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A. CHARALAMBOPOULOS, D.I. FOTIADIS
and C.V. MASSALAS

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Department of Computer Science
University of Ioannina
451 10 Ioannina, Greece

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A. Charalambopoulos[(1)], D.I. Fotiadis[(2)], D. Kourounis[(3)] and C.V. Massalas[(4)]

[(1)]Division of Mathematics, Polytechnic School, Aristotle University of Thessaloniki, GR 54006 Thessaloniki, Greece

[(2)]Dept. of Computer Science, University of Ioannina, GR 45110 Ioannina, Greece

[(3)]Dept. of Electrical Engineering, Polytechnic School, Aristotle University of Thessaloniki, GR 54006 Thessaloniki, Greece

[(4)]Dept. of Mathematics, University of Ioannina, GR 45110 Ioannina, Greece

SUMMARY

In the present work we extend the methodology of constructing the basis of vector Navier spheroidal functions in a form which is suitable for the solution of boundary value problems. Those eigensolutions are produced by the application of vector differential operators on solutions of the scalar Helmholtz equations. This procedure complicates the solution of the problem since the independent coordinates are connected through the scale factors and cannot be separated in their final form. We have tried to minimize the extended analytical burden of the final formulae by exploiting a priori general properties of the vector operators and underlying Helmholtz equation Kernel functions. Thus, the final expressions dispose the simplest possible forms which are adequate for the applications and render analytical facing of the problem efficient. In addition, we formulate the method of using those eigenvectors. The physical vector fields in boundary value problems are expressed in terms of Navier eigenvectors, they satisfy by construction the underlying equations and they are forced to satisfy the boundary conditions. Finally, in two cases, the electromagnetic and the elastic one, which are representative problems with differences occuring in the treatment of boundary conditions, we demonstrate the use of vector Navier spheroidal functions.

Key Words: Navier Functions; Spheroidal Geometry.

AMS Subject Classification: 34K10; 33E99.

1. Introduction

The purpose motivating the current work stems from the necessity of investigating very interesting problems arising in a plethora of scientific areas, in the framework of the spheroidal coordinate system.

Indeed, the main factor contributing to the complexity of the analytical methods constructed to describe, investigate and solve mathematical models simulating physical processes, is due to the underlying geometry fitting suitably to the characteristics of the system under consideration. In fact, the majority of the analytical methods have been developed in order to describe systems and processes, "living" in the spherical geometry. In addition, even in cases that the underlying geometry is not the spherical one, there is the trend to approximate the problem under investigation by its spherical equivalent "copy" and apply then to it the well tested analytical approaches. Of course, this is permittable and actually very effective in case that the studied system is, geometrically, a perturbation of the spherical ideal case or in case that there exists qualitative analysis certifying actually unaffectiveness of the response of the system to geometrical variance. However, the mostpart of the interesting problems do not dispose the symmetry spherical geometry requires and then the extension of the existed analytical knowledge to other coordinate systems must be followed. Actually, this extension has several limitations emerging from the applicability of the analytical procedures to the new systems as this can be testified a priori using basic criteria. As a matter of fact these criteria include the solvability of the equations describing the system behavior, following fundamental mathematical tools as coordinate separability of the underlying equations, possibility of application of Green function techniques and possibility of reduction to the spherical case through simple limit procedures assuring stability.

In addition to the satisfaction of general solvability criteria, the adopted geometrical system must fit geometrically within a satisfactory degree of accuracy, with a whole family of interesting geometrical structures arising in applications. In other words, the adopted system must have parametrical representation permitting dimension scaling to cover a large group of geometrical configurations.

One of the coordinate systems sharing all the previous privilleges is the spheroidal coordinate system. It fits with all the configurations lacking symmetry to one direction and behaves normally when someone tries to adapt analysis to it. In this work, the possibilities offered by this coordinate system are analyzed and used suitably allowing the formation of the general framework for a variety of applications as it will be clarified in the sequel.

In many scientific branches of great interest, including theory of elasticity, biomechanics, scattering theory of acoustic, electromagnetic or elastic waves, the structures under investigation can be well simulated through simple geometrical setups. As an example, the authors have developed an extended research methodology of studying very important problems of biomechanics concerning dynamic characteristics of cranial systems, or problems of wave propagation, adopting simple geometrical assumptions [1-4]. Refering to the above mentioned scientific areas, the crucial functions representing the behavior of the investigated problems satisfy a variety of partial differential equations like the Helmholtz equation, the Maxwell equations or the equation of the linearized elasticity. The formulation of the general framework for the solution of the previous equation is a difficult task even for the spherical coordinate system.

However, special remark must be mentioned to the fact that the majority of equations, supporting mainly electromagnetic and elastic problems, concern vector instead of scalar fields and this implies an extra and serious amount of complication to the resulting analysis. Actually the vector character of the sought solutions is not just an extra analysis burden but incorporates a very important qualitative difference in the general approach of the problem. The essence lies on the fact that it is preferable to construct a basis set, to represent the solution space, consisted of vector eigensolutions instead of imposing the vector character of the solution to the coefficients of equivalent scalar eigensolution expansions. This alternative approach offers the possibility to incorporate in the vector form of the basis solutions some «immediate» properties reflecting their particular nature (ex. solenoidal or irrotational property of electromagnetic fields). In addition it is flexible and more adequate to have a basis space having the same algebraic structure with the original space of the determinable solutions and finally the unknown coefficients of the basis expansion are scalars and more easily recovered in general.

However, the price to pay in order to establish this framework is generally high. In cartesian, cylindrical or spherical geometry, the difference concerning the difficulty level between scalar and vector solution construction is already apparent. Electromagnetics and elasticity are formulated in a very efficient manner through the well defined Navier eigenvectors incorporating a priori special physical properties and constituting a complete set of solutions of the underlying equations [5]. The authors have testified repeatedly the usefulness of these functions in the solution of many interesting application problems permitting spherical geometry assumption. An important complication factor is the fact that differential equation and boundary conditions require the intermediate derivation of several physical quantities based on Navier eigenvectors (ex. stresses in elasticity), interpreted mathematicaly as multiple differential operations on them, contributing essentialy to the augmentation of the analytical burden. These manipulations are extended although the above mentioned systems «cooperate kindly» revealing all their symmetries.

The usefulness and fitness of the above fundamental vector fields in the prementioned simple geometrical systems renders essentially their investigation to other geometrical configurations simulating, in a more realistic manner, specific problems. As mentioned above, the spheroidal system is very adequate to represent most physical situations, with preferred direction, appearing in applications and simultaneously allows the applicability of several analytical methods. However, it results that the extension of the methodology of recovering vector fields to the more realistic spheroidal system turns out to be a really tough task and that is what the present work aims at. The complexity of spheroidal vector fields construction is proved indirectly by the fact that almost all the research applied to vector field problems, arising in applications, in the regime of spheroidal coordinates, has been realized through expansions in terms of scalar solutions, the well known spheroidal wave functions [5]. However, as mentioned above, this approach has disadvantages and fails to affront the general problem in a suitable and unified manner. Moreover, the already existed knowledge [6] about vector solutions in spheroidal geometry includes extended and complicated expressions, whose analytical treatment is, in mostimes, a very difficult task rendering the analytical handling of the underlying problems undesirable.

In the present work then we extend the methodology to construct the basis of vector Navier spheroidal functions under the usefulness point of view. As a matter of fact, these eigensolutions are produced after applying vector differential operators on solutions of the scalar Helmholtz equations. Unfortunately, in contrast to spherical geometry, in spheroidal coordinates the grad, div and curl operators, which give birth to the Navier functions after being apllied to the above mentioned suitable scalar functions, are complicated. Their main disadvantage is that the independent coordinates are connected through the scale factors and cannot be separated in the final form. Actually, this is a rather qualitative difference but someone can not ignore the quantitative extra burden appearing in the spheroidal system due to the fact that the scalar Helmholtz equation has- in separable coordinate forms- much more complicated solutions since the basic ingredients functions are infinite expansions of Legendre and spherical Bessel functions, which constitute the simple basis functions in spherical geometry. In what follows we have tried to overcome these problems by expoiting properties of vector operators and finally formulate the wanted functions in a suitable form. To illustrate the use of those functions we have chosen two cases, the electromagnetic and the elastic one where we have used the proposed methodology to formulate the solution of the problems in terms of the spheroidal Navier vectors.

2. Spheroidal Geometry- Spheroidal Wave Functions.

In this section, we present all the necessary information about the geometrical system under investigation and the scalar solutions of the Helmholtz equation, which fournishes with the basic functions via which are represented the solutions of the differential equations under examination.

The connection between cartesian and spheroidal coordinates as well as the scalar factors are given by the relations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\alpha \sinh \mu \sin \theta \begin{pmatrix} \cos\varphi \\ \sin \varphi \end{pmatrix}$$

$$z = \frac{1}{2}\alpha \cosh \mu \cos\theta$$

$$h_\mu = h_\theta = \frac{1}{2}\alpha\sqrt{\cosh \mu^2 - \cos^2 \theta}$$

$$h_\varphi = \frac{1}{2}\alpha \sinh \mu \sin \theta$$

where the spheroidal coordinates range over the intervals

$\mu \geq 0, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$.

The case $\mu = 0$ corresponds to the line interval connecting the two focii of the spheroidal system located at the points $z = -\frac{1}{2}\alpha$ end $z = \frac{1}{2}\alpha$.

