\frac{\partial}{\partial z}(T_{zz}+T)=0, \tag{26}$$

and Darcy's law becomes

$$\frac{\partial T}{\partial r}=C\frac{\partial}{\partial t}(U_r-u_r),$$
$$\frac{\partial T}{\partial z}=C\frac{\partial}{\partial t}(U_z-u_z). \tag{27}$$

Substituting the expressions given by Eq. (25) into Eqs. (26) and (27) we obtain

$$(2N+A+Q)\mathscr{L}u_r+(Q+R)\mathscr{L}U_r=0,$$
$$Q\mathscr{L}u_r+C\frac{\partial}{\partial t}u_r+R\mathscr{L}U_r-C\frac{\partial}{\partial t}U_r=0, \tag{28}$$

where $\mathscr{L}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}$,

and

$$(2N+A+Q)\frac{\partial^2 u_z}{\partial z^2}+(Q+R)\frac{\partial^2 U_z}{\partial z^2}=0,$$
$$Q\frac{\partial^2 u_z}{\partial z^2}+C\frac{\partial u_z}{\partial z}+R\frac{\partial^2 U_z}{\partial z^2}-C\frac{\partial U_z}{\partial t}=0. \tag{29}$$

Substituting the proposed solution for u_z and U_z into Eq. (29), we obtain an identity and a relationship of the form

$$Cz\frac{\partial}{\partial t}(D_1(t)-D_2(t))=0\,, \tag{30}$$

which is equivalent to

$$D_2(t)=D_1(t)+\Theta\,, \tag{31}$$

where Θ is a constant.

The boundary conditions become

at $r=a$:

$$(2N+A+Q)\frac{\partial u_r}{\partial r}+(A+Q)\left(\frac{u_r}{r}-D_1(t)\right)+(Q+R)\left(\frac{\partial U_r}{\partial r}+\frac{U_r}{r}-D_1(t)-\Theta\right)=-p(t), \tag{32}$$

at $r=b$:

$$(2N+A+Q)\frac{\partial u_r}{\partial r}+(A+Q)\left(\frac{u_r}{r}-D_1(t)\right)+(Q+R)\left(\frac{\partial U_r}{\partial r}+\frac{U_r}{r}-D_1(t)-\Theta\right)=0\,, \tag{33}$$

and at S:

$$\begin{aligned}&2\pi(A+Q)(bu_r(b,t)-au_r(a,t))-\pi(b^2-a^2)(2N+A+Q)D_1(t)+\\&+2\pi(Q+R)(bU_r(b,t)-aU_r(a,t))-\pi(b^2-a^2)(Q+R)(D_1(t)-\Theta)=-P(t).\end{aligned} \tag{34}$$

The problem has non-homogeneous boundary conditions and therefore we employ the following method to solve it. We assume there exists a function $w=w(r,t)$ such that

$$\mathscr{L}w(r,t)=0\;\Rightarrow\;w(r,t)=A_1(t)r+A_2(t)\frac{1}{r}, \tag{35}$$

which satisfies the following equations

at $r=a$:

$$(2N+A+Q)\frac{\partial w}{\partial r}+(A+Q)\left(\frac{w}{r}+D_1(t)\right)+(Q+R)\left(\frac{\partial w}{\partial r}+\frac{w}{r}+D_1(t)+\Theta\right)=p(t), \tag{36}$$

at $r=b$:

$$(2N+A+Q)\frac{\partial w}{\partial r}+(A+Q)\left(\frac{w}{r}+D_1(t)\right)+(Q+R)\left(\frac{\partial w}{\partial r}+\frac{w}{r}+D_1(t)+\Theta\right)=0\,, \tag{37}$$

and at S:

$$2\pi(A+2Q+R)(bw(b,t)-aw(a,t))+\pi(b^2-a^2)(2N+A+2Q+R)D_1(t)+ \\ +\pi(b^2-a^2)(Q+R)\Theta = P(t), \tag{38}$$

or equivalently,

at $r=a$:

$$2(N+A+2Q+R)A_1(t)-2N\frac{1}{a^2}A_2(t)+(A+2Q+R)D_1(t)=p(t)-(Q+R)\Theta, \tag{39}$$

at $r=b$:

$$2(N+A+2Q+R)A_1(t)-2N\frac{1}{b^2}A_2(t)+(A+2Q+R)D_1(t)=-(Q+R)\Theta, \tag{40}$$

and at S:

$$2(A+2Q+R)A_1(t)+(2N+A+2Q+R)D_1(t)=\frac{P(t)}{\pi(b^2-a^2)}-(Q+R)\Theta. \tag{41}$$

