WAVE PROPAGATION IN HUMAN LONG BONES OF ARBITRARY CROSS SECTION WITH A CAVITY FILLED WITH AN INCOMPRESSIBLE VISCOUS FLUID

D.I. Fotiadis, G. Foutsitzi, C.V. Massalas

4-2000

Preprint no. 4-2000/2000

Department of Computer Science
University of Ioannina
451 10 Ioannina, Greece

Wave Propagation in Human Long Bones of Arbitrary Cross Section with a Cavity Filled with an Incompressible Viscous Fluid

D.I. Fotiadis *, G. Foutsitzi, C.V. Massalas †

Abstract

In this work the analysis of wave propagation in human long bones of arbitrary cross section with cavity filled with bone marrow is presented. We approximate the behavior of the cortical bone with a solid piezoelectric cylinder of crystal class 6 and the bone marrow with an incompressible viscous fluid. The three - dimensional theory of piezoelectricity, and the Gauss equation, are used to describe the behavior of the bone. The properties of the bone are introduced by its stiffness, piezoelectric stress and dielectric matrices. The motion of the fluid in the cavity is described by the Navier Stokes equations and the continuity equation. We propose a wave - type solution for the solid and fluid which is combined with the appropriate boundary conditions. The system is considered to be stress free and coated with electrodes which are shorted on the outer surface, while continuity of fields is required on the interface between fluid and solid. The frequency equation, which is applicable for the irregular shape, has been obtained by making use of the Fourier expansion collocation method. The latter gives the relation between angular frequency, attenuation and wavelength which can be solved numerically in the complex plane.

Key words: Bone Biomechanics; Wave Propagation.

AMS subject classifications: 35Q72, 92C10, 92C50.

1 Introduction

The study of wave propagation in human long bones provides with useful data on monitoring and controlling fracture healing or other processes in bone fracture and distraction osteogenesis [1]. It is also known that electromagnetic pulses and ultrasound are used in the acceleration of the above processes [2]. The bone behaves like a piezoelectric material of crystal class 6 with the axis of symmetry corresponding to its long axis and this has been justified experimentally [3]. The wave propagation in long bones has been studied by several authors [4] - [8], but to our knowledge no previous attempt has been taken to account for the existence of bone marrow. Its study introduces an additional difficulty which is related to the fact that it has viscous properties which not only increase the degree of difficulty, but also introduce complex terms. The first increases the mathematical complexity of the problem and the second makes the numerical treatment much more difficult.

In a recent paper [9] we have studied the wave propagation in a piezolectric cylinder of arbitrary cross section with a cavity of arbitrary shape, using the Fourier expansion collocation method proposed in [10]. We have succeeded to have a relation between angular frequency and wavelength for various geometries and system parameters. The results obtained were very close to those reported experimentally.

In the present work we have tried to combine the findings of this previous contribution with the inclusion of bone marrow in the cavity of arbitrary cross section. The mathematical modelling for the solid cylinder is based on the three - dimensional theory of piezolectricity, while the linearized Navier - Stokes equations and the continuity equation have

*Dept. of Computer Science, University of Ioannina, GR 451 10 Ioannina, Greece, fotiadis@cs.uoi.gr

†Dept. of Mathematics, University of Ioannina, GR 451 10 Ioannina, Greece, cmasalas@cc.uoi.gr