The Laplace operator in spheroidal coordinates takes the form:

$$\Delta\psi = \frac{\frac{4}{a^2}}{\cosh\mu^2 - \cos^2\theta}\left[\begin{array}{l}\frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}\left(\sinh\mu\frac{\partial\psi}{\partial\mu}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) \\ + \frac{\sinh\mu^2 + \sin^2\theta}{\sinh\mu^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\end{array}\right] \tag{1}$$

Let us consider the Helmholtz equation

$$\Delta\psi + \mathrm{k}^2\psi = 0 \tag{2}$$

Applying separation of variables techniques we conclude that

$$\psi = \begin{pmatrix}\cos \\ \\ \sin\end{pmatrix}(\mathrm{m}\varphi)R(\xi)S(\eta), \quad \eta = \cos\vartheta,\ \xi = \cosh\mu \tag{3}$$

where the functions R, S satisfy the equations

$$\left.\begin{array}{l}\frac{d}{d\xi}\left[(\xi^2-1)\frac{dR}{d\xi}\right] - \left[\lambda_{mn} - c^2\xi^2 + \frac{m^2}{\xi^2-1}\right]R = 0 \\ \frac{d}{dn}\left[(n^2-1)\frac{dS}{dn}\right] - \left[\lambda_{mn} - c^2n^2 + \frac{m^2}{n^2-1}\right]S = 0\end{array}\right\} \quad c = \frac{1}{2}\alpha k, \tag{4}$$

and λ_{mn} stand for separation of variable constants.

It is proved [5, 7], that the functions R, S are given by the relations

$$S_{mn}(n;c) = \sum_{k=0,1}^{\infty}{}' d_k^{mn}(c)P_{m+k}^m(n) = \begin{cases}\sum_{k=0}^{\infty} d_{2k}^{mn}(c)P_{m+2k}^m(n), & n = m, m+2, \ldots \\ \sum_{k=0}^{\infty} d_{2k+1}^{mn}(c)P_{m+2k+1}^m(n), & n = m+1, m+3, \ldots\end{cases} \tag{5}$$

$$R_{mn}^{(p)}(\xi;c) = \left[\sum_{k=0,1}^{\infty}{}' \frac{(2m+k)!}{k!}d_k^{mn}(c)\right]^{-1}\left(1-\frac{1}{\xi^2}\right)^{m/2}\sum_{k=0,1}^{\infty}{}' i^{k+m-n}\frac{(2m+k)!}{k!}d_k^{mn}(c)Z_{m+k}^{(p)}(c\xi)$$

(6)

where there exist four alternatives for the spherical Bessel functions $Z_{m+k}^{(p)}$

$$Z_n^{(1)}(z) = j_n(z)$$
$$Z_n^{(2)}(z) = y_n(z)$$
$$Z_n^{(3)}(z) = h_n^{(1)}(z) = \left(j_n(z) + iy_n(z)\right)$$
$$Z_n^{(4)}(z) = h_n^{(2)}(z) = (j_n(z) - iy_n(z)$$

while $P_n^m(n)$ denote the Legendre functions. In addition, the symbol $\sum_{k=0,1}^{\infty}{}'$, as it is clear from Equation (5), indicates summation over even or odd indices, depending on the starting index.

The crucial point is the determination of the coefficients $d_k^{mn}(c)$.

Inserting Equations (5), (6) in Equation (2) and exploiting recurrence relations for Legendre functions we conclude to the following recursive scheme:

$$\frac{(2m+k+2)(2m+k+1)}{(2m+2k+5)(2m+2k+3)}c^2 d_{k+2}^{mn}(c) + \begin{bmatrix} (m+k)(m+k+1) - \lambda_{mn}(c) + \\ \dfrac{2(m+k)(m+k+1) - 2m^2 - 1}{(2m+2k-1)(2m+2k+3)}c^2 \end{bmatrix} d_k^{mn}(c) \qquad (7)$$
$$+ \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}c^2 d_{k-2}^{mn}(c) = 0$$

Setting for simplicity

$$\alpha_k = \frac{(2m+k+2)(2m+k+1)}{(2m+2k+5)(2m+2k+3)}c^2$$
$$\beta_k = (m+k)(m+k+1) + c^2\frac{2(m+k)(m+k+1) - 2m^2 - 1}{(2m+2k-1)(2m+2k+3)}$$
$$\gamma_k = \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}c^2$$

we obtain the homogeneous system

$$\begin{aligned}
&\alpha_0 d_2^{mn}(c) + [\beta_0 - \lambda_{mn}(c)] d_0^{mn}(c) = 0 \\
&\alpha_1 d_3^{mn}(c) + [\beta_1 - \lambda_{mn}(c)] d_1^{mn}(c) = 0 \\
&\ldots \\
&\alpha_k d_{k+2}^{mn}(c) + [\beta_k - \lambda_{mn}(c)] d_k^{mn}(c) + \gamma_k d_{k-2}^{mn}(c) = 0.
\end{aligned} \tag{8}$$

We obtain then, separately, the following eigenvalue problems

$$\begin{bmatrix} \beta_0 & \alpha_0 & 0 & 0 & \ldots & \ldots \\ \gamma_2 & \beta_2 & \alpha_2 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & .. & \gamma_{2k} & \beta_{2k} & \alpha_{2k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} d_0^{mn}(c) \\ d_2^{mn}(c) \\ d_4^{mn}(c) \\ \ldots \\ d_{2k}^{mn}(c) \\ \ldots \end{bmatrix} = \lambda_{mn}(c) \begin{bmatrix} d_0^{mn}(c) \\ d_2^{mn}(c) \\ d_4^{mn}(c) \\ \ldots \\ d_{2k}^{mn}(c) \\ \ldots \end{bmatrix} \tag{9a}$$

for $(n-m)$ even.

$$\begin{bmatrix} \beta_1 & \alpha_1 & 0 & 0 & \ldots & \ldots \\ \gamma_3 & \beta_3 & \alpha_3 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & .. & \gamma_{2k+1} & \beta_{2k+1} & \alpha_{2k+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} d_1^{mn}(c) \\ d_3^{mn}(c) \\ d_5^{mn}(c) \\ \ldots \\ d_{2k+1}^{mn}(c) \\ \ldots \end{bmatrix} = \lambda_{mn}(c) \begin{bmatrix} d_1^{mn}(c) \\ d_3^{mn}(c) \\ d_5^{mn}(c) \\ \ldots \\ d_{2k+1}^{mn}(c) \\ \ldots \end{bmatrix} \tag{9b}$$

for $(n-m)$ odd.

Equation (9a) provides with the eigenvalues $\lambda_{m,m}(c), \lambda_{m,m+2}(c), \lambda_{m,m+4}(c), \ldots$ and

Equation (9b) provides with the eigenvalues $\lambda_{m,m+1}(c), \lambda_{m,m+3}(c), \lambda_{m,m+5}(c), \ldots$

For every $\lambda_{mn}(c)$ determined above, the coefficients $d_k^{mn}(c)$ are determined modulo a multiplicative constant. These coefficients are fully determined when a normalization condition is imposed.

In Ref. [7], we can find that

$$\sum_k \frac{(k+2m)!}{k!} d_k^{mn}(c) = \frac{(n+m)!}{(n-m)!} \tag{10}$$

Under the above condition, Equation (6) which expresses the "radial" functions becomes simpler and takes the form

$$R_{mn}^{(P)}(\xi;c) = \frac{(n-m)!}{(n+m)!}\left(1-\frac{1}{\xi^2}\right)^{m/2} \sum_{k=0,1}^{\infty}{}' i^{k+m-n} \frac{(2m+k)!}{k!} d_k^{mn}(c) Z_{m+k}^{(P)}(c\xi). \tag{11}$$

3. A Brief Discussion on the General Theory of Vector Eigenfuctions.

In this section, we present the general theory giving birth to the vector solutions of the vector Helmholtz equation:

$$\nabla^2 \mathbf{F}(\mathbf{r}) + k^2 \mathbf{F}(\mathbf{r}) = \nabla(\nabla \cdot \mathbf{F}(\mathbf{r})) - \nabla \times (\nabla \times \mathbf{F}(\mathbf{r})) + k^2 \mathbf{F}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in R^3. \tag{12}$$

As mentioned above, the vector Equation (12) governs many physical phenomena. Indeed, the harmonic electromagnetic fields [8] or the transverse and longitudinal elastic fields [9] satisfy this equation and the crucial parameter k corresponds to the wave number incorporated to the harmonic character of the specific situation. Although Equation (12) concerns time-reduced configuration, the solutions of the vector wave equation may be derived from the solutions of (12) by the Fourier transform.

For an arbitrary vector function $\mathbf{F}(\mathbf{r})$ satisfying Equation (12), let us define the vector field

$$\mathbf{w}(\mathbf{r}) = \frac{1}{4\pi}\iiint \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\mathrm{d}\mathbf{r}', \quad \mathbf{r} \in R^3 \tag{13}$$

where integration extends to R^3.

It is proved that the vector field $\mathbf{w}(\mathbf{r})$ satisfies the vector Poisson equation

$$\nabla^2 \mathbf{w}(\mathbf{r}) = -\mathbf{F}(\mathbf{r}), \quad \mathbf{r} \in R^3 \ . \tag{14}$$

Using, then, the vector formula

$$\nabla \times (\nabla \times) = \nabla(\nabla \cdot) - \nabla^2 \tag{15}$$

we conclude that

$$\mathbf{F} = -\nabla(\nabla \cdot \mathbf{w}) + \nabla \times (\nabla \times \mathbf{w}). \tag{16}$$

This is exactly Helmholtz decomposotion theorem, valid for any function $\mathbf{F}(\mathbf{r})$, not necessarily satisfying Equation (12), and states that any vector field may be expresed as the sum of the gradient of a scalar potential and the curl of a vector potential.

Consequently, Equation (16) takes the form

$$\mathbf{F}(\mathbf{r}) = \nabla\varphi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r}) \tag{17}$$

where $\varphi(\mathbf{r})$ stands for the scalar potential, $\mathbf{A}(\mathbf{r})$ stands for the vector potential and are given by the relations

$$\varphi(\mathbf{r}) = -\nabla \cdot \mathbf{w}(\mathbf{r}) \tag{18a}$$

$$\mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{w}(\mathbf{r}). \tag{18b}$$

The first of the fields given in representation (17) is usually called the *longitudinal* (or lamellar) component, since a gradient points in the direction of greatest rate of change of the scalar potential. The second is called the *transverse* (or solenoidal) component, since the curl of a vector is usually transverse to the direction of greatest change.