Then the functions

$$\tilde{u}_r(r,t)=u_r(r,t)+w(r,t), \\ \tilde{U}_r(r,t)=U_r(r,t)+w(r,t), \tag{42}$$

satisfy the system of equations

$$(2N+A+Q)\mathscr{L}\tilde{u}_r+(Q+R)\mathscr{L}\tilde{U}_r=0, \\ Q\mathscr{L}\tilde{u}_r+C\frac{\partial}{\partial t}\tilde{u}_r+R\mathscr{L}\tilde{U}_r-C\frac{\partial}{\partial t}\tilde{U}_r=0, \tag{43}$$

with homogeneous boundary conditions

at $r=a$:

$$(2N+A+Q)\frac{\partial\tilde{u}_r}{\partial r}+(A+Q)\frac{\tilde{u}_r}{r}+(Q+R)\left(\frac{\partial\tilde{U}_r}{\partial r}+\frac{\tilde{U}_r}{r}\right)=0, \tag{44}$$

at $r = b$:

$$\left(2N + A + Q\right)\frac{\partial \tilde{u}_r}{\partial r} + \left(A + Q\right)\frac{\tilde{u}_r}{r} + \left(Q + R\right)\left(\frac{\partial \tilde{U}_r}{\partial r} + \frac{\tilde{U}_r}{r}\right) = 0, \tag{45}$$

and at S:

$$\left(A + Q\right)\left(b\tilde{u}_r(b,t) - a\tilde{u}_r(a,t)\right) + \left(Q + R\right)\left(b\tilde{U}_r(b,t) - a\tilde{U}_r(a,t)\right) = 0\,. \tag{46}$$

The system of Eqs. (43) can be written as

$$\boldsymbol{D}\begin{bmatrix}\tilde{u}_r \\ \tilde{U}_r\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{47}$$

where

$$\boldsymbol{D} = \begin{bmatrix} Q\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) + C\frac{\partial}{\partial t} & R\left(\frac{\partial^2}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) - C\frac{\partial}{\partial t} \\ \left(2N + A + Q\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) & \left(Q + R\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) \end{bmatrix}.$$

Introducing a function h such that $\det(\boldsymbol{D})\,h = 0$, [17, 18], we obtain

$$\left\{ \begin{aligned} &\left(Q + R\right)\left[Q\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) + C\frac{\partial}{\partial t}\right] \\ &-\left(2N + A + Q\right)\left[R\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) - C\frac{\partial}{\partial t}\right] \end{aligned} \right\}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)h = 0\,. \tag{48}$$

Assuming that $h = h_1 + h_2$, Eq. (48) is equivalent to the system of equations

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)h_1=0, \tag{49}$$

and

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)h_2+k\frac{\partial}{\partial t}h_2=0, \tag{50}$$

where

$$k=\frac{C(2N+2Q+R+A)}{Q^2-2NR-AR}. \tag{51}$$

The solution of Eq. (49) is given as

$$h_1(r,t)=A_3(t)r+\frac{A_4(t)}{r}. \tag{52}$$

Using the method of separation of variables, h_2 can be written as

$$h_2(r,t)=R(r)T(t). \tag{53}$$

Then, Eq. (50) becomes

$$\frac{R(r)^{(2)}}{R}+\frac{1}{r}\frac{R(r)}{R}-\frac{1}{r^2}=-k\frac{T(t)^{(1)}}{T}=m^2,$$

or equivalently,

$$kT^{(1)}+m^2T=0, \tag{54}$$

with solution

$$T(t)=B_1e^{-\frac{m^2t}{k}}, \tag{55}$$

and

$$r^2R(r)^{(2)}+rR(r)^{(1)}-(r^2m^2+1)R(r)=0, \tag{56}$$

with solution

$$R(r)=B_2I_1(mr)+B_3K_1(mr), \tag{57}$$

where $I_1(mr)$ and $K_1(mr)$ are the modified Bessel functions of the first and second kind, respectively, of order one.

