ROUTING AND PATH COLORING IN RINGS: NP - COMPLETENESS

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Abstract

In this technical report we study path routing and coloring problems in ring graphs and present some NP-completeness results. In the *Path Coloring* problem (PC), a collection of paths in an undirected graph is given and the goal is to color all paths using a minimum number of colors, so that overlapping paths are assigned different colors. A variation of PC is the *Routing and Path Coloring* problem (RPC), in which pairs of vertices are given instead of paths and path specification is part of the problem.

The decision version of PC for rings is know to be NP-complete [1]. In this report we prove that it remains NP-complete in some special cases. Using this fact, we prove that the decision version of RPC for rings is also an NP-complete problem.

1 Introduction

A (proper) coloring of a collection[1] of paths P, is an assignment of colors to paths in P, such that overlapping paths have different colors (we say that two paths overlap if they pass through the same edge).

In the *Path Coloring* problem (PC), an undirected graph $G(V, E)$ and a collection of paths P are given and the goal is to find a coloring of P that uses a minimum number of colors. A variation of PC is the *Routing and Path Coloring* problem (RPC). In this problem instead of a collection of paths we are given a collection of pairs of vertices R. The goal is to choose a path between every pair of vertices in R and find a coloring of the resulting collection of paths, so that the number of colors is minimum among all the possible choices of paths and colorings.

PC and RPC have been studied for several graph topologies that appear in networks. In this report we consider these problems for ring graphs (RING-PC and RING-RPC problems). The RING-PC problem was first formulated as coloring of circular arcs. The NP-completeness of its decision version follows from [1]. An approximation algorithm for RING-PC is presented in [4]. Approximation algorithms for RING-RPC appear in [2, 3]. In this report we prove that the decision version of RING-RPC (denoted d-RING-RPC) is also NP-complete.

In section 2 we give some definitions and technical preliminaries. In section 3 we prove the NP-completeness of some special cases of d-RING-PC. Finally, in section 4 we prove the NP-completeness d-RING-RPC.

2 Definitions and Technical Preliminaries

A *ring* is a graph that consists of a single cycle. The vertices of a ring are labeled $1, 2, \ldots n$ in clockwise direction. Formally a ring is a graph $R = (V_r, E_r)$, where

$$V_r = \{1, 2, \ldots, n\}$$

$$E_r = \{\{i, i+1\} | 1 \leq i \leq n-1\} \cup \{\{n, 1\}\}.$$

Notice that a ring is completely defined by the number of its vertices.

In a ring there are two alternative paths between any two vertices. We denote the path $(i, (i \bmod n) + 1, \ldots, j)$, that connects i and j in clockwise direction, by (i, j). A path is *short* if its length is less than $\frac{n}{2}$, or equivalently at most $\lceil \frac{n}{2} \rceil - 1$. A path is *long* if it is not short.

[1] We use the term collection instead of set, since a path may appear many times in P.

An instance of RING-PC is a pair $\langle n, P \rangle$, where n is the number of vertices in the ring and P is a collection of paths. Similarly an instance of RING-RPC is a pair $\langle n, R \rangle$, where R is a collection of unordered pairs of vertices.

Given a RING-PC instance the *load* of edge e (denoted by $k(e, P)$) is the number of paths in P that pass through e. The load of the instance is $k(P) = \max_{e \in E_r} k(P, e)$. Notice that $k(P)$ is a lower bound for the number of colors needed to color all paths in P.

In the following sections we consider the decision problems d-RING-PC and d-RING-RPC. An instance of d-RING-PC is a triple $\langle n, P, w \rangle$, where n is the number of vertices in the ring, P is a collection of paths and w is a positive integer. $\langle n, P, w \rangle$ is a "yes" instance of d-RING-PC iff there exists a coloring of P with at most w colors. d-RING-RPC is defined in a similar way.

An instance $\langle n, P, w \rangle$ of d-RING-PC is *canonical* if for every edge e in the ring $k(e, P) = w$.

3 Coloring Short Paths in a Ring

In this section we prove that d-RING-PC restricted to canonical instances with only short paths remains NP-complete.