In addition, in case that $\mathbf{F}(\mathbf{r})$ satisfies Equation (12), then it is straightforward to show that the potentials φ and $\mathbf{A}$ satisfy the equations

$$\nabla^2 \varphi(\mathbf{r}) + k^2 \varphi(\mathbf{r}) = 0 , \quad \mathbf{r} \in R^3 \tag{19a}$$

$$\nabla^2 \mathbf{A}(\mathbf{r}) + k^2 \mathbf{A}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in R^3 . \tag{19b}$$

Eq. (19a) reveals the first advantage of the separation of the vector fields to longitudinal and transverse components. In fact, the determination of the longitudinal component is totally equivalent to the determination of the scalar potential $\varphi(\mathbf{r})$, which solves the much easier scalar Helmholtz equation, whose examination has been presented in previous section.

So, the longitudinal solutions have the form

$$\mathbf{L} = \nabla\varphi \tag{20}$$

where φ solves the scalar Helmholtz function.

In the sequel, our effort must be devoted to the transverse component. This field may always be derived from a pair of scalar fields, fact implied by its free divergence property, reducing then by one the initial general three scalars representing it.

Under a general point of view, we consider the curvilinear coordinates ξ_1, ξ_2, ξ_3, with scale factors h_1, h_2, h_3 and suppose that this coordinate system adapts the physical problem thanks to the fact, for example, that some boundary of the problem coincides with the coordinate surface $\xi_1 = C$.

It would be advantageous to choose the two scalars describing the transverse field so that the part of the field derived from one scalar be tangential to the surface $\xi_1 = C$ and the other to be normal to it. A vector normal to the ξ_1 surface is $\mathbf{a_1}f$, where f stands for some scalar function of ξ_1, ξ_2, ξ_3, which has to be determined. This is not, however, always a transverse field, its divergence is seldom zero, even if f is independent of ξ_1.

However, the vector

$$\mathbf{M} = \nabla \times (\mathbf{a_1}f) = \frac{\mathbf{a_2}}{\mathrm{h_1 h_3}} \frac{\partial}{\partial \xi_3}(\mathrm{h_1} f) - \frac{\mathbf{a_3}}{\mathrm{h_1 h_2}} \frac{\partial}{\partial \xi_2}(\mathrm{h_1} f) \tag{21}$$

is tangential to the surface $\xi_1 = C$ and since is, a priori, divergence free, constitutes the one sought solution after the scalar f has been determined.

Extended manipulations [5] lead to the conclusion that for six from the eleven coordinate systems which allow separation of the scalar Helmholtz equation, it is possible to set up a solution of the type introduced by Eq. (20) in the general form

$$\mathbf{M} = \nabla \times \left(\mathbf{a}_1 w(\xi_1)\psi\right) \tag{22}$$

where ψ is a solution of the scalar Helmholtz equation, $\xi_1 = z$, $w = 1$ for the cylindrical coordinates (including cartesian) and where $\xi_1 = r$, $w = r$ for the spherical and conical coordinates.

Consequently the one independent transverse vector solution is based on one scalar field satisfying the scalar Helmholtz equation. The other transverse component would be desirable to be normal to the coordinate surface $\xi_1 = C$. As mentioned above a function of the form $\mathbf{a}_1 f$ can not be expected to play this role. Nevertheless, every function of the form

$$\mathbf{N} = \nabla \times \nabla \times \left(\mathbf{a}_1 w \chi\right), \tag{23}$$

where χ satisfies also scalar Helmholtz equation and w has the same form as in Equation (21), constitutes a transverse solution, which can be proved to be normal to the coordinate surface $\xi_1 = C$. Actually, $\mathbf{N}$ is not identical to $\mathbf{M}$ even if $\chi = \psi$. In fact $\mathbf{N}$ is often perpendicular to $\mathbf{M}$ when $\chi = \psi$.

Summarizing all these results, we conclude that we can define three scalar functions, φ, ψ, χ, all satisfying the scalar Helmholtz equation and leading to the construction of a basis for the most general solutions of the vector Helmholtz equation and this basis consists of the vector fields given by Equations (20), (22), (23) whose form allows a suitable application of the various boundary conditions.

4. Spheroidal Eigenvectors

The analysis presented in last chapter reveals the limited cases in which the construction of vector eigenvectors satisfies the necessary criteria for the «reasonable» solvability of boundary value problems.

At first glance, the restriction in the choice of the possible crucial functions $w(\xi_1)$ brings bad news for the spheroidal geometry. Indeed, flexibility in the treatment of the boundary conditions on spheroidal surfaces would require "candidate" transverse eigenvectors of the following type:

$$\mathbf{M} = \nabla \psi \times \hat{\xi} \ ,$$
$$\mathbf{N} = \nabla \times \left(\nabla \psi \times \hat{\xi}\right). \qquad (24)$$

In fact, these vectors live in the tangent and normal space of the surface and adapt very well to several types of boundary conditions.

Unfortunately, these vector functions do not satisfy vector Helmholtz equation! This can be testified directly but it can be deduced from the general theory presented in last section, given that no permitable function $w(\xi_1)$ leads to expressions (24).

Consequently, we are obliged to restrict ourselves to vector eigenvectors constructed via auxiliary vectors not coinciding with the unit vectors of the spheroidal geometry. This necessary assumption is accompanied with the disadvantage that the boundary conditions can not have the most possible easily handled form, encountered in cartesian, spherical and conical coordinates.

In contrast, the vectors

$$^{\mathbf{a}}\mathbf{M} = \nabla \psi \times \mathbf{a} \quad \text{and}$$
$$^{\mathbf{a}}\mathbf{N} = \nabla \times {}^{\mathbf{a}}\mathbf{M} = \nabla \times (\nabla \psi \times \mathbf{a}) \qquad (25)$$

where $\mathbf{a} \in \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \mathbf{r}\}$ ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are the unit vectors in the x, y, z directions respectively), do satisfy vector Helmholtz equation and constitute eight choices to represent the transverse components of the vector fields, forcing ψ to run over the countable basis set of scalar Helmholtz equation solutions. In addition, they have the most symmetric form and fournish with the least burden to the complication of the boundary conditions.

The transverse solutions (25) together with the longitudinal solutions $\mathbf{L} = \nabla \psi$ have to be expressed in the spheroidal coordinate system. To accomplish this, we use the spheroidal representation of *grad* operator

$$\nabla = \frac{2}{a} \frac{1}{\left(\sinh^2 \mu + \sin^2 \theta\right)^{1/2}} \left[\hat{\mu} \frac{\partial}{\partial \mu} + \hat{\theta} \frac{\partial}{\partial \theta} \right] + \frac{2\hat{\varphi}}{a \sinh \mu \sin \theta} \frac{\partial}{\partial \varphi}$$

$$\left(= \frac{2}{a}\frac{1}{\left(\xi^2-\eta^2\right)^{1/2}}\left[\hat{\xi}\left(\xi^2-1\right)^{1/2}\frac{\partial}{\partial\xi}+\hat{\eta}\left(1-\eta^2\right)^{1/2}\frac{\partial}{\partial\eta}\right]+\frac{2\hat{\varphi}}{a\left(\xi^2-1\right)^{1/2}\left(1-\eta^2\right)^{1/2}}\frac{\partial}{\partial\varphi}\right).$$

(26)

The longitudinal basis functions $\mathbf{L}_{\substack{e\\o}mn}^{(i)}$ are obtained after applying *grad* operator (26) on scalar functions

$$\psi_{\substack{e\\o}mn}^{(i)} = R_{mn}^{(i)}(\xi,c)S_{mn}(\eta,c)\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}.$$

We obtain then

$$\mathbf{L}_{\substack{e\\o}mn}^{(i)} = \frac{2}{a}\frac{1}{\left(\xi^2-\eta^2\right)^{1/2}}\begin{bmatrix}\hat{\xi}\left(\xi^2-1\right)^{1/2}S_{mn}(\eta,c)\dfrac{d}{d\xi}R_{mn}^{(i)}(\xi,c)\\ +\hat{\eta}\left(1-\eta^2\right)^{1/2}R_{mn}^{(i)}(\xi,c)\dfrac{d}{d\eta}S_{mn}(\eta,c)\end{bmatrix}\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}$$
$$+\frac{2m\hat{\varphi}}{a\left(\xi^2-1\right)^{1/2}\left(1-\eta^2\right)^{1/2}}R_{mn}^{(i)}(\xi,c)S_{mn}(\eta,c)\begin{Bmatrix}-\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}. \qquad (27)$$

Let us now treat the transverse vector fields expressed by relations (25).

We start with the vectors ${}^{\mathbf{r}}\mathbf{M}$ and ${}^{\mathbf{r}}\mathbf{N}$, which correspond to the «right» choice in the spherical case. Actually they constitute the most "clever" selection when the spheroid under consideration is a perturbed sphere.

In order to determine ${}^{\mathbf{r}}\mathbf{M}$ we use the relation

$${}^{\mathbf{r}}\mathbf{M}_{\substack{e\\o}mn}^{(i)} = \nabla\psi_{\substack{e\\o}mn}^{(i)}\times\mathbf{r} \qquad (28)$$

which leads to the expression

$${}^{\mathbf{r}}\mathbf{M}_{\substack{e\\o}mn}^{(i)} = \frac{m}{\left(\xi^2-\eta^2\right)^{1/2}}R_{mn}^{(i)}(\xi,c)S_{mn}(\eta,c)\left(\frac{\eta}{\left(\xi^2-1\right)^{1/2}}\hat{\xi}-\frac{\xi}{\left(1-\eta^2\right)^{1/2}}\hat{\eta}\right)\begin{Bmatrix}-\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}$$
$$+\frac{\left(\xi^2-1\right)^{1/2}\left(1-\eta^2\right)^{1/2}}{\left(\xi^2-\eta^2\right)}\hat{\varphi}\left(\xi R_{mn}^{(i)}(\xi,c)\frac{d}{d\eta}S_{mn}(\eta,c)-\eta S_{mn}(\eta,c)\frac{d}{d\xi}R_{mn}^{(i)}(\xi,c)\right)\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}. \qquad (29)$$