Thus, Eq. (53) becomes

$$h_2(r,t) = (B_6 I_1(mr) + B_7 K_1(mr))e^{-\frac{m^2 t}{k}}. \tag{58}$$

The functions $\widetilde{u}_r$ and $\widetilde{U}_r$ can be expressed as

$$\widetilde{u}_r = (Q+R)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)(h_1 + h_2), \tag{59}$$

$$\widetilde{U}_r = -(2N + A + Q)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)(h_1 + h_2). \tag{60}$$

Given that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)h_1 = 0, \tag{61}$$

and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)h_2 = -k\frac{\partial}{\partial t}h_2 = -k\frac{\partial}{\partial t}\left[(B_6 I_1(mr) + B_7 K_1(mr))e^{-\frac{m^2 t}{k}}\right] = \\ (B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{-\frac{m^2 t}{k}}, \tag{62}$$

the expressions for $\widetilde{u}_r$ and $\widetilde{U}_r$ be written as

$$\widetilde{u}_r = (Q+R)(B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{-\frac{m^2 t}{k}}, \\ \widetilde{U}_r = -(2N + A + Q)(B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{-\frac{m^2 t}{k}}. \tag{63}$$

Introducing (63) into the boundary conditions (44) and (45), we obtain

$$B_6 I_1(ma) + B_7 K_1(ma) = 0,$$
$$B_6 I_1(mb) + B_7 K_1(mb) = 0. \tag{64}$$

The system of equations (64) has a non-zero solution only if its determinant is zero, i.e.,

$$I_1(ma)K_1(mb) - I_1(mb)K_1(ma) = 0, \tag{65}$$

and then

$$B_6 = -B_7 \frac{K_1(ma)}{I_1(ma)}. \tag{66}$$

Equation (65) must be solved for m. If Eq. (65) holds, the boundary condition (46) is automatically satisfied.

Using Eq. (42), the radial displacements for the solid and the fluid part can be written as

$$u_r = (Q+R)(B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{-\frac{m^2 t}{k}} - A_1(t)r - A_2(t)\frac{1}{r},$$
$$U_r = -(2N+A+Q)(B_6 I_1(mr) + B_7 K_1(mr))m^2 e^{-\frac{m^2 t}{k}} - A_1(t)r - A_2(t)\frac{1}{r}. \tag{67}$$

Using Eq. (67) and the expressions for the axial displacements

$$u_z = -D_1(t)z,$$
$$U_z = -D_2(t)z, \tag{68}$$

the stress components are calculated from Eq. (25) as

$$\begin{aligned} T_{rr} = &-2N(Q+R)m^2 e^{-\frac{m^2 t}{k}}\left(B_6 \frac{1}{r} I_1(mr) + B_7 \frac{1}{r} K_1(mr)\right) \\ &-(Q^2 - 2NR - AR)m^2 e^{-\frac{m^2 t}{k}}(B_6 m I_0(mr) - B_7 m K_0(mr)) \\ &-2(N+A+Q)A_1(t) + 2NA_2(t)\frac{1}{r^2} - (A+Q)D_1(t) - Q\Theta, \end{aligned}$$

$$T_{zz} = -\left(Q^2 + 2QN - AR\right)m^2 e^{-\frac{m^2 t}{k}}\left(B_6 m I_0(mr) - B_7 m K_0(mr)\right) \\ -2(A+Q)A_1(t) - (2N + A + Q)D_1(t) - Q\Theta, \tag{69}$$

and

$$T = \left(Q^2 - 2RN - AR\right)m^2 e^{-\frac{m^2 t}{k}}\left(B_6 m I_0(mr) - B_7 m K_0(mr)\right) \\ -2(Q+R)A_1(t) - (Q+R)D_1(t) - R\Theta .$$

where $I_0(mr)$ and $K_0(mr)$ are the modified Bessel functions of the first and second kind, respectively, of order zero.