16th IMACS World Congress (

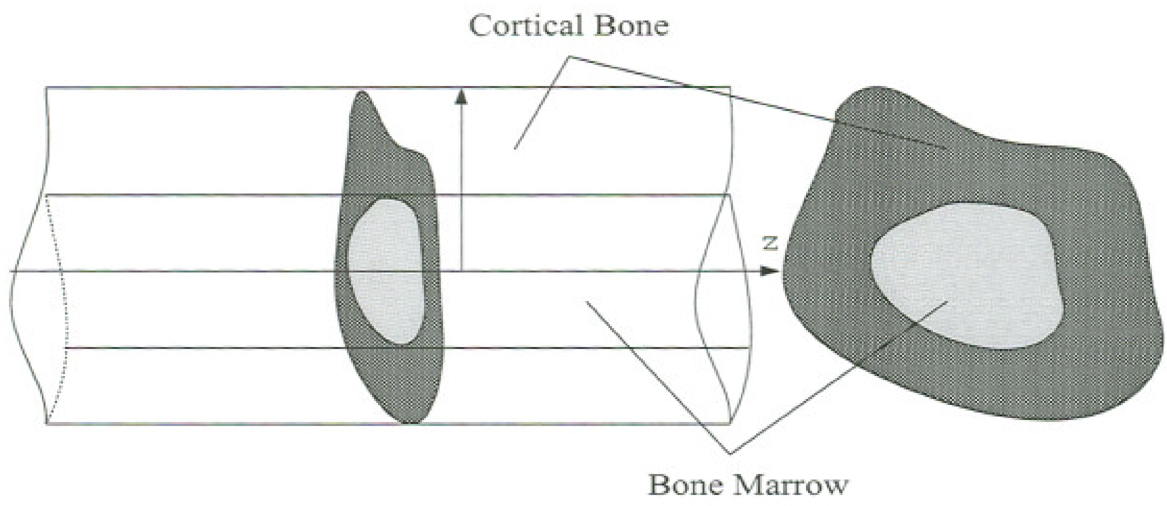

Figure 1: The Problem Geometry.

been employed to describe the dynamic behavior of the fluid. The harmonic wave - type solution to the field equation of the solid and the fluid was found following a method used in [9]. The coefficients introduced must be determined using the boundary conditions. We consider that the outer surface of the bone is stress free and coated with electrodes which are shorted. For the inner surface we consider continuity of stresses, displacements and electrostatic potential. Those boundary conditions introduce certain simplifications to the problem which are related with the surroundings of the bone. In our approach we do not consider that the bone is supported by muscles or ligaments. For the boundary conditions on those surfaces which are of arbitrary shape, we have implemented the Fourier expansion collocation method [10]. This leads to the determination of the frequency equation which must be solved numerically in the complex plane.

The approach presented is the most general one, since it can be applied to various bone shapes and bone dimensions when the bone cross - section does not change along the bone axis. The common practice is that the geometry of the bone is obrained from CT using 3-D reconstruction.

2 Mathematical Description

The geometry of the problem is shown in Fig. 1. Since long bones (like the human femur) have a trabecular structure and the experimental observations did not show any wave reflections at the end points [5], the long human bone can be modelled as an infinite cylinder of arbitrary cross section which contains the bone marrow.

The piezolectrical material (cortical bone) behavior can be described using the three - dimensional theory of piezoelectricity with the equations of motion

(1) $$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = \rho_s \frac{\partial^2 u_r}{\partial t^2},$$

(2) $$\frac{\partial T_{\theta z}}{\partial z} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{2}{r}T_{r\theta} + \frac{\partial T_{r\theta}}{\partial r} = \rho_s \frac{\partial^2 u_\theta}{\partial t^2},$$

(3) $$\frac{\partial T_{zz}}{\partial z} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{2}{r}T_{rz} + \frac{\partial T_{r\theta}}{\partial r} = \rho_s \frac{\partial^2 u_z}{\partial t^2},$$

the Gauss equation

(4) $$\frac{1}{r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial \theta} + \frac{\partial D_z}{\partial z} = 0,$$

and the constitutive equations

(5) $$\mathbf{T} = \mathbf{c} : \nabla_s \mathbf{u} + \mathbf{e} \cdot \nabla V,$$

(6) $$\mathbf{D} = \mathbf{e} : \nabla_s \mathbf{u} - \epsilon \cdot \nabla V,$$

where ρ_s is the mass density of the cortical bone, T_{mn} are the elements of the stress tensor $\mathbf{T}$, D_k are the elements of the electric displacement vector $\mathbf{D}$, u_r, u_θ, u_z are the displacement vector ($\mathbf{u}$) components, V is the electrostatic potential and $\nabla_s \mathbf{u}$ denotes the symmetric gradient of $\mathbf{u}$.