The construction of the vector field ${}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)}$ can be based on the definition

$$
{}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)} = \nabla \times {}^{\mathbf{r}}\mathbf{M}_{\substack{e\\o}mn}^{(i)}
$$

and realized after applying the operator *curl* - expressed in spheroidal coordinates - to the previously constructed eigenvector ${}^{\mathbf{r}}\mathbf{M}_{\substack{e\\o}mn}^{(i)}$. We obtain then, after some calculations that

$$
\begin{aligned}
{}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)} = {} & \frac{2\left(\xi^2-1\right)^{1/2}}{a\left(\xi^2-\eta^2\right)^{1/2}}\Bigg[-\frac{d}{d\xi}R_{mn}^{(i)}(\xi,c)\frac{d}{d\eta}\left(\frac{\eta\left(1-\eta^2\right)S_{mn}(\eta,c)}{\left(\xi^2-\eta^2\right)}\right) \\
& +\xi R_{mn}^{(i)}(\xi,c)\frac{d}{d\eta}\left(\frac{\left(1-\eta^2\right)}{\left(\xi^2-\eta^2\right)}\frac{d}{d\eta}S_{mn}(\eta,c)\right)-\frac{m^2\xi}{\left(\xi^2-1\right)\left(1-\eta^2\right)}R_{mn}^{(i)}(\xi,c)S_{mn}(\eta,c)\Bigg]\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}\hat{\xi} \\
& +\frac{2\left(1-\eta^2\right)^{1/2}}{a\left(\xi^2-\eta^2\right)^{1/2}}\Bigg[\frac{d}{d\xi}\left(\frac{\xi\left(\xi^2-1\right)R_{mn}^{(i)}(\xi,c)}{\left(\xi^2-\eta^2\right)}\right)\frac{d}{d\eta}S_{mn}(\eta,c) \\
& -\eta S_{mn}(\eta,c)\frac{d}{d\xi}\left(\frac{\left(\xi^2-1\right)}{\left(\xi^2-\eta^2\right)}\frac{d}{d\xi}R_{mn}^{(i)}(\xi,c)\right)+\frac{m^2\eta}{\left(\xi^2-1\right)\left(1-\eta^2\right)}R_{mn}^{(i)}(\xi,c)S_{mn}(\eta,c)\Bigg]\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}\hat{\eta} \\
& -\frac{2m\left(\xi^2-1\right)^{1/2}\left(1-\eta^2\right)^{1/2}}{a\left(\xi^2-\eta^2\right)}\Bigg[-\frac{R_{mn}^{(i)}(\xi,c)}{\left(\xi^2-1\right)}\frac{d}{d\eta}\left(\eta S_{mn}(\eta,c)\right) \\
& -\frac{d}{d\xi}\left(\xi R_{mn}^{(i)}(\xi,c)\right)\frac{S_{mn}(\eta,c)}{\left(1-\eta^2\right)}\Bigg]\begin{Bmatrix}-\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}\hat{\varphi}\ . \qquad (30)
\end{aligned}
$$

Actually, this is the straightforward construction of the eigenvector ${}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)}$ and previous relation is the preferable form, under some possible slight modifications used in a few works treating spheroidal problems [6, 12, 13]. However the expression (30) is a very complicated expression and becomes much more complex if someone tries to apply boundary differential operators on it. As a matter of fact, it contains a lot of differentiations of second order and this creates the question whether some terms can be simplified after combining this expression with the differential equation itself. In other words there exists the feeling that expression (30) contain some ficticious terms, which should be rearranged suitably to lightening the burden of the equation. Actually these terms stem from the fact that operator *curl* does not "know" the differential equation satisfied by the functions on which it acts. Nevertheless, instead of rearranging terms, it would be preferable to follow a simple different procedure to obtain an alternative «minimal» expression of ${}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)}$.

Indeed, following some simple arguments based on differential equation properties, we begin with the definition equation

$$ {}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)} = \nabla \times {}^{\mathbf{r}}\mathbf{M}_{\substack{e\\o}mn}^{(i)} \tag{31} $$

and we obtain

$$ {}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)} = \nabla \times \left(\nabla \times {}^{\mathbf{r}}\mathbf{M}_{\substack{e\\o}mn}^{(i)} \right) = \nabla \times \left[\nabla \times \left(\psi_{\substack{e\\o}mn}^{(i)} \mathbf{r} \right) \right] = $$
$$ \nabla \left[\nabla \cdot \left(\psi_{\substack{e\\o}mn}^{(i)} \mathbf{r} \right) \right] - \nabla^2 \left(\psi_{\substack{e\\o}mn}^{(i)} \mathbf{r} \right) = \nabla \left[\nabla \psi_{\substack{e\\o}mn}^{(i)} \cdot \mathbf{r} \right] + 3 \nabla \psi_{\substack{e\\o}mn}^{(i)} - \nabla^2 \left(\psi_{\substack{e\\o}mn}^{(i)} \right) \mathbf{r} - 2 \nabla \psi_{\substack{e\\o}mn}^{(i)} = $$
$$ \nabla \psi_{\substack{e\\o}mn}^{(i)} + k^2 \psi_{\substack{e\\o}mn}^{(i)} \mathbf{r} + \nabla \left[(\mathbf{r} \cdot \nabla) \psi_{\substack{e\\o}mn}^{(i)} \right] = 2 \nabla \psi_{\substack{e\\o}mn}^{(i)} + k^2 \psi_{\substack{e\\o}mn}^{(i)} \mathbf{r} + (\mathbf{r} \cdot \nabla) \nabla \psi_{\substack{e\\o}mn}^{(i)} \tag{32} $$

The first term of the last part of representation (32) is equal to $\mathbf{L}_{\substack{e\\o}mn}^{(i)}$, while the third term is acquired after applying the differential operator $\mathbf{r} \cdot \nabla$ on the same function. As far as the second term is concerned, it has a very simple expression.

In other words, we have

$$ {}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)} = 2\mathbf{L}_{\substack{e\\o}mn}^{(i)} + (\mathbf{r} \cdot \nabla) \mathbf{L}_{\substack{e\\o}mn}^{(i)} + k^2 \psi_{\substack{e\\o}mn}^{(i)} \mathbf{r} . \tag{33} $$

Another useful representation of ${}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)}$ is the intermediate step of (32) fournishing the formula

$$ {}^{\mathbf{r}}\mathbf{N}_{\substack{e\\o}mn}^{(i)} = \mathbf{L}_{\substack{e\\o}mn}^{(i)} + \nabla \left(\mathbf{r} \cdot \mathbf{L}_{\substack{e\\o}mn}^{(i)} \right) + k^2 \Psi_{\substack{e\\o}mn}^{(i)} r . \tag{34} $$

It is true that expressions (32-34) dispose as well as second order differentiations (all appearing in the term $(\mathbf{r} \cdot \nabla) \mathbf{L}_{\substack{e\\o}mn}^{(i)}$), but now no retractable differentiation is appearing.

Similar treatment can be applied to the construction of vector solutions ${}^{x}\mathbf{M}, {}^{y}\mathbf{M}, {}^{z}\mathbf{M}, {}^{x}\mathbf{N}, {}^{y}\mathbf{N}, {}^{z}\mathbf{N}$. Together with the eigenvectors $\mathbf{L}, {}^{r}\mathbf{M}, {}^{r}\mathbf{N}$, we have nine sets of eigenvectors and it depends on the particular nature of the problem, what specific choice must be adopted. We present, here, the formation of vector solutions ${}^z\mathbf{M}$ and ${}^z\mathbf{N}$ and simply remark that the other eigenvectors based on cartesian vectors $\hat{\mathbf{x}},\hat{\mathbf{y}}$ are constructed similarly.

The derivation of ${}^z\mathbf{M}_{\substack{e\\o}mn}^{(i)}$ is based on the definition expression

$$ {}^z\mathbf{M}_{\substack{e\\o}mn}^{(i)} = \nabla\times\left(\Psi_{\substack{e\\o}mn}^{(i)}\hat{z}\right) = {}^z\mathbf{L}_{\substack{e\\o}mn}^{(i)}\times\hat{z}\,. \tag{35} $$

Expressing suitably the vectors $\hat{\xi}\times\hat{z}$, $\hat{\eta}\times\hat{z}$ and $\hat{\varphi}\times\hat{z}$ in spheroidal coordinates and combining (34) and (35), we obtain easily that

$$ \begin{aligned} {}^z\mathbf{M}_{\substack{e\\o}mn}^{(i)} &= -\frac{2}{\alpha}\frac{1}{\xi^2-\eta^2}\sqrt{\xi^2-1}\sqrt{1-\eta^2}\begin{bmatrix} \xi\dfrac{d}{d\xi}\left[R_{mn}^{(i)}(\xi,c)\right]S_{mn}(\eta,c) \\ -\eta R_{mn}^{(i)}(\xi,c)\dfrac{\partial}{\partial\eta}\left[S_{mn}(\eta,c)\right] \end{bmatrix}\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}\hat{\varphi} \\ &+\frac{2}{\alpha}\frac{m}{(\xi^2-\eta^2)^{1/2}}R_{mn}^{(i)}(\xi,c)S_{mn}(\eta,c)\begin{Bmatrix}-\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}\left[\frac{\xi}{\sqrt{\xi^2-1}}\hat{\xi}-\frac{\eta}{\sqrt{1-\eta^2}}\hat{\eta}\right]. \end{aligned} \tag{36} $$

To derive ${}^z\mathbf{N}_{\substack{e\\o}mn}^{(i)}$, we could use the relation ${}^z\mathbf{N}_{\substack{e\\o}mn}^{(i)} = \nabla\times{}^z\mathbf{M}_{\substack{e\\o}mn}^{(i)}$ and express everything in spheroidal coordinates. We obtain then a rather complicated expression and the above procedure is proved to be a bad idea. We could, however, proceed as in the ${}^r\mathbf{N}$ construction case.

Using differential properties, we obtain

$$ \begin{aligned} {}^z\mathbf{N}_{\substack{e\\o}mn}^{(i)} &= \nabla\times{}^z\mathbf{M}_{\substack{e\\o}mn}^{(i)} = \nabla\times\left[\nabla\times\left(\Psi_{\substack{e\\o}mn}^{(i)}\right)\hat{z}\right] = \nabla\left(\nabla\cdot\left(\Psi_{\substack{e\\o}mn}^{(i)}\hat{z}\right)\right) - \nabla^2\left(\Psi_{\substack{e\\o}mn}^{(i)}\hat{z}\right) = \\ &\nabla\left(\frac{\partial\Psi_{\substack{e\\o}mn}^{(i)}}{\partial z}\right) + k^2\Psi_{\substack{e\\o}mn}^{(i)}\hat{z} \end{aligned} \tag{37} $$

where we have profited again that $\Psi_{\substack{e\\o}mn}^{(i)}$ satisfies Helmholtz equation.