Theorem 1 *d-RING-PC restricted to canonical instances is NP-complete.*

Proof. We will reduce d-RING-PC for arbitrary instances to d-RING-PC restricted to canonical instances. Let $\langle n, P, w \rangle$ be an instance of d-RING-PC. We construct a collection of paths P' as follows. P' contains all paths in P. Moreover, for every edge $e = \{i, j\}$ such that $k(e, P) < w$, P' contains $w - k(e, P)$ paths of length 1, connecting i and j through e. Obviously $\langle n, P', w \rangle$ is a canonical instance.

If paths in P' can be colored with w colors, then paths in P can also be colored with w colors, since $P \subseteq P'$. Conversely, if there exist a coloring of paths in P with w colors, this coloring can be extended to paths in $P' - P$, by assigning the $w - k(e, P)$ colors that are unused in an edge e to the $w - k(e, P)$ uncolored paths of length one that pass through e. ∎

Theorem 2 *d-RING-PC restricted to canonical instances with only short paths is NP-complete.*

Proof. We will reduce d-RING-PC for canonical instances to d-RING-PC for canonical instances with only short paths. Let $\langle n, P, w\rangle$ be a canonical instance of d-RING-PC. We will construct a sequence of instances $\langle n_0, P_0, w\rangle, \langle n_1, P_1, w\rangle, \ldots, \langle n_m, P_m, w\rangle$ such that:

- For $i = 0, 1, \ldots, m$:
  - n_i is an odd number
  - $\langle n_i, P_i, w\rangle$ is canonical instance.
  - $\langle n_i, P_i, w\rangle$ is a "yes" instance iff $\langle n, P, w\rangle$ is a "yes" instance.
- P_m contains only short paths.

If n is an odd number then $n_0 = n$ and $P_0 = P$. Otherwise, in order to achieve an odd number of vertices, we split edge $\{n, 1\}$ into two edges $\{n, n+1\}$ and $\{n+1, 1\}$. Thus, if n is even, then $n_0 = n+1$ and paths in P_0 are obtained from paths in P by replacing each subpath $(n, 1)$ by the subpath $(n, n+1, 1)$.

Assume that instance $\langle n_i, P_i, w\rangle$ contains a long path p_i of length l_i. Without loss of generality we may assume that $p_i = (n_i - \lfloor \frac{l_i}{2} \rfloor, \lceil \frac{l_i}{2} \rceil)$ (otherwise we can rename vertices). Notice that

$$\lceil \frac{l_i}{2} \rceil \leq \lceil \frac{n_i - 1}{2} \rceil = \frac{n_i - 1}{2}$$

and

$$n_i - \lfloor \frac{l_i}{2} \rfloor \geq n_i - \lfloor \frac{n_i - 1}{2} \rfloor = n_i - \frac{n_i - 1}{2} = \frac{n_i + 1}{2}$$

since $l_i \leq n_i - 1$ and n_i is odd. This implies that p_i does not pass through edge $\{\frac{n_i-1}{2}, \frac{n_i+1}{2}\}$.

We construct a new instance $\langle n_{i+1}, P_{i+1}, w\rangle$, in which p_i is replaced by two short paths. The ring in the new instance contains two new vertices: a vertex x is inserted between n_i and 1 and edge $\{n_i, 1\}$ is replaced by edges $\{n_i, x\}$ and $\{x, 1\}$; similarly a vertex z is inserted between $\frac{n_i-1}{2}$ and $\frac{n_i+1}{2}$ and edge $\{\frac{n_i-1}{2}, \frac{n_i+1}{2}\}$ is replaced by edges $\{\frac{n_i-1}{2}, z\}$ and $\{z, \frac{n_i+1}{2}\}$.