The solution of the system of equations (39) to (41) leads to expressions for the unknowns, $A_1(t)$, $A_2(t)$, $D_1(t)$ given in Appendix A. Taking

$$\Theta = \frac{3A + 2N + 6Q + 3R}{Q + R} \quad \text{and} \quad B_7 = 1,$$

and substituting $A_1(t)$, $A_2(t)$, $D_1(t)$ into Eqs. (67), (68) and (69), we obtain the displacements and stresses in the hollow cylinder, respectively. The strains for the solid part can then be estimated from Eq. (4) and for the fluid part from the corresponding relations for the fluid.

4. The forced fit of the hollow cylinder about an isotropic solid elastic cylinder

For an isotropic solid elastic cylinder subjected to an external pressure $p(t)$, the non-zero stress is $T_{rr} = -p(t)$. The displacement in the radial direction is given [2] as

$$u = \frac{-\left(2\mu_p + \lambda_p\right)p(t)r}{2\mu_p\left(3\lambda_p + 2\mu_p\right)}, \tag{70}$$

where λ_p and μ_p are Lamè' s constants for the isotropic solid elastic cylinder.

We calculate the pressure of interaction $p(t)$ which occurs when an isotropic solid cylinder of radius $a_0 + \frac{\rho}{2}$ is force-fitted into a hollow adaptive poroelastic cylinder of radius a_0.

Let a and b denote the inner and outer radii, respectively, of the hollow adaptive poroelastic cylinder at the instant after the solid isotropic cylinder has been forced into the hollow cylinder. Although the radii of the hollow cylinder will actually change during the adaptation process, the deviation of these quantities from a and b is assumed to be negligible in small strain theory.

At some instant after the two cylinders have been force-fitted together the pressure of interaction is $p(t)$. The radial displacement of the solid cylinder at its surface is

$$u_1 = \frac{-(2\mu_p + \lambda_p)p(t)a}{2\mu_p(3\lambda_p + 2\mu_p)} \quad . \tag{71}$$

The radial displacement of the bone at its inner surface is

$$u_2 + U_2 = -(2N + A - R)(B_6 I_1(ma) + B_7 K_1(ma))m^2 e^{\frac{-m^2 t}{k}} - 2A_1(t)a - 2A_2(t)\frac{1}{a} \quad . \tag{72}$$

Since the two surfaces are at the same radius after insertion of the rod, that is,

$$a = a_0 + \frac{\rho}{2} + u_1 = a_0 + u_2 + U_2 , \tag{73}$$

we find

$$\rho = 2(u_2 + U_2 - u_1) , \tag{74}$$

or

$$\rho = 2(\Lambda_1 P(t) + \Lambda_2 + \Lambda_3 p(t)), \tag{75}$$

where Λ_i, $i = 1,2,3$ are given in Appendix B.

Solving Eq. (75) for $p(t)$ we obtain,

$$p(t) = \frac{1}{\Lambda_3}\left(\frac{\rho}{2} - \Lambda_1 P(t) - \Lambda_2\right). \tag{76}$$

When Eq. (76) is substituted into Eqs. (67), (68), (4) and (25), the displacement, strain and stress fields are determined. They are functions of the porosity f or, equivalently, of the change in the volume fraction e.

The remodeling rate equation is obtained by substituting the expressions for E and ε into

$$\dot{e} = A_1(f) + A_2(f)\left(E - \frac{3(Q+R)}{2N+3A+3Q}\varepsilon\right). \tag{77}$$

In order to solve Eq. (77) explicit forms for $A_1(f)$, $A_2(f)$, $E(f)$, $\varepsilon(f)$, $Q(f)$, $R(f)$, $N(f)$ and $A(f)$ must be inserted.