Expression (37) is elegant and actually, commenting exactly as in the ${}^r\mathbf{N}$ case, we mention this expression is the preferable form for boundary condition treatment. Even if we try to express Eq. (37) in spheroidal coordinates, we obtain a rather complicated form again, given by the following equation

$$
\begin{aligned}
{}^{z}\mathbf{N}^{(i)}_{\substack{e\\o}mn} &= k^2 \frac{1}{\left(\xi^2-\eta^2\right)^{1/2}} R^{(i)}_{mn} S_{mn}(\eta,c)\left[\sqrt{\xi^2-1}\,\eta\hat{\xi}+\xi\sqrt{1-\eta^2}\,\hat{\eta}\right]\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix} \\
&+\left(\frac{2}{\alpha}\right)^2 \frac{1}{(\xi^2-\eta^2)}\left\{\begin{aligned}
&\frac{2(1-\eta^2)\xi\eta\sqrt{\xi^2-1}}{\left(\xi^2-\eta^2\right)^{3/2}}\frac{\partial}{\partial\xi}R^{(i)}_{mn}(\xi,c)S_{mn}(\eta,c) \\
&-\frac{\sqrt{\xi^2-1}(1-\eta^2)(\xi^2+\eta^2)}{\left(\xi^2-\eta^2\right)^{3/2}}R^{(i)}_{mn}(\xi,c)\frac{\partial}{\partial\eta}S_{mn}(\eta,c) \\
&+\frac{\left(\xi^2-1\right)^{3/2}\eta}{\left(\xi^2-\eta^2\right)^{1/2}}\frac{\partial^2}{\partial\xi^2}R^{(i)}_{mn}(\xi,c)S_{mn}(\eta,c) \\
&+\frac{\xi\sqrt{\xi^2-1}(1-\eta^2)}{\left(\xi^2-\eta^2\right)^{1/2}}\frac{\partial}{\partial\xi}R^{(i)}_{mn}(\xi,c)\frac{\partial}{\partial\eta}S_{mn}(\eta,c)
\end{aligned}\right\}\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}\hat{\xi} \\
&+\left(\frac{2}{\alpha}\right)^2 \frac{1}{(\xi^2-\eta^2)}\left\{\begin{aligned}
&\frac{-2(\xi^2-1)\xi\eta\sqrt{1-\eta^2}}{\left(\xi^2-\eta^2\right)^{3/2}}R^{(i)}_{mn}(\xi,c)\frac{\partial}{\partial\eta}S_{mn}(\eta,c) \\
&+\frac{\sqrt{\xi^2-1}\sqrt{1-\eta^2}(\xi^2+\eta^2)}{\left(\xi^2-\eta^2\right)^{3/2}}\frac{\partial}{\partial\xi}R^{(i)}_{mn}(\xi,c)S_{mn}(\eta,c) \\
&+\frac{\xi\left(1-\eta^2\right)^{3/2}}{\left(\xi^2-\eta^2\right)^{1/2}}\frac{\partial^2}{\partial\eta^2}S_{mn}(\eta,c)R^{(i)}_{mn}(\xi,c) \\
&+\frac{\eta\left(\xi^2-1\right)\sqrt{1-\eta^2}}{\left(\xi^2-\eta^2\right)^{1/2}}\frac{\partial}{\partial\xi}R^{(i)}_{mn}(\xi,c)\frac{\partial}{\partial\eta}S_{mn}(\eta,c)
\end{aligned}\right\}\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}\hat{\eta} \\
&+\left(\frac{2}{\alpha}\right)^2 \frac{1}{\left(\xi^2-\eta^2\right)}\frac{m}{\sqrt{\xi^2-1}\sqrt{1-\eta^2}}\begin{bmatrix}\eta\left(\xi^2-1\right)\dfrac{\partial}{\partial\xi}R^{(i)}_{mn}(\xi,c)S_{mn}(\eta,c)+ \\ \left(1-\eta^2\right)\xi R^{(i)}_{mn}(\xi,c)\dfrac{\partial}{\partial\eta}S_{mn}(\eta,c)\end{bmatrix}\begin{Bmatrix}-\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}\hat{\varphi}.
\end{aligned}
\tag{38}
$$

As a matter of fact, relation (38) could obtain a more condensed form if we gathered all terms corresponding to the same unit vector but we preferred to keep this form to reveal the origin of every specific term.

Expression (38) is much more complicated than (36) and we understand, as in the ${}^{r}\mathbf{N}$ case, the treatment of boundary conditions on ${}^{z}\mathbf{N}$, should be more sophisticated than straightforward. Before proceeding to the handling of eigenvectors, it should be useful to make some remarks concerning the criteria determining the specific choice between the nine available vector solutions. The complexity of $\mathbf{N}$ functions compared to the rather simple form $\mathbf{M}$ would imply, at first sight, the suggestion to avoid functions $\mathbf{N}$ at any cost. Unfortunately, this is not always possible. For instant, it would be really enthousiastic to work with $\mathbf{L},{}^{r}\mathbf{M}$ and ${}^{z}\mathbf{M}$ instead of the set $\mathbf{L},{}^{r}\mathbf{M},{}^{r}\mathbf{N}$. However, it is easily proved and can be testified by simple inspection that for problems with azimouthal symmetry – the case $m = 0$ - the vectors ${}^{r}\mathbf{M}$ and ${}^{z}\mathbf{M}$ become parallel and lose their independence. Consequently, they fail to represent an arbitrary solenoidal field. In addition, under the same framework, azimouthal symmetry could be a natural property of the problem as the underlying geometry disposes this property. In conclusion, the most of times, it is not an easy task to avoid including in the eigenvector basis functions of kind $\mathbf{N}$. An additional remark is that the assurance of dimensionless eigenvectors demands suitable scaling of the previous functions and usually this is accomplished by suitable division with powers of wave number. So usually, we are talking about $\mathbf{L} = \frac{1}{k}\nabla\Psi$, $\mathbf{M} = \frac{1}{k}\nabla\times(\Psi\mathbf{r})$ and $\mathbf{N} = \frac{1}{k}(\nabla\times\mathbf{M})$. The special treatment of these functions will be apparent in the next section.

5. Formulation and Solution of Boundary Value Problems based on Spheroidal Eigenvectors

As it has been stated in the previous sections, what orientates the suitable forms of spheroidal eigenvectors is the applicability and usefulness of these functions to several scientific fields. Two main areas have attracted our attention in our work. The electromagnetic and the elastic case. Both regions of mathematical physics contain the most part of interesting boundary value problems and reveal the powerfulness of the already constructed eigenvectors.

Electromagnetic Case

In this framework, the goal is the determination of the electromagnetic field in finite and infinite regions given the boundary conditions occuring on discontinuity surfaces. The electromagnetic field obeys to the differential Maxwell equations, while the boundary conditions stem from suitable treatment of the integral forms of Maxwell equations. We conclude then that every discontinuity surface must compensate tangential components of electric field $\mathbf{E}$ and normal components of the magnetic flux density $\mathbf{B}$. In addition, the surface charge density ρ_s balances the difference of the normal components of the electric flux density $\mathbf{D}$, while the surface current density $\mathbf{k}$ coincides with the difference of tangential components of the magnetic field $\mathbf{H}$. Consequently, boundary condition investigation imposes the treatment of functions $\hat{\mathbf{n}} \cdot \mathbf{D}$, $\hat{\mathbf{n}} \times \mathbf{E}$, $\hat{\mathbf{n}} \cdot \mathbf{B}$ and $\hat{\mathbf{n}} \times \mathbf{H}$ on discontinuity surfaces ($\hat{\mathbf{n}}$ stands for the unit normal vector on the surface under consideration).

In many interesting applications, we have isotropic and homogeneous media and the stimulation of the system obeys to harmonic time – dependence. In this case, problems referring to scattering theory, electromagnetic oscillations and so on, the examination of only one field, e.g. the electric one, is necessary and sufficient for the determination of the whole electromagnetic field. Under these assumptions, the treatment of the scalar function $\hat{\mathbf{n}} \cdot \mathbf{E}$ and vector function $\hat{\mathbf{n}} \times \mathbf{E}$ is sufficient on boundary surfaces.

The usual framework, defining the spectral analysis of the mentioned problem, is to expand the electric field $\mathbf{E}_i$ occuring in every component V_i of the system, separated from the other adjacent components by the mentioned above discontinuity surfaces S_i, in terms of spheroidal eigenvectors $\mathbf{M}_i$ and $\mathbf{N}_i$. The free curl eigenvectors $\mathbf{L}_i$ are excluded from this representation as the electric field is a free divergence field in the free – charge region and is then represented through solenoidal basis vector functions. The eigenvectors $\mathbf{L}_i, \mathbf{M}_i, \mathbf{N}_i$ depend on the physical parameters of region V_i and so are labeled through index i.

Thus, we lead to the expression

$$\mathbf{E}_i = \sum \left(\alpha_i \mathbf{M}_i + \beta_i \mathbf{N}_i \right) \tag{39}$$

where summation runs over all denumerable sets of parameters originated from separation constants in the coordinate separation approach.

In every boundary value problem the determination of coefficients α_i, β_i is sought, fact coinciding with the determination of the electromagnetic fields of the problem.

This is accomplished through the boundary condition satisfaction. In oscillation problems, the boundary conditions are homogeneous and so the coefficients α_i, β_i are determined modulo some arbitrary constants as expected, while scattering problems are connected with non – homogeneous boundary conditions leading to unique determination of the underlying fields.

The method of handling the boundary conditions is the following. Every boundary condition satisfied on a spheroidal surface S_i, described by $\xi = C_i$, must be projected on a complete set of functions on η, φ - space leading after suitable orthogonalization arguments to algebraic linear systems having as unknowns the sought coefficients α_i, β_i.