For a piezoelectric material of crystal class 6, the stiffness matrix $\mathbf{c}$, the piezoelectric stress matrix $\mathbf{e}$ and the dielectric matrix ϵ are given as follows

(7) $$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix},$$

(8) $$\mathbf{e} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & -e_{14} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix},$$

and

(9) $$\epsilon = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

respectively, where $c_{66} = \frac{1}{2}(c_{11} - c_{12})$.

A wave type solution of the equations (1) - (4) in cylindrical coordinates can be taken as

(10) $$u_r = \sum_{m=0}^{\infty} \varepsilon_m \big(G^m_{,r} + \frac{1}{r}\Psi^m_{,\theta}\big) e^{i(\gamma z - \omega t)},$$

(11) $$u_\theta = \sum_{m=0}^{\infty} \varepsilon_m \big(\frac{1}{r} G^m_{,\theta} - \Psi^m_{,r}\big) e^{i(\gamma z - \omega t)},$$

(12) $$u_z = i \sum_{m=0}^{\infty} \varepsilon_m W^m e^{i(\gamma z - \omega t)},$$

(13) $$V = i \sum_{m=0}^{\infty} \varepsilon_m \Phi^m e^{i(\gamma z - \omega t)},$$

where G^m, Ψ^m, W^m and Φ^m are functions of r and θ, $i = \sqrt{-1}$, $\varepsilon_m = \frac{1}{2}$ for $m = 0$ and $\varepsilon_m = 1$ for $m \geq 1$, γ is the wavenumber and ω is the angular frequency.

In that case the solution of the governing equations (1) - (4) is given as

(14) $$\begin{aligned} u_r \;=\; & \sum_{m=0}^{\infty} \varepsilon_m \sum_{j=1}^{4} \sum_{l=1}^{2} \big\{ \big[\alpha_j^{m,l} \delta_j^{p1} \frac{\partial}{\partial r} \varsigma^{m,l}(k_j r) + \beta_j^{m,l} \delta_j^{p2} \frac{m}{r} \varsigma^{m,l}(k_j r)\big] \cos(m\theta) \\ + \; & \big[-\alpha_j^{m,l} \delta_j^{p2} \frac{m}{r} \varsigma^{m,l}(k_j r) + \beta_j^{m,l} \delta_j^{p1} \frac{\partial}{\partial r} \varsigma^{m,l}(k_j r)\big] \sin(m\theta) \big\} e^{i(\gamma z - \omega t)}, \end{aligned}$$

(15) $$\begin{aligned} u_\theta \;=\; & \sum_{m=0}^{\infty} \varepsilon_m \sum_{j=1}^{4} \sum_{l=1}^{2} \big\{ \big[-\alpha_j^{m,l} \delta_j^{p2} \frac{\partial}{\partial r} \varsigma^{m,l}(k_j r) + \beta_j^{m,l} \delta_j^{p1} \frac{m}{r} \varsigma^{m,l}(k_j r)\big] \cos(m\theta) \\ - \; & \big[\alpha_j^{m,l} \delta_j^{p1} \frac{m}{r} \varsigma^{m,l}(k_j r) + \beta_j^{m,l} \delta_j^{p2} \frac{\partial}{\partial r} \varsigma^{m,l}(k_j r)\big] \sin(m\theta) \big\} e^{i(\gamma z - \omega t)}, \end{aligned}$$

(16) $$u_z \;=\; i \sum_{m=0}^{\infty} \varepsilon_m \sum_{j=1}^{4} \sum_{l=1}^{2} \big\{ \big[\alpha_j^{m,l} \delta_j^{p3} \varsigma^{m,l}(k_j r)\big] \cos(m\theta) + \big[\beta_j^{m,l} \delta_j^{p3} \varsigma^{m,l}(k_j r)\big] \sin(m\theta) \big\} e^{i(\gamma z - \omega t)},$$

$$V = i\sum_{m=0}^{\infty}\varepsilon_m\sum_{j=1}^{4}\sum_{l=1}^{2}\{[\alpha_j^{m,l}\delta_j^{p4}\zeta^{m,l}(k_jr)]\cos(m\theta)+[\beta_j^{m,l}\delta_j^{p4}\zeta^{m,l}(k_jr)]\sin(m\theta)\}e^{i(\gamma z-\omega t)}, \tag{17}$$

