CONVERGENCE OF A TIME DISCRETE GALERKIN METHOD FOR SEMILINEAR PARABOLIC EQUATIONS

G. AKRIVIS and Ch. MAKRIDAKIS

28-99

Preprint no. 28-99/1999

Department of Computer Science
University of Ioannina
451 10 Ioannina, Greece

CONVERGENCE OF A TIME DISCRETE GALERKIN METHOD FOR SEMILINEAR PARABOLIC EQUATIONS

Georgios Akrivis, and Charalambos Makridakis

Abstract. We approximate the solution of initial boundary value problems of semilinear parabolic equations by time-discrete discontinuous finite element methods. We propose a method for obtaining optimal order-regularity a priori error bounds. Our approach is based on techniques developed for the numerical approximation in bifurcation theory for mildly nonlinear elliptic equations and on the stability analysis of the method for linear parabolic problems with mesh dependent norms.

12 − 10 − 1999

1. Introduction

In this note we propose a method for obtaining optimal order-regularity a priori error bounds for the discontinuous Galerkin approximation of evolution problems of the following form: Let $(H, (\cdot,\cdot))$ be a Hilbert space, and F a (possibly) nonlinear operator, $F : D(F) \to H$. We seek $u : (0,T) \to D(F)$

$$\begin{aligned} &u' + F(u) = 0, \quad 0 < t < T,\\ &u(0) = u^0. \end{aligned} \tag{1.1}$$

We will assume that (1.1) possesses a unique solution. More specific assumptions on the abstract problem (1.1) will be cited in the sequel. Our results apply, in particular, when (1.1) is a formulation of a semilinear parabolic problem. We have chosen to first present them in an abstract setting because other applications are possible and also in order to better illustrate the proposed approach.

The *discontinuous Galerkin method* is formulated as follows: Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$, $I_n := (t_{n-1}, t_n]$, and $k_n := t_n - t_{n-1}$. We seek approximations to u in the space of piecewise polynomial functions of degree at most q,

$$S_k := \{\varphi : [0,T] \to H / \ \varphi|_{I_n} = \sum_{j=0}^{q} \chi_j \ t^j, \ \chi_j \in H\};$$

1991 *Mathematics Subject Classification.* Primary 65M60, 65M12; Secondary 65L06.

the elements of S_k are allowed to be discontinuous at the nodal points t_n, but are taken to be continuous to the left there.

Given a Banach space V, we denote by $L^p(V)$ the space $L^p((0,T);V), 1 \le p \le \infty$, with the standard definition of the corresponding norm; $H^1(V)$ is defined analogously.

The Galerkin approximation $u_k \in S_k$ to the solution u is defined by

$$b(u_k, \varphi) = 0 \qquad \forall \varphi \in S_k, \tag{1.2}$$

with

$$\begin{aligned} b(v,\varphi) := \sum_{n=1}^{N} \int_{I_n} \left[(v', \varphi) + (F(v), \varphi)\right] dt \\ + \sum_{n=1}^{N-1} (v^{n+} - v^n, \varphi^{n+}) + (v^{0+}, \varphi^{0+}) \end{aligned} \tag{1.3}$$

where $v^n := v(t_n)$ and $v^{n+} := \lim_{s \downarrow 0} v(t_n + s)$. Let $v^{N+} := 0$.

The discontinuous Galerkin method for dissipative evolution problems, cf. [J], [EJT], is a time finite element discretization method. When combined with appropriate integration rules it reduces to the classical Runge-Kutta-Radau time discretization. For the analysis of this method when (1.1) is linear and dissipative cf. [Th] and the references theirin. This method can be used as a model for the investigation of properties related to adaptive computations for parabolic problems, [EJL]. In this direction, one of the results we are interested in is the derivation of optimal (*order and regularity*) a priori error estimates, cf. [EJL] and in the fully discrete case [EJ 2], [MB]. Such optimal results are established in [EJL] for linear parabolic problems for any q. In the fully discrete case cf. [EJT 2] for a priori bounds in $L^\infty(L^2)$ and in $L^\infty(L^\infty)$ ($q = 0$ or $q = 1$), and [MB] for estimates in $L^2(L^2)$. (All available results in $L^\infty(L^2)$, as well as the results in Section 3 below, are optimal up to a logarithmic factor.) For nonlinear problems such estimates are available in the case of a nonlinear parabolic problem but only for $q = 0$, [EJ1]. In [EL] a semilinear parabolic problem is considered with the discontinuous Galerkin method combined with numerical integration and, therefore, the a priori estimates are not optimal in the above sense.