The complete set of functions in the η, φ - space can be selected through several choices. However, the most adaptable choice to the specific forms of underlying functions is the set $P_n^m(\eta)\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}$, $m = 0,1,2,...; n \geq m$, denoted for simplicity in the sequel as $P(\eta)F(\varphi)$. However, special attention must be assigned to the fact that the necessity to introduce a weight function $\omega(\xi, \eta, \varphi)$ usually occurs in applications and this appears in the sequel.

As mentioneed above, boundary conditions involve the functions $\hat{\xi} \cdot \mathbf{E}_i$ and $\hat{\xi} \times \mathbf{E}_i$. Transforming everything in scalar functions, the above functions are equivalent to the terms $\hat{\xi} \cdot \mathbf{E}_i$, $\hat{\eta} \cdot \mathbf{E}_i$ and $\hat{\varphi} \cdot \mathbf{E}_i$. It results that the treatment of the boundary conditions is equivalent to the determination of the "brackets"

$\left\langle \hat{\xi} \cdot \mathbf{M}_i, P(\eta)F(\varphi) \right\rangle$, $\left\langle \hat{\xi} \cdot \mathbf{N}_i, P(\eta)F(\varphi) \right\rangle$,

$\left\langle \hat{\eta} \cdot \mathbf{M}_i, P(\eta)F(\varphi) \right\rangle$, $\left\langle \hat{\eta} \cdot \mathbf{N}_i, P(\eta)F(\varphi) \right\rangle$

and

$\left\langle \hat{\varphi} \cdot \mathbf{M}_i, P(\eta)F(\varphi) \right\rangle, \left\langle \hat{\varphi} \cdot \mathbf{N}_i, P(\eta)F(\varphi) \right\rangle$.

At this point, the suitable representation of the eigenvectors $\mathbf{M}_i, \mathbf{N}_i$ plays important role. Particularly, the integrals referring to $\mathbf{N}_i$ would be very complicated expressions if the definition representation is used without any special treatment.

We present, now briefly, the derivation of the above crucial "inner" products. To guarantee simplicity, the scalar function Ψ on which the construction of eigenvectors is based are denoted as $\Psi(\xi,\eta,\varphi)=R(\xi)S(\eta)\Phi(\varphi)$, omitting the several indices appearing in the definition formulae.

We now begin with the handling of the term $\hat{\xi}\cdot\mathbf{E}_i$ postponing the determination of the introduced weight function ω.

More precisely, we obtain

$$\left\langle\hat{\xi}\cdot\mathbf{N},PF\right\rangle=\int_S\hat{\xi}\cdot\mathbf{N}PF\omega dS=\frac{1}{k}\int_S\hat{\xi}\cdot\nabla\times(\nabla\times\mathbf{M})PF\omega dS=$$
$$\frac{1}{k}\int_S\hat{\xi}\cdot\nabla\times(\mathbf{M}PF\omega)dS-\frac{1}{k}\int_S\hat{\xi}\cdot[\nabla(PF\omega)\times\mathbf{M}]dS.$$

But for every vector field $\mathbf{A}$ we have

$$\int_S\hat{\xi}\cdot\nabla\times\mathbf{A}dS=0. \tag{40}$$

Consequently

$$\int_S\hat{\xi}\cdot\mathbf{N}PF\omega dS=-\int_S\hat{\xi}\cdot[\nabla(PF\omega)\times\mathbf{M}]dS=\int_S\left(\hat{\xi}\times\mathbf{M}\right)\cdot\nabla(PF\omega)dS. \tag{41}$$

In addition

$$\begin{aligned}\hat{\xi}\times\mathbf{M}&=\frac{1}{k}\hat{\xi}\times(\nabla\Psi\times\mathbf{r})=\frac{1}{k}\nabla\Psi\left(\mathbf{r}\cdot\hat{\xi}\right)-\frac{1}{k}r\left(\hat{\xi}\cdot\nabla\Psi\right)=\\ &\frac{\alpha}{2k}\frac{\xi\sqrt{\xi^2-1}}{\sqrt{\xi^2-\eta^2}}\nabla\Psi-\frac{1}{k}\mathbf{r}\left(\hat{\xi}\cdot\nabla\Psi\right)\end{aligned} \tag{42}$$

Inserting (42) into (41), expressing the differential operators and the surface element in spheroidal coordinates, we obtain

$$k\left(\frac{2}{\alpha}\right)^2 \int_S \hat{\xi}\cdot \mathbf{N}PF\omega dS = \frac{2}{\alpha k}\left(\xi^2-1\right)\xi \int_0^{2\pi} d\varphi \int_{-1}^{1} d\eta \frac{1}{\left(\xi^2-\eta^2\right)} \begin{bmatrix} \frac{\partial \Psi}{\partial \xi}\left(\xi^2-1\right)\frac{\partial \omega}{\partial \xi}PF \\ +\left(1-\eta^2\right)\frac{\partial \Psi}{\partial \eta}\frac{\partial(\omega P)}{\partial \eta}F \end{bmatrix}$$

$$+\frac{2}{\alpha k}\xi \int_0^{2\pi} d\varphi \int_{-1}^{1} d\eta \frac{\partial \Psi}{\partial \varphi}\frac{\partial F}{\partial \varphi}P\omega \frac{1}{\left(1-\eta^2\right)} \tag{43}$$

$$-\frac{2}{\alpha k}\left(\xi^2-1\right)\int_0^{2\pi} d\varphi \int_{-1}^{1} d\eta \frac{1}{\left(\xi^2-\eta^2\right)}\frac{\partial \Psi}{\partial \xi}\left[\left(1-\eta^2\right)\eta\frac{\partial}{\partial \eta}+\left(\xi^2-1\right)\xi\frac{\partial}{\partial \xi}\right](PF\omega).$$

Notice that Equation (43) does not contain second order terms, which have been eliminated through the integral law (40).

The presence of the denominator $\left(\xi^2-\eta^2\right)$ in the integrals of Equation (43) is a complicator factor and would be useful this term to be compensated through the weight function ω. A posteriori information justifies that a good choice for this auxiliary function is

$$\omega = \left(1-\eta^2\right)\left(\xi^2-\eta^2\right)^2. \tag{44}$$

This assumption, of course, creates some inevitable calculus burden, but it offers the great advantage to avoid the scale factor $\left(\xi^2-\eta^2\right)$ mixing the spheroidal coordinates. A priori, we notice that we are obliged to use the same weight function in the product $\left\langle \hat{\xi}\cdot \mathbf{M}, PF\right\rangle$ as these terms appear simultaneously.

Combining Equations (43) and (44) and following extended and elaborate analysis, we obtain

$$k\left(\frac{2}{\alpha}\right)^2\int_S \hat{\xi}\cdot\mathbf{N}PF\omega dS =$$

$$\frac{2}{\alpha k}\left(\xi^2-1\right)\xi R(\xi)\left(\int_0^{2\pi}\Phi F d\varphi\right)\left\{\begin{array}{l} -6\int_{-1}^{1}\left(1-\eta^2\right)\left[\left(1-\eta^2\right)\frac{\partial}{\partial\eta}S\right]Pd\eta \\ -2\left(\xi^2-1\right)\int_{-1}^{1}\eta\left[\left(1-\eta^2\right)\frac{\partial}{\partial\eta}S\right]Pd\eta \\ +\left(\xi^2-1\right)\int_{-1}^{1}\left(1-\eta^2\right)\frac{\partial}{\partial\eta}S\left(1-\eta^2\right)\frac{\partial}{\partial\eta}Pd\eta \\ +\int_{-1}^{1}\left(1-\eta^2\right)\left[\left(1-\eta^2\right)\frac{\partial}{\partial\eta}S\right]\left[\left(1-\eta^2\right)\frac{\partial}{\partial\eta}P\right]d\eta \end{array}\right\}$$

$$+\frac{2}{\alpha k}\xi R(\xi)\left(\int_0^{2\pi 1}\Phi' F' d\varphi\right)\left\{\left(\xi^2-1\right)^2\int_{-1}^{1}SPd\eta+2\left(\xi^2-1\right)\int_{-1}^{1}\eta^2 SPd\eta+\int_{-1}^{1}\eta^4 SPd\eta\right\}$$

$$+\frac{2}{\alpha k}\left(\xi^2-1\right)R'(\xi)\left(\int_0^{2\pi}\Phi F d\varphi\right)\left\{\begin{array}{l} 6\int_{-1}^{1}\left(1-\eta^2\right)^2 SPd\eta+2\left(\xi^2-1\right)\int_{-1}^{1}\left(1-\eta^2\right)\eta^2 SPd\eta \\ -\int_{-1}^{1}\left(1-\eta^2\right)^2\eta\left[\left(1-\eta^2\right)\frac{\partial P}{\partial\eta}\right]Sd\eta \\ -\left(\xi^2-1\right)\int_{-1}^{1}\left(1-\eta^2\right)\eta\left[\left(1-\eta^2\right)\frac{\partial}{\partial\eta}P\right]Sd\eta \end{array}\right\}. \quad (45)$$

In the previous expression the integrations have been separated and all the η - integrals are easily calculated. Indeed, function S contains associated Legendre functions of the same azimouthal number with function P, while there exist well known recurrence formulae to express the influence of the differential operator $\left(1-\eta^2\right)\frac{\partial}{\partial\eta}$ or of η on Legendre functions in terms of the Legendre functions themselves. So, the well known orthogonalization arguments of the associated Legendre functions can be used to define the quantities appeared in (45). The precise values of these integrals are not within the scope of this paper, but their presentation is used to the application paper [10].

In the same framework we can find that

$$k\left(\frac{2}{\alpha}\right)^2 \int_S \hat{\xi}\cdot \mathbf{M}PF\omega dS =$$

$$-R(\xi)\left(\int_0^{2\pi}\Phi F d\varphi\right)\left\{\xi^4\int_{-1}^{1}\left(1-\eta^2\right)\eta SPd\eta - 2\xi^2\int_{-1}^{1}\left(1-\eta^2\right)\eta^3 SPd\eta + \int_{-1}^{1}\left(1-\eta^2\right)\eta^5 SPd\eta\right\}. \tag{46}$$

We handle now the integrals incorporating projections on the $\hat{\eta}$ - direction. We denote the weight function as ω_1 since there is no reason to be related to ω.