where $\alpha_j^{m,l}, \beta_j^{m,l}$ are arbitrary constants, $\zeta^{m,l}(k_jr)$ are the Bessel and modified Bessel functions of the first and second kind, $\delta_j^{pq} = -d_1^{pq}k_j^6 + d_2^{pq}k_j^4 - d_3^{pq}k_j^2 + d_4^{pq}$, $p, q, j = 1, 2, 3, 4$, d_s^{pq} depend on bone properties, frequency and wavelength, and k_j are the roots of the algebraic equation [11]

$$\alpha k_j^8 - \beta k_j^6 + \gamma k_j^4 - \delta k_j^2 + \varepsilon = 0. \tag{18}$$

The motion of the fluid in the cavity is governed by the Navier - Stokes equations

$$\rho_f\frac{\partial v_r}{\partial t} = -\frac{\partial p}{\partial r} + \eta(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{1}{r^2}v_r), \tag{19}$$

$$\rho_f\frac{\partial v_\theta}{\partial t} = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \eta(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{1}{r^2}v_\theta), \tag{20}$$

$$\rho_f\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2}), \tag{21}$$

and the continuity equation

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}v_r + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} = 0, \tag{22}$$

where v_r, v_θ, v_z are the velocity components of the fluid, ρ_f is its mass density of the fluid, p is the pressure and η is the viscocity.

Following the method proposed in [8] the analytical wave - type solution of the system of equations (19) - (22), in cylindrical coordinates, can be written as

$$v_r = \sum_{m=0}^{\infty}\varepsilon_m(U_{,r}^m + \frac{1}{r}\psi_{,\theta}^m)e^{i(\gamma z-\omega t)}, \tag{23}$$

$$v_\theta = \sum_{m=0}^{\infty}\varepsilon_m(\frac{1}{r}U_{,\theta}^m - \psi_{,r}^m)e^{i(\gamma z-\omega t)}, \tag{24}$$

$$v_z = i\sum_{m=0}^{\infty}\varepsilon_m w^m e^{i(\gamma z-\omega t)}, \tag{25}$$

$$p = i\sum_{m=0}^{\infty}\varepsilon_m \phi^m e^{i(\gamma z-\omega t)}, \tag{26}$$

where U^m, ψ^m, w^m and ϕ^m are functions of r and θ. The expressions (23) - (26) are used in Equations (19) - (22) and the resulting system can be solved using an auxilliary function g [11]. The obtained solution is

$$\begin{aligned} v_r &= \sum_{m=0}^{\infty}\varepsilon_m\sum_{l=1}^{2}\{[A_l^m\sigma_l^{i1}\frac{\partial}{\partial r}I^m(\mu_l r) + A_3^m\frac{m}{r}I^m(\mu_3 r)]\cos(m\theta) \\ &+ [B_l^m\sigma_l^{i1}\frac{\partial}{\partial r}I^m(\mu_l r) - B_3^m\frac{m}{r}I^m(\mu_3 r)]\sin(m\theta)\}e^{i(\gamma z-\omega t)}, \end{aligned} \tag{27}$$

$$\begin{aligned} v_\theta &= \sum_{m=0}^{\infty}\varepsilon_m\sum_{l=1}^{2}\{[B_l^m\sigma_l^{i1}\frac{m}{r}\frac{\partial}{\partial r}I^m(\mu_l r) - B_3^m\frac{\partial}{\partial r}I^m(\mu_3 r)]\cos(m\theta) \\ &- [A_l^m\sigma_l^{i1}\frac{m}{r}I^m(\mu_l r) + A_3^m\frac{\partial}{\partial r}I^m(\mu_3 r)]\sin(m\theta)\}e^{i(\gamma z-\omega t)}, \end{aligned} \tag{28}$$

$$v_z = \sum_{m=0}^{\infty}\varepsilon_m\sum_{l=1}^{2}\{\sigma_l^{i2}[A_l^m\cos(m\theta) + B_l^m\sin(m\theta)]I^m(\mu_l r)\}e^{i(\gamma z-\omega t)}, \tag{29}$$

$$p = \sum_{m=0}^{\infty} \varepsilon_m \sum_{l=1}^{2} \{\sigma_l^{i3} [A_l^m \cos(m\theta) + B_l^m \sin(m\theta)] I^m(\mu_l r)\} e^{i(\gamma z - \omega t)}, \tag{30}$$

where A_l^m, B_l^m are arbitrary constants and $I^m(z)$ are the modified Bessel functions of the first kind.