In [Jo] Johnson analyzes the discontinuous Galerkin Method for nonlinear dissipative ode's, see also [E]. For the convergence of the discontinuous Galerkin method in the non parabolic case cf. [KM].

In the sequel we will present an approach yielding a priori estimates for the discontinuous Galerkin approximation for the nonlinear problem (1.1) in abstract form. In Section 2 we establish the abstract results and in Section 3 apply them to a semilinear parabolic equation; in Section 4 we briefly discuss limitations of the method and also possible extensions. The main idea, motivated by the finite element analysis in the Bifurcation theory, [CR], [RP], [CaR], is based on the central Lemma 2.1, [CR], and on the analysis of the stability of discontinuous Galerkin methods by mesh dependent norms [MB].

2. Abstract formulation

Our intention is to rewrite the variational problem (1.2) in operator form:

$$\text{Find } u_k \in X \text{ such that} \quad G(u_k) = 0. \tag{2.1}$$

For appropriate Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and $Z = Y'$, G will be considered a nonlinear operator from X to Z, $G : X \to Z$. We assume here that b can be defined in $X \times Y$, such that, for $v \in X$,

$$|b(v, w)| \leq C(v)\|w\|_Y \qquad \forall w \in Y .$$

Under appropriate assumptions on $b(\cdot,\cdot)$, X, Z, and G, we will use the following well-known result from the error analysis in bifurcation theory for mildly nonlinear elliptic equations, cf. [CR], to show existence of u_k and to derive estimates for $\|u - u_k\|_X$.

Lemma 2.1. *Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Let a map $G : X \to Z$ be differentiable, $DG(v) : X \to Z$ be continuous (for $v \in X$), and, for a given $\widetilde{v} \in X, DG(\widetilde{v})$ be an isomorphism of X onto Z. Let*

$$\varepsilon := \|G(\widetilde{v})\|_Z, \quad \gamma := \|DG(\widetilde{v})^{-1}\|_{L(Z,X)},$$

and, with $\alpha := 2\gamma\varepsilon$, let $B(\widetilde{v}, \alpha) := \{v \in X : \|\widetilde{v} - v\|_X \leq \alpha\}$ and

$$L(\alpha) := \sup_{v \in B(\widetilde{v},\alpha)} \|DG(\widetilde{v}) - DG(v)\|_{L(X,Z)}.$$

Assume that $2\gamma L(\alpha) < 1$. Then, there exists a unique solution u of the equation $G(u) = 0$ in the ball $B(\widetilde{v}, \alpha)$; moreover, we have

$$\|u - \widetilde{v}\|_X \leq \frac{\gamma}{1 - \gamma L(\alpha)} \|G(\widetilde{v})\|_Z. \quad \square$$

Let $\widetilde{v} \in S_k$ denote the interpolant of u defined by

$$\begin{aligned} &\widetilde{v}(t_n) = u(t_n) \\ &\int_{I_n} (\widetilde{v}, \varphi)dt = \int_{I_n} (u, \varphi)dt \quad \forall \varphi \in \mathbb{P}_{q-1}(I_n; H) \end{aligned} \tag{2.2}$$

with $\mathbb{P}_{q-1}(I_n; H)$ the space of polynomials on I_n of degree at most $q-1$ with values in H.