All paths in P_i except for p_i are inserted in P_{i+1}, after replacing each subpath $(n, 1)$ by $(n, x, 1)$ and each subpath $(\frac{n-1}{2}, \frac{n+1}{2})$ by $(\frac{n-1}{2}, z, \frac{n+1}{2})$. P_{i+1} also contains the following subpaths of p_i: $p'_i = (n_i - \lfloor \frac{l_i}{2} \rfloor, n_i - \lfloor \frac{l_i}{2} \rfloor + 1, \ldots, n_i, x)$ and $p''_i = (x, 1, \ldots, \lceil \frac{l_i}{2} \rceil)$. (Actually at this point a renaming of vertices must be performed to achieve the standard vertex naming $1, 2, \ldots, n_{i+1}$).

It is easy to see that n_i is odd and $\langle n_i, P_i, w\rangle$ is canonical, for $i = 0, 1, \ldots, j$. We next prove by induction the following fact:

$\langle n_i, P_i, w\rangle$ is a "yes" instance iff $\langle n, P, w\rangle$is a "yes" instance.

The possible insertion of a new vertex in the construction of $\langle n_0, P_0, w\rangle$ does not affect path overlapping. Thus, the fact holds for $i = 0$. Suppose that the fact holds for the instance $\langle n_i, P_i, w\rangle$. It suffices to show that $\langle n_{i+1}, P_{i+1}, w\rangle$ is a "yes" instance if and only if $\langle n_i, P_i, w\rangle$ is a "yes" instance. If paths in P_i can be colored with w colors, then we can get a coloring of paths in P_{i+1} by assigning the color of p_i to p_i' and p_i''. The colors of the remaining paths are the same as in the coloring of P_i. Conversely assume that paths in P_{i+1} can be colored with w colors. Paths p_i' and p_i'' must have the same color; otherwise the remaining $w-2$ colors do not suffice for the remaining $w-1$ paths passing through both edges $\{n_i, x\}$ and $\{x, 1\}$. Thus, we can get a coloring of paths in P_i by assigning the common color of p_i' and p_i'' to p_i.

The length of p_i' and p_i'' is $\lfloor \frac{l_i}{2} \rfloor + 1$ and $\lceil \frac{l_i}{2} \rceil$ respectively. It follows that p_i' and p_i'' are short paths, since

$$\lceil \frac{l_i}{2} \rceil \le \lfloor \frac{l_i}{2} \rfloor + 1 \le \frac{l_i + 2}{2} < \frac{n_i + 2}{2} = \frac{n_{i+1}}{2}$$

Moreover, any short path in P_i has length $l \le \frac{n_i - 1}{2}$, which implies that it cannot contain both edges $\{n, 1\}$ and $\{\frac{n_i-1}{2}, \frac{n_i+1}{2}\}$. Thus, its length in the new instance is at most $l + 1 \le \frac{n_{i+1}-1}{2}$, i.e. it remains short.

Consequently the number of long paths is decreased by one at every step. If P_0 contains m long paths, then P_m contains only short paths. ∎

4 Routing and Path Coloring in Rings

In this section we prove that d-RING-PC is an NP-complete problem.

Theorem 3 *d-RING-RPC in NP-complete.*

Proof. We will reduce d-RING-PC for canonical instances with only short paths to d-RING-RPC. Let $\langle n, P, w\rangle$ be a canonical instance of d-RING-PC, in which all paths are short. Let R be the collection of requests, which contains a request $\{i, j\}$ for every path (i, j) in P. We will prove that $\langle n, P, w\rangle$ is a "yes" instance of d-RING-PC if and only if $\langle n, P, w\rangle$ is a "yes" instance of d-RING-PC.

Assume that paths in P can be colored with w colors. Then we can assign to each request $\{i, j\}$ the shortest path between i and j, i.e. satisfy the requests using exactly paths in P, and color the resulting paths with w colors.

For the other direction assume that there exists a routing P' for R and a coloring of paths in P' with w colors. We claim every path in P' must be the shortest path between its endpoints, i.e. $P' = P$. In order to prove this claim notice that the sum of the lengths of all paths in P is exactly nw. If P' contains a long path then the sum of the lengths of all paths in P' will be greater than nw. This implies that for some edge e in the ring $k(e, P') > w$. But in that case paths in P' cannot be colored with w colors. Hence, P' does not contain any long path, i.e. $P' = P$. This implies that paths in P can be colored with w colors. ∎

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