5. An approximate solution

The remodeling rate equation is given as

$$\dot{e} = A_1(f) + A_2(f)\left(E - \frac{3(Q+R)}{2N+3A+3Q}\varepsilon\right), \tag{78}$$

where

$$E = -(Q+R)m^3 e^{\frac{-m^2 t}{k}} \left(\frac{K_1(ma)I_0(mr) - I_1(ma)K_0(mr)}{I_1(ma)} \right) - D_1(t) - 2A_1(t),$$

and (79)

$$\varepsilon = (2N + A + Q)m^3 e^{\frac{-m^2 t}{k}} \left(\frac{K_1(ma)I_0(mr) - I_1(ma)K_0(mr)}{I_1(ma)} \right) - (D_1(t) + \Theta) - 2A_1(t).$$

We assume that $A_1(f)$ and $A_2(f)$ can be approximated as

$$A_1(f) = C_0 - C_1 f - C_2 f^2,$$

and (80)

$$A_2(f) = A_2^0 - A_2^1 f,$$

where C_0, C_1, C_2, A_2^0 and A_2^1 are constants.

The dependence of Q, A, R, N on f is given by Biot and Willis [14],

$$Q = \frac{f\left(1 - f - \frac{\delta}{\kappa}\right)}{\gamma + \delta - \frac{\delta^2}{\kappa}},$$

$$R = \frac{f^2}{\gamma + \delta - \frac{\delta^2}{\kappa}}, \qquad (81)$$

$$A = \frac{\frac{\gamma}{\kappa} + f^2 + (1-2f)\left(1 - \frac{\delta}{\kappa}\right)}{\gamma + \delta - \frac{\delta^2}{\kappa}} - \frac{2}{3}\mu,$$

and

$N = \mu_b$,

where κ is the coefficient of jacketed compressibility, δ the coefficient of unjacketed compressibility, γ the coefficient of fluid content and μ_b the shear modulus of bone.

Introducing Eqs. (80) and (81) into the expressions for $A_1(t), A_2(t), D_1(t), \Theta$ and then to Eqs. (79) and (78) we obtain an equation of the form

$$\dot{e} = g(f, m, p(t), P(t)), \tag{82}$$

that is, the remodeling rate is a function of f, m, $p(t)$ and $P(t)$. The parameter m can be found from Eq. (65) and $p(t)$ is related to $P(t)$ as indicated by Eq. (76).

Equation (82) is solved numerically. For a constant value of P at a specific time t, we obtain p as a function of f, using Eq. (76). Finally, we end up with an expression of $\dot{e}$ as a function of the porosity f that is numerically solved.

6. A numerical example

To illustrate the numerical solution of the remodeling rate equation we present a numerical example with chosen data. We examine the case of a stainless steel medullary pin being force-fitted into a femur with an internal radius $a = 10mm$ and an external radius $b = 15mm$. The ratio of ρ to a is 0.005. Then, m is calculated from Eq. (65) and found equal to $-1.52151\times10^{-13} + 632.187\,\mathrm{i}$. The material properties of the pin are $\lambda_p = 120GPa$ and $\mu_p = 80GPa$ [3].

The material properties for bone are obtained as described below. According to Biot and Willis [14], A, N, Q, R are related to μ, λ, κ and f as follows,

$$N=\mu \ , \ A-\frac{Q^2}{R}=\lambda \ , \ 2\mu+3\lambda=\frac{3}{\kappa} \ , \ \frac{Q+R}{R}f=1-\frac{\delta}{\kappa} \ . \tag{83}$$

Taking $\mu_b = 5.5GPa$, $\lambda_b = 40GPa$ [19, 20, 3], we obtain $\kappa = 0.023(GPa)^{-1}$.Assuming that for bone $\delta = 0.02(GPa)^{-1}$ and $\gamma = 4.68(GPa)^{-1}$, Eqs. (81) are written as

$$Q=\frac{f(0.131-f)}{4.682}GPa \ ,$$

$$R=\frac{f^2}{4.682}GPa \ ,$$

$$A=\left(\frac{203.478+f^2+(1-2f)0.130}{4.682}-3.667\right)GPa \ , \tag{84}$$

and

$$N=5.5GPa \ .$$

We further assume that the fluid occupying the pores is water, whose viscosity is $n=1\times10^{-12}GPa \sec$ [21], and that the permeability of bone is [15], $\kappa_b = 10^{-14}m^2$, so by using Eqs. (10), (84) and (51), the parameter k can be expressed as a function of the porosity f.