Stability assumptions. We assume that $DF(\widetilde{v}) : X \to Z := Y'$ is well defined and the linearized problem around $\widetilde{v} \in S_k$ has the following stability properties: The bilinear form $b'(\widetilde{v}; \cdot, \cdot) : X \times Y \to \mathbb{R}$,

$$\begin{aligned} b'(\widetilde{v}; w, \varphi) = \sum_{n=1}^{N} \int_{I_n} \big[(w', \varphi) + (DF(\widetilde{v})w, \varphi)\big]\, dt \\ + \sum_{n=1}^{N-1} (w^{n+} - w^n, \varphi^{n+}) + (w^{0+}, \varphi^{0+}), \end{aligned}$$

is continuous,

$$(\mathrm{A}\alpha) \qquad |b'(\widetilde{v}; w, \varphi)| \leq C_\alpha \|w\|_X \, \|\varphi\|_Y,$$

and satisfies the inf $-$ sup condition,

$$(\mathrm{A}\beta) \qquad \sup_{\varphi \in S_k,\ \varphi \neq 0} \frac{b'(\widetilde{v}; w, \varphi)}{\|\varphi\|_Y} \geq C_\beta \|w\|_X.$$

Then we have

Theorem 2.1. *Let (A) be satisfied, and assume that*

$$\|DF(\widetilde{v}) - DF(v)\|_{L(X,Y')} \leq C_\gamma \|\widetilde{v} - v\|_X \tag{2.3}$$

and

$$C_\gamma \, C_\beta^{-2} \, \|F(\widetilde{v}) - F(u)\|_{Y'} \leq \frac{1}{4} \quad \text{for} \quad k \leq k_0. \tag{2.4}$$

Then, for $k \leq k_0$, there exists a locally unique solution u_k of (1.2) such that

$$\|u_k - \widetilde{v}\|_X \leq \frac{2}{C_\beta} \, \|F(\widetilde{v}) - F(u)\|_{Y'}. \tag{2.5}$$

Proof. We will use Lemma 2.1. For $w \in X$, let $G(w) \in Y'$ be defined by

$$< G(w), \varphi >= b(w, \varphi) \quad \forall \varphi \in Y$$

with $< \cdot, \cdot >$ denoting both, the inner product in $L^2(H)$, $< v, w >:= \int_0^T (v, w)dt$, and the duality pairing between Y' and Y. Then, G is differentiable and the derivative $DG(\widetilde{v})$ satisfies the relation

$$< DG(\widetilde{v})w, \varphi >= b'(\widetilde{v}; w, \varphi) \quad \forall w \in X \ \varphi \in Y.$$

In view of assumption (Aβ) we have

$$\|DG(\widetilde{v})w\|_{Y'} = \sup_{\varphi \in Y,\ \varphi \neq 0} \frac{< DG(\widetilde{v})w, \varphi >}{\|\varphi\|_Y} \geq C_\beta \|w\|_X,$$

i.e., $DG(\widetilde{v})$ is invertible and

$$\|DG(\widetilde{v})^{-1}\|_{L(Z,X)} \leq \frac{1}{C_\beta} \tag{2.6}$$

with $Z := Y'$.

Next, we will evaluate $\|G(\widetilde{v})\|_Z$. For $\varphi \in Y$, we have

$$
\begin{aligned}
< G(\widetilde{v}), \varphi > &= b(\widetilde{v}, \varphi) \\
&= \sum_{n=1}^{N} \int_{I_n} \left[(\widetilde{v}', \varphi) + (F(\widetilde{v}), \varphi)\right] dt \\
&\quad + \sum_{n=1}^{N-1} (\widetilde{v}^{n+} - \widetilde{v}^{n}, \varphi^{n+}) + (\widetilde{v}^{0+} - u^0, \varphi^{0+}) \\
&= -\sum_{n=1}^{N} \int_{I_n} (\widetilde{v}, \varphi')\, dt + (\widetilde{v}^{n}, \varphi^{n}) - (\widetilde{v}^{n-1}, \varphi^{n-1+}) + \int_0^T (F(\widetilde{v}), \varphi)\, dt \\
&= -\sum_{n=1}^{N} \int_{I_n} (u, \varphi')\, dt + (u^{n}, \varphi^{n}) - (u^{n-1}, \varphi^{n-1+}) + \int_0^T (F(\widetilde{v}), \varphi)\, dt \\
&= \int_0^T (u', \varphi)\, dt + \int_0^T (F(\widetilde{v}), \varphi)\, dt \\
&= \int_0^T (F(\widetilde{v}) - F(u), \varphi)\, dt
\end{aligned}
$$