Next, we assume that the outer S_1 and the inner S_0 surfaces are of arbitrary shape and that the following boundary conditions hold:

(i) $T_{qq} = T_{qs} = T_{qz} = 0, \quad V = 0, \quad \text{on} \quad S_1,$

(ii) $T_{qq} = \tau_{qq}, T_{qs} = \tau_{qs}, T_{qz} = \tau_{qz}, \dot{u}_q = v_q, \dot{u}_s = v_s, \dot{u}_z = v_z, V = 0, \quad \text{on} \quad S_0,$

where q is the coordinate normal to the boundary, s is the tangent coordinate, T_{qq} is the normal stress, $T_{qi}, i = s, z$ are the shear stresses of the solid cylinder, and τ_{qq} is the normal stress, $\tau_{qi}, i = s, z$ are the shear stresses of the fluid and the "dot" denotes differentiation with respect to time.

Those boundary conditions correspond to outer surface which is stress free (in reality this is affected by the existence of muscles and ligaments) and covered with electrodes which are shorted. The latter is valid for the inner surface, since in this surface we have continuity of displacements, normal and shear stresses and electrostatic potential.

The coordinates q and s vary with the angle θ. Thus, to satisfy the boundary conditions, we follow Nagaya's procedure [10]. First the curved boundaries S_1 and S_0 are divided into small segments S_1^k and $S_0^k, k = 1, 2, \ldots, I$ for the outer and inner surface of the solid cylinder respectively. Then each of these directions is assumed to be constant although each magnitude varies along the segment. Thus, we can take approximately the curvilinear coordinate q as the orthogonal coordinate X^k and the curvilinear coordinate s as the orthogonal coordinate Y^k.

For the k - th segment the boundary conditions can be written in Cartesian coordinates as:

(i) $T_{XX} = T_{YY} = T_{Xz} = 0, \quad V = 0, \quad \text{on} \quad S_1^k$

(ii) $T_{XX} = \tau_{XX}, T_{XY} = \tau_{XY}, T_{Xz} = \tau_{Xz}, \dot{u}_X = v_X, \dot{u}_Y = v_Y, \dot{u}_z = v_z, V = 0, \quad \text{on} \quad S_0^k.$

Since the solution of the problem is expressed in terms of the coordinates r and θ, we transform the fields appearing in the boundary conditions in terms of these coordinates instead of the Cartesian coordinates X and Y.

However, the coordinate r at the boundaries is expressed as a function of θ. Thus, the boundary conditions cannot be satisfied directly. To overcome this problem, we perform the Fourier expansion of those equations along the inner and outer boundaries of the cross section. If the series appearing in the transformed expressions were truncated up to $N+1$, one has

$$det\{D_{rs}(\omega, \gamma, k_{ij}, c_{ij}, e_{ij}, \epsilon_{ij}, \eta)\} = 0. \tag{31}$$

For given material parameters $(c_{ij}, e_{ij}, \epsilon_{ij}, \eta)$, the relation (31) is a transcendental relation of the frequency ω, the wave number γ and the roots k_j of the equation (18).

The elements $D_{rs}, r, s = 1, 2, \ldots, 22N + 11$ of the matrix $\mathbf{D}$ are the Fourier coefficients of the Fourier expansion of the boundary conditions. If the geometric relations for the coordinate r at the boundaries are known, the integrals occured can be calculated numerically and thus the frequency equation is given by (31).