The remodeling rate coefficients are taken [3] as,

$$C_0 = 10^{-9}\sec^{-1}, \ C_1 = 10^{-7}\sec^{-1}, C_2 = 10^{-6}\sec^{-1} \ ,$$

and (85)

$$A_2^0 = 10^{-5}\sec^{-1} \ , \ A_2^1 = 10^{-5}\sec^{-1} \ .$$

We investigate the situation when remodeling starts. At $t = 0$, the medullary pin is forced into the medulla. The internal pressure $p(t)$ is calculated from Eq. (76), where $P(t)$ is assumed to have a uniform value of $1631N$ [3]. An expression of $\dot{e}$ as a function of the porosity f can be obtained, which is numerically evaluated for a range of porosities at $r = 0.0125m.$. Figure 3 shows the variation of the remodeling rate with porosity. It can be seen that as the porosity increases the remodeling rate decreases.

7. Discussion

A theoretical analysis of internal remodeling induced by the forced fit of a medullary pin in a hollow cylindrical poroelastic bone model has been presented. In the proposed model, bone is treated as a poroelastic medium, consisting of a solid elastic bone matrix and interstitial fluid flowing through the interconnected pores inside the matrix. The formulation of the problem is based on the three-dimensional theory of consolidation for poroelastic media introduced by Biot [11, 12, 13]. For simplicity, bone has been treated as isotropic. A constitutive relation for the poroelastic bone, which incorporates the internal bone remodeling process is proposed. The contribution of the fluid term to the remodeling process is clearly indicated. Using Biot and Willis approximation [14] for the dependence of the elastic constants on porosity, and a second-order approximation for the dependence of the unknown material functions on porosity, a new remodeling rate equation is proposed. Numerical values of the remodeling rate coefficients have been taken from the literature [3]. The rate of internal remodeling, which is defined as the temporal derivative of the change in the bone volume fraction of the bone matrix material, is expressed as a function of porosity. Our results show that as the porosity of bone increases, the rate of internal remodeling decreases.

To the authors' knowledge, the present work constitutes the first attempt to use the poroelasticity theory in the formulation of the problem of internal remodeling induced by a medullary pin. The advantages of using the Biot formulation for the material description of bone are: (a) it offers a direct visualisation of the contribution of the fluid term, and (b) the dependence of the elastic constants on porosity is known.

The suggested solution in our model allows an estimate of the relative rate of remodeling at different locations inside the bone model. The present model is subject to further improvement. An alternative solution could be sought for, so that more realistic predictions might be reached. An insight into the remodeling rate equation could possibly lead to an alternative approximation. The work presented here could be extended to include another symmetry for bone such as transverse isotropy or orthotropy. In addition, the cavity of the hollow poroelastic cylinder could be assumed to be filled with fluid which is free to flow in and out of the pores of the bone matrix. The inherent complexity of the physical problem and the mathematical difficulties encountered are to be taken into consideration but should in no case be preventive. An improvement of the present model might offer a more realistic approach to the internal bone remodeling problem. The analysis for surface bone remodeling induced by the force-fitting of a medullary pin is in preparation.

Appendix A

$$A_1(t)=\frac{(A+2Q+R)P(t)+a^2\pi(A+2N+2Q+R)p(t)+2N\pi(b^2-a^2)(Q+R)\Theta}{2(a^2-b^2)N\pi(3A+2N+6Q+3R)},$$

$$A_2(t)=-\frac{a^2b^2p(t)}{2(b^2-a^2)N},$$

$$D_1(t)=\frac{-(A+2Q+R+N)P(t)-a^2\pi(A+2Q+R)p(t)+N\pi(b^2-a^2)(Q+R)\Theta}{(a^2-b^2)(3A+2N+6Q+3R)}.$$