i.e.,

$$\|G(\widetilde{v})\|_Z = \|F(\widetilde{v}) - F(u)\|_{Y'}. \tag{2.7}$$

It remains to verify that for

$$\alpha \le \frac{2}{C_\beta} \|F(\widetilde{v}) - F(u)\|_{Y'},$$

with

$$B(\widetilde{v}, \alpha) := \{v \in X : \|\widetilde{v} - v\|_X \le \alpha\}$$

and

$$L(\alpha) := \sup_{v \in B(\widetilde{v}, \alpha)} \|DG(\widetilde{v}) - DG(v)\|_{L(X,Z)},$$

there holds

$$\frac{2}{C_\beta}\, L(\alpha) < 1. \tag{2.8}$$

Indeed, in view of (1.6),

$$\frac{2}{C_\beta}\, L(\alpha) \le \frac{2}{C_\beta}\, C_\gamma \alpha \le \left(\frac{2}{C_\beta}\right)^2 C_\gamma \|F(\widetilde{v}) - F(u)\|_{Y'},$$

and (2.8) follows from (2.4). □

3. A SEMILINEAR PARABOLIC EQUATION

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded domain with smooth boundary $\partial\Omega$, and $f : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz continuous, smooth function. We consider the following initial and boundary value problem: seek a real-valued function u, defined on $\bar{\Omega} \times [0, T]$, satisfying

$$
\begin{aligned}
&u_t - \Delta u = f(u) && \text{in } \Omega \times [0,T],\\
&u = 0 && \text{on } \partial\Omega \times [0,T],\\
&u(\cdot, 0) = u^0 && \text{in } \Omega,
\end{aligned}
\tag{3.1}
$$

with $u^0 : \Omega \to \mathbb{R}$ a given initial value. We assume that the data are smooth and compatible such that (3.1) possesses a sufficiently regular solution.

Problem (3.1) is of the form (1.1) with

$$
F(v) := -\Delta v - f(v),
$$

$$
H = L^2(\Omega), V = H_0^1(\Omega), D(F) = H_0^1(\Omega) \cap H^2(\Omega).
$$

Let

$$
S_k := \{\varphi : [0,T] \to H / \ \varphi|_{I_n} = \sum_{j=0}^{q} \chi_j \ t^j, \ \chi_j \in H_0^1(\Omega)\}.
$$

Then, the discontinuous Galerkin method is to find $u_k \in S_k$ satisfying

$$
b(u_k, \varphi) = 0 \qquad \forall \varphi \in S_k,
\tag{3.2}
$$

with b given in (1.3) whereby

$$
(F(v), \varphi) = (\nabla v, \nabla \varphi) - (f(v), \varphi) \quad v \in H_0^1(\Omega).
$$

To derive estimates in the $L^\infty((0,T); L^2(\Omega))$–norm for the error $u - u_k$ we have to appropriately select the spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and then apply the results of the previous section.

To this end, let $X := S_k$ and $Y := S_k \cap L^1((0,T); H^2(\Omega))$ be equipped with the norms

$$
\|v\|_X := \sup_{0<t<T} \|v\|_{L^2(\Omega)}
\tag{3.3}
$$

and

$$
\begin{aligned}
\|v\|_Y := &\sum_{n=1}^{J} \int_{I_n} \|v_t\|_{L^2(\Omega)} dt + \int_0^T \|v\|_{H^2(\Omega)} dt\\
&+ \sum_{n=1}^{J-1} \|v^n - v^{n+}\|_{L^2(\Omega)} + \|v^J\|_{L^2(\Omega)},
\end{aligned}
\tag{3.4}
$$

respectively. Of course, $(X, \|\cdot\|_X)$ is a Banach space, and due to the fact that S_k is piecewise polynomial space in t it is easily seen that $(Y, \|\cdot\|_Y)$ is a Banach space as well. Since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$, relation (3.2) can be equivalently written in the form:

$$\text{Find } u_k \in X \text{ such that} \qquad b(u_k, \varphi) = 0 \quad \forall \varphi \in Y. \tag{3.5}$$