The determinant is shown in (32). In this representation $j = 1, 2, 3, 4, l = 1, 2$ and $\mathbf{D}^{r,s}$ are submatrices of order $N \times N$, $N \times (N+1)$, $(N+1) \times (N+1)$ or $(N+1) \times N$. The terms $D_{(1)}^{r,s}$ are the same as the terms $\mathbf{D}_{(0)}^{r,s}$ for $r = 1, 2, 3, \ldots, 8, s = j, j+4, j+8, j+12$ with the subscript $_{(1)}$ to correspond to $r = \widetilde{r}_1$ and the subscript $_{(0)}$ to $r = \widetilde{r}_{(0)}$.

The frequency equation can be solved numerically to obtain the frequency and attenuation coefficients as a function of wavelength.

3 Concluding Remarks

We have proposed a method to find the frequency relation for the wave propagation problem in human long bones filled with bone marrow. The piezoelectric behavior of the cortical bone itself as well as the viscous nature of the bone marrow introduce the peculiarity of the problem. We have led to a complicated expression for the frequency equation using the Fourier collocation method to overcome difficulties originated from the arbitrary cross section. This approach is very general and can be applied to any geometrical configuration.

References

[1] C. Malizos, M. Hantes, A. Mavrodontides, M. Plissiti, D.I. Fotiadis, C.V. Massalas, T. Bakas, G. Evangelakis and A. Moukarika, *Quantitative monitoring and prognosis of osteogenesis*, Med. Biol. Engng. Comp., 37(2) (1999), pp. 288–289.

[2] W.J.W. Sharrard, M.L. Sutcliffe, M.J. Robson, and A.G. Maceachern, *The treatment of fibrous non - union of fractures by pulsing electromagnetic stimulation*, J. Bone Joint Surg., 64-B (1982), pp. 189–193.

[3] A. Damask, *Bone: mechanical and electrical properties*, Academic Press, New York (1978).

[4] A. Ambardar and C.D Ferris, *Wave propagation in a piezoelectric two - layer cylindrical shell with hexagonal symmetry: Some implications for long bones*, J. Acoust. Soc. Am., 63 (1978), pp. 781–792.

[5] N. Guzelsu and S. Saha, *Electro - mechanical wave propagation in long bones*, J. Biomechanics, 14 (1981), pp. 19–33.

[6] H.S. Paul and M. Venkatensan, *Wave propagation in a piezoelectric solid cylinder of arbitrary cross section of class 6 (Bone)*, Int. J. Engng. Sci., 27 (1989), pp. 847–853.

[7] H.S. Paul and M. Venkatensan, *Wave propagation in a piezoelectric bone with cylindrical cavity of arbitrary shape*, Int. J. Engng. Sci., 29 (1991), pp. 1601–1607.

[8] D.I. Fotiadis, G. Foutsitzi and C.V. Massalas, *Wave propagation modelling in human long bones*, Acta Mechanica, XXX (1999), pp. 1–17.

[9] D.I. Fotiadis, G. Foutsitzi and C.V. Massalas, *Wave propagation in human long bones of arbitrary cross section*, Int. J. Engng. Sci., (2000).

[10] K. Nagaya, *Stress wave propagation in a bar of arbitrary cross section*, ASME J. Appl. Mech., 49 (1982), pp. 157–164.

[11] D.I. Fotiadis, G. Foutsitzi and C.V. Massalas, *The effect of the bone marrow presence in the wave propagation in human long bones*, in preparation.