We have

$$\int_S \hat{\eta}\cdot \mathbf{M}PF\omega_1 dS = \frac{1}{k}\int_S \hat{\eta}\cdot(\nabla\Psi\times\mathbf{r})PF\omega_1 dS =$$

$$= \frac{1}{k}\int_S (\mathbf{r}\times\hat{\eta})\cdot\nabla\Psi(PF\omega_1)dS = \frac{1}{k}\int_S \left(\mathbf{r}\cdot\hat{\xi}\right)\left(\hat{\xi}\times\hat{\eta}\right)\cdot\nabla\Psi(PF\omega_1)dS =$$

$$= -\frac{1}{k}\int_S \left(\mathbf{r}\cdot\hat{\xi}\right)\hat{\varphi}\cdot\nabla\Psi(PF\omega_1)dS.$$

Expressing everything in spheroidal coordinates, we obtain

$$k\left(\frac{2}{\alpha}\right)^2 \int_S \hat{\eta}\cdot \mathbf{M}PF\omega_1 dS = -\xi R(\xi)\left(\xi^2-1\right)^{1/2}\int_0^{2\pi}d\varphi\int_{-1}^{1}\frac{\Phi' FSP\omega_1}{\sqrt{1-\eta^2}}d\eta. \tag{47}$$

The right choice for ω_1 is $\omega_1 = \sqrt{1-\eta^2}\left(\xi^2-\eta^2\right)^2$ and this seems useless now, but it is invoked by the treatment of $\mathbf{N}$ - type eigenvector.

Consequently

$$k\left(\frac{2}{\alpha}\right)^2 \int_S \hat{\eta}\cdot \mathbf{M}PF\omega_1 dS =$$

$$-\xi\sqrt{\xi^2-1}R(\xi)\left(\int_0^{2\pi}\Phi' F d\varphi\right)\left\{\int_{-1}^{1}\eta^4 SPd\eta - 2\xi^2\int_{-1}^{1}\eta^2 SPd\eta + \xi^4\int_{-1}^{1}SPd\eta\right\}. \tag{48}$$

To handle the corresponding integral concerning vector $\mathbf{N}$ we use representation (34) according to which

$$k^2 N = \nabla\Psi + \nabla(\mathbf{r}\cdot\nabla)\Psi + k^2\Psi\mathbf{r}, \tag{49}$$

where we have adopted the suitable dimensional analysis.

We infer that

$$\begin{aligned} k\int_S \hat{\eta}\cdot\mathbf{N}PF\omega_1 dS &= -\frac{2}{\alpha k}\int_S \frac{\frac{\partial\Psi}{\partial\theta}}{\left(\xi^2-\eta^2\right)^{1/2}}PF\omega_1 dS \\ &+\frac{1}{k}\int_S (\hat{\eta}\cdot\nabla)[(\mathbf{r}\cdot\nabla)\Psi]PF\omega_1 dS + \frac{k\alpha}{2}\int_S \frac{\eta\sqrt{1-\eta^2}}{\left(\xi^2-\eta^2\right)^{1/2}}\Psi PF\omega_1 dS. \end{aligned} \tag{50}$$

Only the second term of the right hand part of (50) contains second – order derivatives and merits then some special treatment.

Indeed, we obtain

$$\begin{aligned} &\frac{1}{k}\int_S (\hat{\eta}\cdot\nabla)[(\mathbf{r}\cdot\nabla)\Psi]PF\omega_1 dS \\ &= \frac{2}{\alpha k}\left(\frac{\alpha}{2}\right)^2 \sqrt{\xi^2-1}\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\eta^2}\,\frac{\partial}{\partial\eta}[(r\cdot\nabla)\Psi](PF\omega_1)d\eta \\ &= -\frac{2}{\alpha k}\left(\frac{\alpha}{2}\right)^2 \sqrt{\xi^2-1}\int_0^{2\pi} d\varphi \int_{-1}^{1} (r\cdot\nabla)\Psi\,\frac{\partial}{\partial\eta}\left(\sqrt{1-\eta^2}PF\omega_1\right)d\eta \end{aligned} \tag{51}$$

where the integration by parts in η - integral has simplified the analysis eliminating the second order derivatives.

Following extended analytical manipulation of the underlying spheroidal functions we obtain that

$$k\left(\frac{2}{\alpha}\right)^2 \int_S (\hat{\eta}\cdot N) PF\omega_1 dS =$$

$$-\frac{2}{\alpha k}\sqrt{\xi^2-1}R(\xi)\int_0^{2\pi}\Phi F d\varphi \begin{Bmatrix} \int_{-1}^{1}\eta^4\left(1-\eta^2\right)\left(\frac{\partial S}{\partial \eta}\right)P d\eta - 4\int_{-1}^{1}\left(1-\eta^2\right)\eta^2\frac{\partial S}{\partial \eta}P d\eta \\ -\left(\xi^2-1\right)^2\int_{-1}^{1}\left(1-\eta^2\right)\frac{\partial S}{\partial \eta}P d\eta + \left(\xi^2-1\right)\int_{-1}^{1}\eta\left(1-\eta^2\right)^2\frac{\partial S}{\partial \eta}\frac{\partial P}{\partial \eta}d\eta \\ -\int_{-1}^{1}\eta^3\left(1-\eta^2\right)\frac{\partial S}{\partial \eta}\left(1-\eta^2\right)\frac{\partial P}{\partial \eta}d\eta \end{Bmatrix}$$

$$+\frac{k\alpha}{2}\sqrt{\xi^2-1}R(\xi)\int_0^{2\pi}\Phi F d\varphi \begin{Bmatrix} \left(\xi^2-1\right)^2\int_{-1}^{1}\left(1-\eta^2\right)\eta S P d\eta - 2\left(\xi^2-1\right)\int_{-1}^{1}\left(1-\eta^2\right)\eta^3 S P d\eta \\ +\int_{-1}^{1}\eta^5(1-\eta^2) S P d\eta \end{Bmatrix}$$

$$+\frac{2}{\alpha k}\left(\xi^2-1\right)^{3/2}\xi R'(\xi)\int_0^{2\pi}\Phi F d\varphi \begin{Bmatrix} -\left(\xi^2-1\right)\int_{-1}^{1}\left(1-\eta^2\right)\frac{\partial S}{\partial \eta}P d\eta \\ +\int_{-1}^{1}\eta^2\left(1-\eta^2\right)\frac{\partial S}{\partial \eta}P d\eta + 2\left(\xi^2-1\right)\int_{-1}^{1}\eta S P d\eta \\ -6\int_{-1}^{1}\eta^3 S P d\eta + 4\int_{-1}^{1}\eta S P d\eta \end{Bmatrix}. \quad (52)$$

Similar remarks are valid for the η - integrals appearing in Equation (52).

As far as the projection on the $\hat{\varphi}$ - direction, the resulting integrals are treated similarly. Selecting $\omega_2 = \left(1-\eta^2\right)^{1/2}\left(\xi^2-\eta^2\right)^{1/2}$ we obtain

$$k\left(\frac{2}{\alpha}\right)^2 \int_S \hat{\varphi}\cdot \mathbf{M}\left(PF\omega_2\right)dS = \left(\xi^2-1\right)\xi R(\xi)\int_0^{2\pi}\Phi F d\varphi\int_{-1}^{1}\left(1-\eta^2\right)\frac{\partial S}{\partial \eta}P d\eta$$
$$-\left(\xi^2-1\right)R'(\xi)\int_0^{2\pi}\Phi F d\varphi\int_{-1}^{1}\left(1-\eta^2\right)\eta S P d\eta, \quad (53)$$

$$k\left(\frac{2}{\alpha}\right)^2 \int_S \hat{\varphi} \cdot \mathbf{N}(PF\omega_2) dS = -\frac{2}{k\alpha}\left(\xi^2 - 1\right)\xi R'(\xi) \int_0^{2\pi} \Phi F' d\varphi \int_{-1}^{1} SPd\eta$$

$$-\frac{2}{k\alpha} R(\xi) \int_0^{2\pi} \Phi F' d\varphi \left(\begin{array}{l} \int_{-1}^{1} \eta\left(1-\eta^2\right)\frac{\partial S}{\partial \eta} Pd\eta \\ +\left(\xi^2 - 1\right)\int_{-1}^{1} SPd\eta + \int_{-1}^{1}\left(1-\eta^2\right) SPd\eta \end{array} \right) \quad (54)$$

The Elastic Case

A very interesting scientific area from the theoretical point of view is the elastic boundary value problem. The elastic medium under a certain simulation regime, propagates the data permitting the propagation of elastic waves, which interfere with discontinuity surfaces creating secondary distrurbances establishing the scattering procedures. The elastic fields, in accordance with the electromagnetic ones, can be explained through the eigenvector basis.

However, elastic propagation has the particular nature that every initial distrurbance creates two elastic waves travelling with different wave numbers. The "slower" wave is the so called "transverse" part having solenoidal nature and is expressed completely in terms of Navier eigenvectors $\mathbf{M}, \mathbf{N}$ introduced in the previous sections. The "faster" wave is irrotational and constitutes the "longitudinal" component of the elastic wave. It can be expanded in the eigenvectors of $\mathbf{L}$ - type with different wave number.

Keeping the same terminology as in the electromagnetic case, we summarize that every elastic field in the region V_i is represented as

$$\mathbf{u}_i = \mathbf{u}_i^p + \mathbf{u}_i^s$$

where

$$\mathbf{u}_i^p = \sum \alpha_i \mathbf{L}_i, \quad \mathbf{u}_i^s = \sum \left(\beta_i \mathbf{M}_i + \gamma_i \mathbf{N}_i\right),$$

and

$$\mathbf{L}_i = \frac{1}{k_p}\nabla \Psi_i^p, \quad \mathbf{M}_i = \frac{1}{k_s}\nabla \times \left(\Psi_i^s \mathbf{r}\right), \quad \mathbf{N}_i = \frac{1}{k_s}\nabla \times \mathbf{M}_i.$$

Notice that $\Psi_i^t, t = p, s$ satisfy the scalar Helmholtz equation with wave number $k_i^t, t = p, s$ with $k_i^p > k_i^s$.