Clearly, $G : X \to Y' =: Z$ is well defined by

$$< G(w), \varphi >= b(w, \varphi) \quad \forall \varphi \in Y. \tag{3.6}$$

The bilinear form $b'(\widetilde{v}; \cdot, \cdot) : X \times Y \to \mathbb{R}$,

$$\begin{aligned} b'(\widetilde{v}; w, \varphi) := \sum_{n=1}^{N} \int_{I_n} \big[(w_t, \varphi) + (\nabla w, \nabla \varphi) - (f'(\widetilde{v})w, \varphi)\big]\, dt \\ + \sum_{n=1}^{N-1} (w^{n+} - w^n, \varphi^{n+}) + (w^{0+}, \varphi^{0+}), \end{aligned} \tag{3.7}$$

can also be written as

$$\begin{aligned} b'(\widetilde{v}; w, \varphi) := - \sum_{n=1}^{N} \int_{I_n} \big[(w, \varphi_t) + (w, \Delta\varphi) + (f'(\widetilde{v})w, \varphi)\big]\, dt \\ + \sum_{n=1}^{N-1} (w^n, \varphi^n - \varphi^{n+}) + (w^J, \varphi^J) \quad \forall w \in X, \ \forall \varphi \in Y. \end{aligned}$$

Consequently, (Aα) is satisfied. Further, a modification of the proof of Lemma 12.3 of [Th], see also [EJT], yields

Lemma 3.1. *Assume that $k_{n+1} \geq c k_n$ with a positive constant c. With*

$$L_N := \big(\log \frac{T}{k_N}\big)^{1/2} + 1, \quad \text{and} \quad E_N := e^{\,c_\star \lambda T} C_\star,$$

where $\lambda = \lambda(f)$ is an appropriate constant, condition (Aα) is satisfied with

$$C_\beta = \frac{1}{L_N E_N} C$$

and an appropriate positive constant C.

Proof. To prove the inf − sup condition (Aβ), as in [MB], let v be a given element of X. It suffices to find $\Phi \in Y$ and two positive constants β_0 and β_1 such that

$$b(\widetilde{v}; v, \Phi) \geq \beta_0 \|v\|_X^2 \tag{3.8α}$$

and

$$\|\Phi\|_Y \le \beta_1 \|v\|_X. \tag{3.8β}$$

Since $\|v(t)\|_{L^2(\Omega)}$ is piecewise polynomial in time, $\|v(t^\star)\|_{L^2(\Omega)} = \|v\|_X$ with $t^\star \in I_n$ (or $\|v(t^{n-1+})\|_{L^2(\Omega)} = \|v\|_X$) for some n. Let $\Phi \in S_k$ be the solution of the dual discrete problem: Find $\Phi \in S_k$ such that

$$b(\widetilde{v}; \chi, \Phi) = (v(t^\star), \Phi(t^\star))_{L^2(\Omega)}.$$

Then, $\Phi \in Y$ and (3.8α) is satisfied with $\beta_0 = 1$. An appropriate modification of the proof of Lemma 12.3 of Thomée [Th], in view of the fact that f' and f'' are bounded, yields (3.8β) with $\beta_1 = CL_N E_N$. □

Remark. In the general case considered in this paper E_N above will increase exponentially with T. Under special assumptions on f though E_N may not be present in C_β. E.g., this is the case when $f, \widetilde{v}$ are such that $DF(\widetilde{v})v = -\Delta v - f'(\widetilde{v})v$ is coercive in $V = H_0^1(\Omega)$. In any case E_N is only involved in the proof of the inf-sup condition for $b(\widetilde{v}; \cdot, \cdot)$ and thus attempts to improve its dependence on T in special cases should be concentrated on proving this inf-sup condition with better constant. Compare with the discussion in [JRB, Section 5] where this method for $q = 0$ is considered for model problems of fluid flow.