$$
\left[\begin{array}{cccccccc}
\mathbf{D}^{1,j}_{(1)} & \mathbf{D}^{1,j+4}_{(1)} & \mathbf{D}^{1,j+8}_{(1)} & \mathbf{D}^{1,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{2,j}_{(1)} & \mathbf{D}^{2,j+4}_{(1)} & \mathbf{D}^{2,j+8}_{(1)} & \mathbf{D}^{2,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{3,j}_{(1)} & \mathbf{D}^{3,j+4}_{(1)} & \mathbf{D}^{3,j+8}_{(1)} & \mathbf{D}^{3,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{4,j}_{(1)} & \mathbf{D}^{4,j+4}_{(1)} & \mathbf{D}^{4,j+8}_{(1)} & \mathbf{D}^{4,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{5,j}_{(1)} & \mathbf{D}^{5,j+4}_{(1)} & \mathbf{D}^{5,j+8}_{(1)} & \mathbf{D}^{5,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{6,j}_{(1)} & \mathbf{D}^{6,j+4}_{(1)} & \mathbf{D}^{6,j+8}_{(1)} & \mathbf{D}^{6,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{7,j}_{(1)} & \mathbf{D}^{7,j+4}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{D}^{8,j+8}_{(1)} & \mathbf{D}^{8,j+12}_{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{1,j}_{(0)} & \mathbf{D}^{1,j+4}_{(0)} & \mathbf{D}^{1,j+8}_{(0)} & \mathbf{D}^{1,j+12}_{(0)} & \mathbf{D}^{9,l+16}_{(0)} & \mathbf{D}^{9,19}_{(0)} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{2,j}_{(0)} & \mathbf{D}^{2,j+4}_{(0)} & \mathbf{D}^{2,j+8}_{(0)} & \mathbf{D}^{2,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{10,l+19}_{(0)} & \mathbf{D}^{10,22}_{(0)} \\
\mathbf{D}^{3,j}_{(0)} & \mathbf{D}^{3,j+4}_{(0)} & \mathbf{D}^{3,j+8}_{(0)} & \mathbf{D}^{3,j+12}_{(0)} & \mathbf{D}^{11,l+16}_{(0)} & \mathbf{D}^{11,19}_{(0)} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{4,j}_{(0)} & \mathbf{D}^{4,j+4}_{(0)} & \mathbf{D}^{4,j+8}_{(0)} & \mathbf{D}^{4,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{12,l+19}_{(0)} & \mathbf{D}^{12,22}_{(0)} \\
\mathbf{D}^{5,j}_{(0)} & \mathbf{D}^{5,j+4}_{(0)} & \mathbf{D}^{5,j+8}_{(0)} & \mathbf{D}^{5,j+12}_{(0)} & \mathbf{D}^{13,l+16}_{(0)} & \mathbf{D}^{13,19}_{(0)} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{6,j}_{(0)} & \mathbf{D}^{6,j+4}_{(0)} & \mathbf{D}^{6,j+8}_{(0)} & \mathbf{D}^{6,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{14,l+19}_{(0)} & \mathbf{D}^{14,22}_{(0)} \\
\mathbf{D}^{7,j}_{(0)} & \mathbf{D}^{7,j+4}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{D}^{8,j+8}_{(0)} & \mathbf{D}^{8,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{17,j}_{(0)} & \mathbf{D}^{17,j+4}_{(0)} & \mathbf{D}^{17,j+8}_{(0)} & \mathbf{D}^{17,j+12}_{(0)} & \mathbf{D}^{17,l+16}_{(0)} & \mathbf{D}^{17,19}_{(0)} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{18,j}_{(0)} & \mathbf{D}^{18,j+4}_{(0)} & \mathbf{D}^{18,j+8}_{(0)} & \mathbf{D}^{18,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{18,l+19}_{(0)} & \mathbf{D}^{18,22}_{(0)} \\
\mathbf{D}^{19,j}_{(0)} & \mathbf{D}^{19,j+4}_{(0)} & \mathbf{D}^{19,j+8}_{(0)} & \mathbf{D}^{19,j+12}_{(0)} & \mathbf{D}^{19,l+16}_{(0)} & \mathbf{D}^{19,19}_{(0)} & \mathbf{0} & \mathbf{0} \\
\mathbf{D}^{20,j}_{(0)} & \mathbf{D}^{20,j+4}_{(0)} & \mathbf{D}^{20,j+8}_{(0)} & \mathbf{D}^{20,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{20,l+19}_{(0)} & \mathbf{D}^{20,22}_{(0)} \\
\mathbf{D}^{21,j}_{(0)} & \mathbf{D}^{21,j+4}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{21,l+16}_{(0)} & \mathbf{D}^{21,19}_{(0)} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{D}^{22,j+8}_{(0)} & \mathbf{D}^{22,j+12}_{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{D}^{22,l+19}_{(0)} & \mathbf{D}^{22,22}_{(0)}
\end{array}\right] \tag{32}
$$