The transverse and longitudinal components of the elastic fields are not independent and the boundary conditions are responsible for this. In this case, the displacement fields and stresses have to be balanced on the interfaces. The elastic stress is acquired after the application of the stress operator

$$T_i = 2\mu_i \hat{n} \cdot \nabla + \lambda_i \hat{n} \nabla \cdot + \mu_i \hat{n} \times \nabla \times \tag{55}$$

on the displacement fields, where λ_i, μ_i are the Lamé constants characterizing the elastic properties of the medium V_i, and $\hat{n}$ is the normal unit vector which coincides with $\hat{\xi}$ on every spheroidal surface. Consequently, we have to treat the vector functions $T_i \mathbf{u}_i^p$ and $T_i \mathbf{u}_i^s$ on the surfaces $\xi = C$.

Handling $T_i \mathbf{u}_i^s$ is equivalent with treating $T_i \mathbf{M}_i$ and $T_i \mathbf{N}_i$. Let us begin discussing on these solenoidal elastic components.

Clearly

$$T_i \mathbf{M}_i = 2\mu_i \hat{\xi} \cdot \nabla \mathbf{M}_i + \mu_i \hat{\xi} \times \nabla \times \mathbf{M}_i = 2\mu_i \left(\hat{\xi} \cdot \nabla\right) \mathbf{M}_i + \mu_i k_i^s \hat{\xi} \times \mathbf{N}_i, \tag{56}$$

$$\begin{aligned} T_i \mathbf{N}_i &= 2\mu_i \hat{\xi} \cdot \nabla \mathbf{N}_i + \mu_i \hat{\xi} \times \left(\nabla \times \mathbf{N}_i\right) \\ &= 2\mu_i \hat{\xi} \cdot \nabla \mathbf{N}_i + \mu_i \hat{\xi} \times \left[\nabla \times \frac{1}{k_i^s}\left(\nabla \times \mathbf{M}_i\right)\right] = 2\mu_i \hat{\xi} \cdot \nabla \mathbf{N}_i + \mu_i k_i^s \hat{\xi} \times \mathbf{M}_i. \end{aligned} \tag{57}$$

Consequently, in the framework of treatment of inner products $< \hat{\alpha} \cdot T_i \mathbf{M}_i, PF >$ and $< \hat{\alpha} \cdot T_i \mathbf{N}_i, PF >$, $\hat{\alpha} = \hat{\xi}, \hat{r}, \hat{\varphi}$, the second terms of the right hand side of Equations (56) and (57) have already examined in the electromagnetic case and only the terms $< \hat{\alpha} \cdot \left(\hat{\xi} \cdot \nabla\right) \mathbf{M}_i, PF >$ and $< \hat{\alpha} \cdot \left(\hat{\xi} \cdot \nabla\right) \mathbf{N}_i, PF >$ make their appearance for the first time.

Postponing the determination of those terms, we examine also the influence of the stress operator on the $\mathbf{L}_i$ eigenvectors. More precisely

$$T_i \mathbf{L}_i = 2\mu_i \hat{\xi} \cdot \nabla \mathbf{L}_i + \lambda_i \hat{\xi} \nabla \cdot \mathbf{L}_i = 2\mu_i \hat{\xi} \cdot \nabla \mathbf{L}_i - \lambda_i k_i^p \hat{\xi} \Psi_i^p. \tag{58}$$

Similarly to transverse solutions, boundary conditions handling of longitudinal wave is reduced to the determination of inner products of the form $<\hat{\alpha}\cdot\left(\hat{\xi}\cdot\nabla\right)\mathbf{L}_i, PF>$, $\hat{\alpha}=\hat{\xi},\hat{r},\hat{\varphi}$.

Let us begin the investigation of these integrals from the irrotational component. Using differential identity

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot\nabla\mathbf{a}-(\nabla\cdot\mathbf{a})\mathbf{b}+\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{a}\cdot\nabla\mathbf{b} \tag{59}$$

we infer that

$$\begin{aligned}k\hat{\xi}\cdot\nabla\mathbf{L}=\hat{\xi}\cdot\nabla\nabla\Psi=\nabla\times\left(\nabla\Psi\times\hat{\xi}\right)+(\nabla\cdot\nabla\Psi)\hat{\xi}-\nabla\Psi\left(\nabla\cdot\hat{\xi}\right)+\nabla\Psi\cdot\nabla\hat{\xi}=\\ \nabla\times\left(\nabla\Psi\times\hat{\xi}\right)-k^2\Psi\hat{\xi}-\nabla\Psi\left(\nabla\cdot\hat{\xi}\right)+\nabla\Psi\cdot\nabla\hat{\xi}\end{aligned} \tag{60}$$

where, for simplicity, we omit in the sequel any kind of indices.

Given that $\hat{\xi}\cdot\hat{\xi}=1$, we find that $\nabla\hat{\xi}\cdot\hat{\xi}=0$. Projecting Equation (60) on $\hat{\xi}$ we obtain

$$k\hat{\xi}\cdot\left(\hat{\xi}\cdot\nabla L\right)=\xi\cdot\nabla\times\left(\nabla\Psi\times\hat{\xi}\right)-k^2\Psi-\left(\nabla\cdot\hat{\xi}\right)\left(\hat{\xi}\cdot\nabla\Psi\right). \tag{61}$$

Consequently, exploiting integral law (40), we can show that

$$\begin{aligned}k<\hat{\xi}\cdot\left(\hat{\xi}\cdot\nabla L\right),PF>=\int_S\nabla\Psi\cdot\nabla(PF\omega)dS-\int_S\left(\hat{\xi}\cdot\nabla\Psi\right)\hat{\xi}\cdot\nabla(PF\omega)dS\\ -k^2\int_S\Psi PF\omega dS-\int_S\left(\hat{\xi}\cdot\nabla\Psi\right)\left(\nabla\cdot\hat{\xi}\right)PF\omega dS.\end{aligned} \tag{62}$$

Every term of Equation (62) contains only first – order derivatives and has similar form to that encountered in electromagnetism. It is not within the purpose of this work to give all the terms appearing in the inner products. Actually, this is accomplished in our forthcoming work [11], where interesting elastic problems occuring in spheroidal geometry are studied. However, the most difficult terms expected, are the integrals of the form $<\hat{\alpha}\cdot\left(\hat{\xi}\cdot\nabla\right)\mathbf{N}_i, PF>$, $\hat{\alpha}=\hat{\xi},\hat{r},\hat{\varphi}$. Straightforward calculation would be a mess. In contrast, using (59) we find that

$$\begin{aligned}\left(\hat{\xi}\cdot\nabla\right)\mathbf{N}=\nabla\times\left(\mathbf{N}\times\hat{\xi}\right)+(\nabla\cdot\mathbf{N})\hat{\xi}-\mathbf{N}\left(\nabla\cdot\hat{\xi}\right)+\mathbf{N}\cdot\nabla\xi\\ =\nabla\times\left(\mathbf{N}\times\hat{\xi}\right)-\mathbf{N}\left(\nabla\cdot\hat{\xi}\right)+\mathbf{N}\cdot\nabla\xi.\end{aligned} \tag{63}$$

Consequently, taking for example $\mathbf{a}=\hat{\xi}$,

$$\hat{\xi}\cdot\left(\hat{\xi}\cdot\nabla\right)\mathbf{N}=\hat{\xi}\cdot\nabla\times\left(\mathbf{N}\times\hat{\xi}\right)-\left(\hat{\xi}\cdot\mathbf{N}\right)\left(\nabla\cdot\hat{\xi}\right). \tag{64}$$

Then, using again the integral law (40), we would obtain a very useful expression for the desired integral reduceable to already determined integrals. Suitable use of representation (49) is necessary for handling the projections of $\left(\hat{\xi}\cdot\nabla\right)\mathbf{N}$ to the directions $\hat{\eta}$ and $\hat{\varphi}$.

6. Concluding Remarks

The goal of this work is to construct the vector Navier functions in spheroidal geometry in a form which is suitable for the solution of boundary value problems. Instead of following a direct and straightforward calculation of the Navier functions by defining the application of the differential operators on Helmholtz equation kernel, we try, in this work, to minimize the extended analytical burden of the final formulae. This is accomplished by exploiting a priori general properties of the vector operators and the underlying Helmholtz equation kernel functions. In other words, we absorb the complexity of the analytical treatment using all the symmetries of the problem. So, the final expressions do not have unexploited hidden information and dispose the simplest possible form which turns out to be adequate for the applications and renders analytical facing of the problem efficient.

However, the construction of Navier eigenvectors presented in this work is only the one side of the problem. We have to render the expressions of these functions suitable for the applications and this is actually the most interestig task as it justifies the whole approach. Indeed, these eigenfunctions are going to be used as basis functions in boundary value problems of electromagnetics and elasticity. The physical vector fields in these problems are expressed in terms of the Navier eigenvectors and satisfy then by construction the underlying equations. In the sequel, they are forced to satisfy the boundary conditions of the problem on the boundaries of the system, which are coordinate surfaces. The implication of the boundary conditions vary on the particular physical problem. Mathematically, the boundary conditions are treated by imposing specific differential operators on the vector solution of the problem and force the new expressions to take specific forms on the boundary surfaces. These differential operators could be simple as the curl operator (electromagnetics) or very complicated as the differential stress tensor (elasticity).

However, the linearity of Navier representations renders sufficient the determination of the boundary condition differential operator acting on the kernel Navier eigenvectors. In other words the second more essential task of this work must be the determination of the fundamental vector functions constituting the basis of the transformed solutions after the application of the boundary condition operators. For the spheroidal case, this is a very difficult job, particularly for the elastic case, when the boundary conditions have complicated differential representations. In this framework, all the necessary material is constructed in this work and special mention must be assigned to the determination of the application of stress operator on Navier eigenvectors, in the elastic case.

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