Next, we will verify assumptions (2.3) and (2.4) of Theorem 2.1. To this end let $v, w \in X$ and $\psi \in Y$; we have

$$\begin{aligned}
| < [DF(\widetilde{v}) - DF(v)]w, \psi > | &\le \int_0^T |([f'(\widetilde{v}) - f'(v)]w, \psi)| \, dt \\
&\le \|w\|_X \int_0^T \|[f'(\widetilde{v}) - f'(v)]\psi\|_{L^2(\Omega)} \, dt \\
&\le \|w\|_X \int_0^T \|f'(\widetilde{v}) - f'(v)\|_{L^2(\Omega)} \, \|\psi\|_{L^\infty(\Omega)} \, dt \\
&\le \|w\|_X \, \|f'(\widetilde{v}) - f'(v)\|_{L^\infty((0,T);L^2(\Omega))} \int_0^T \|\psi\|_{L^\infty(\Omega)} \, dt \\
&\le C\|w\|_X \, \|\widetilde{v} - v\|_X \int_0^T \|\psi\|_{H^2(\Omega)} \, dt \\
&\le C\|\widetilde{v} - v\|_X \, \|w\|_X \, \|\psi\|_Y,
\end{aligned}$$

where we have used the Lipschitz continuity of f' and Sobolev's imbedding theorem. Thus, (2.3) is satisfied. Moreover, using the approximation properties of the interpolant $\widetilde{v}$, [Th], we have

$$\begin{aligned}
\|F(\widetilde{v}) - F(v)\|_{Y'} &\le \|\widetilde{v} - u\|_{L^\infty((0,T);L^2(\Omega))} + \|f(\widetilde{v}) - f(u)\|_{L^\infty((0,T);H^{-2}(\Omega))} \\
&\le Ck^{q+1} \|u^{(q+1)}\|_{L^\infty((0,T);L^2(\Omega))}
\end{aligned}$$

and, in view of Lemma 3.1, we easily conclude that (2.4) is satisfied.

We have thus established our main result in this section:

Theorem 3.1. *Let C_β be as in Lemma 3.1. There exists a positive k_0 such that for $k \le k_0$ there exists a locally unique solution u_k of (3.2) such that*

$$\|u_k - \widetilde{v}\|_X \le \frac{2}{C_\beta}\Big(\|\widetilde{v} - u\|_X + \|f(\widetilde{v}) - f(u)\|_{L^\infty((0,T);H^{-2}(\Omega))}\Big)$$

and

$$\|u - u_k\|_{L^\infty((0,T);L^2(\Omega))} \le CC_\beta^{-1} \max_n \Big(k_n^{q+1}\|u^{(q+1)}\|_{L^\infty(I_n;L^2(\Omega))}\Big).$$

4. Extensions-remarks

4.1 Fully discrete analysis. The same technique can be applied in the fully discrete (with space discretization) case. The analysis will be more technical and in addition one has to use mesh dependent norms in the space variable as well, as in [MB], see also [BO], [BOP]. We will not present this case in this note.

4.2 Nonlinear parabolic problems. If one considers nonlinear parabolic problem as in (1.1), (3.1) with nonlinearity in the principal part,

$$F(u) = -\text{div}\,(a(u)\nabla u) - f(u),$$

then similar difficulties as in [PR] arrise. In particular, if, e.g., we work with the spaces of Section 3, $DF(v)$ in (2.3) is not even well defined. One possibility in this case is to follow the ideas in [PR] and to finally derive estimates in spaces of the form $L^\infty(W^{1,p})$. An alternative however is provided in *the fully discrete case* by the use of inverse inequalities in space. In fact, in this case one can check that choosing the fully discrete analogs of the norms of Section 3, cf. [MB], and applying the theory of Section 2 in the discrete setting, $DF_h(\widetilde{v}), DF_h(v)$ are well defined but C_γ in (2.3) will grow like $C\underline{h}^{-1}$ as the minimum spatial mesh size $\underline{h} \to 0$. Therefore, the verification of (2.4) will require a mesh condition of the form $(h^r + k^{p+1})\underline{h}^{-1}$ small, r being the optimal order provided by the space discretization.