Appendix B

$$\Lambda_1=\frac{-a(A+2Q+R)}{(a^2-b^2)N\pi(3A+2N+6Q+3R)},$$

$$\Lambda_2=\frac{2a(Q+R)\Theta}{(3A+2N+6Q+3R)},$$

$$\Lambda_3=\left(\frac{ab^2}{(b^2-a^2)N}-\frac{a^3(N+A+2Q+R)}{(a^2-b^2)N(3A+2N+6Q+3R)}+\frac{(\lambda+2\mu)a}{2\mu(3\lambda+2\mu)}\right).$$

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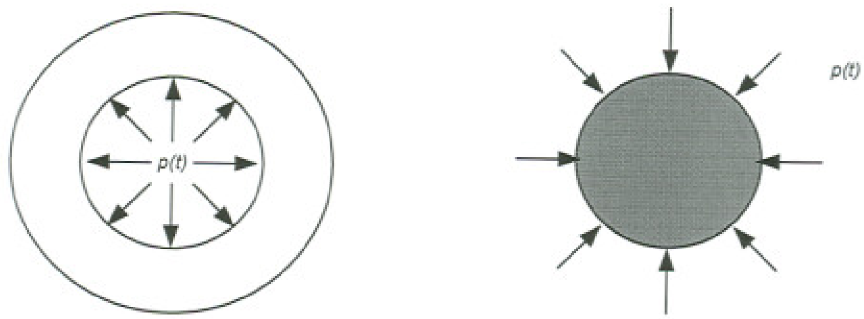

Fig. 1

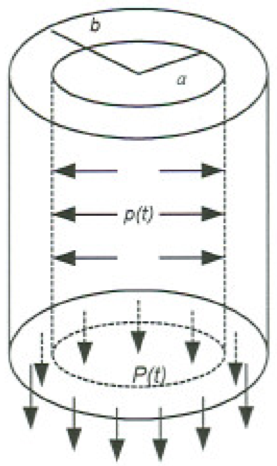

Fig. 2

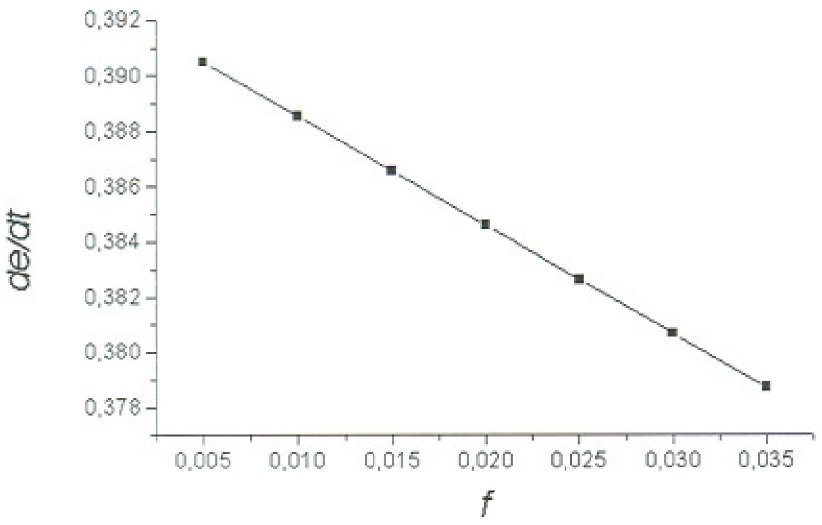
de/dt
0,392
0,390
0,388
0,386
0,384
0,382
0,380
0,378
0,005
0,010
0,015
0,020
0,025
0,030
0,035
f

Fig.3

Figure 1 The decomposition of the medullary pin problem into two sub-problems:

(a) the hollow isotropic poroelastic cylinder is subjected to an internal radial pressure $p(t)$.
(b) the elastic isotropic solid cylinder is subjected to an external radial pressure $p(t)$.

Figure 2 The poroelastic hollow cylinder is subjected to an axial load $P(t)$ and a radial internal pressure $p(t)$.

Figure 3 The variation of the remodeling rate with porosity.