References

[BO] I. Babuška and J. E. Osborn, *Analysis of finite element methods for second order boundary value problems using mesh dependent norms*, Numer. Math. **34** (1980), 41–62.

[BOP] I. Babuška, J. E. Osborn and J. Pitkäranta, *Analysis of mixed methods using mesh dependent norms*, Math. Comp. **35** (1980), 1039–1062.

[CaR] G. Caloz and J. Rappaz, *Numerical analysis for nonlinear and bifurcation problems*, Handbook of Numerical Analysis, Vol. V (P. Ciarlet and J.-L. Lions, eds.), North-Holland, Amsterdam, pp. 487–637, 1997.

[CR] M. Crouzeix and J. Rappaz, *On Numerical Approximation in Bifurcation Theory*, Masson, Paris, 1990.

[EJ1] K. Eriksson and C. Johnson, *Error estimates and automatic time step control for nonlinear parabolic problems*, SIAM J. Numer. Anal. **24** (1987), 12–23.

[EJ2] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems II: Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$*, SIAM J. Numer. Anal. **32** (1995), 706–740.

[EJL] K. Eriksson, C. Johnson and S. Larsson, *Adaptive finite element methods for parabolic problems VI: Analytic semigroups*, SIAM J. Numer. Anal. **35** (1998), 1315–1325.

[EJT] K. Eriksson, C. Johnson and V. Thomée, *Time discretization of parabolic problems by the discontinuous Galerkin method*, RAIRO Modél. Mathém. Anal. Numér. **19** (1985), 611–643.

[E] D. Estep, *A posteriori error bounds and global error control for approximation of ordinary differential equations*, SIAM J. Numer. Anal. **32** (1995), 1–48.

[EL] D. Estep and S. Larsson, *The discontinuous Galerkin method for semilinear parabolic problems*, RAIRO Modél. Mathém. Anal. Numér. **27** (1993), 35–54.

[J] P. Jamet, *Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain*, SIAM J. Numer. Anal. **15** (1978), 912–928.

[Jo] C. Johnson, *Error estimates and adaptive time-step control for a class of one-step methods for stiff ordinary differential equations*, SIAM J. Numer. Anal. **25** (1988), 908–926.

[JRB] C. Johnson, R. Rannacher and M. Boman, *Error control in computational fluid dynamics*, SIAM J. Numer. Anal. **32** (1995), 1058–1079.

[KM] O. Karakashian and Ch. Makridakis, *A space-time finite element method for the nonlinear Schrödinger equation: the discontinuous Galerkin method*, Math. Comp. **67** (1998), 479–499.

[MB] Ch. G. Makridakis and I. Babuška, *On the stability of the discontinuous Galerkin method for the heat equation*, SIAM J. Numer. Anal. **34** (1997), 389–401.

[PR] J. Pousin and J. Rappaz, *Consistency, stability, a priori and a posteriori errors for Petrov–Galerkin methods applied to nonlinear problems*, Numer. Math. **69** (1994), 213–231.

[Th] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer–Verlag, Berlin, 1997.

G. Akrivis
COMPUTER SCIENCE DEPARTMENT, UNIVERSITY OF IOANNINA, 451 10 IOANNINA, GREECE

Ch. Makridakis
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRETE, 714 09 HERAKLION, CRETE, GREECE, AND IACM, FOUNDATION FOR RESEARCH AND TECHNOLOGY - HELLAS, 711 10 HERAKLION, CRETE, GREECE