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# Scattering of a Point Generated Field by a Multilayered Spheroid

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## SUMMARY

The point source excitation acoustic scattering problem by a multilayer isotropic and homogeneous spheroidal body is presented. The multilayer spheroidal body is reached by an acoustic wave emanated by an external point source. The core spheroidal region is inpenetrable and rigid. The exterior interface and the interfaces separating the interior layers are penetrable. The scattered field is determined given the geometrical and physical characteristics of the spheroidal body, the location of the point source and the form of the incident field. The approach is not limited in a certain region of frequencies.

KEY WORDS: Point Source Excitation; Multilayer Spheroidal Scatterer; Resonance Frequencies.

## 1. Introduction

The examination of the scattering problem of acoustic waves from ellipsoidal and spheroidal scatterers has attracted the scientific interest as it constitutes a generalization of the spherical case and gives birth to models simulating more interesting realistic scattering problems than these "living" in the spherical geometry. A lot of effort has been devoted to this direction, especially under the low-frequency regime and under the basic assumption of the plane wave excitation (Burke 1966a, 1966b, 1968, Bowman et. al. 1969, Dassios 1977, 1981, 1982, Kleinman 1965). Indeed, in many interesting cases, the dimension of the scatterer is very small compared with the wavelength of the incident field. This event permits the exploitation of the low-frequency theory leading to the replacement of the scattering problem by a sequence of corresponding potential problems referring to the same geometry. This approximation simplifies a lot the whole procedure as it gives us the possibility to be occupied with Laplace equation solutions instead of handling solutions of the wave or Helmholtz equation, which become more and more complicated when geometry becomes more complex.

Unfortunately, there exist acoustic scattering procedures a priori referring to the resonance, even the ultrasound region - as far as the wave number is concerned - fact rendering the low-frequency approach inadequate for many interesting applications. This is the case for an interesting scattering problem studied simultaneously by the authors (Charalambopoulos et al. 1999a). The problem under consideration concerns acoustical scattering from kidney stones and is inspired by the necessity of identifying the morphology and possible pathological features in the kidneys by solving the direct and inverse scattering problem in the suitable model system.

The frequency range of the acoustic excitation for this particular problem belongs to the resonance region and in many cases, should be considered closer to the high frequency region. Consequently, it is necessary to develop a method independent of the wavenumber, handling directly the undergoing Helmholtz equation. From the geometrical point of view, the best approximation fitting very accurately the specific human organ is a stratified spheroidal medium where the several layers correspond to particular physical components of the structure.

The other important issue of the scattering procedure is the particular form of the wave excitation. In previous works (Dassios 1977, 1981, 1982), the hypothesis of the plane wave incidence has been adopted, referring to the fact that in many scattering procedures, the acoustical wave source can be considered far from the scatterer and having suitable energy in order for the incident wave to interfere with the scatterer. Our motivating problem indicates that this is not the case now, as in medical treatment and monitoring, the sound emitter is located in touch with the human body and as a consequense, in the resonance region framework, the distance between the source and the scatterer is not large at all. We infer that the plane wave excitation assumption is not allowed in our case and that we are obliged to consider the case of a point source excitation. This is of course another serious complication factor. Indeed, the spherical wave emanating from the point source adapts to the spherical coordinate system. But the scattered wave, which is the result of the interference between the incident field and the scatterer, must be expressed in the spheroidal coordinate system in order for it to be adapted suitably to the boundary conditions induced on scatterer surface. The two different geometries must be adapted as the underlying fields are connected through the boundary conditions. Thus, suitable addition theorems must be used in order to fit the two geometries. These additional formulae are more complex in nature to that concerning the fitness between the spheroidal and the plane geometry induced by the plane wave. In the low – frequency regime, the problem of point source excitation has been faced for the spheroidal (Dassios et al. 1995) as well as the ellipsoidal case (Charalambopoulos et al. 1999b), where the wavenumber asymptotic analysis has been proved very helpful.

Summarizing the previous remarks, we state the problem in its well posed mathematical form. The aim of the direct scattering problem is the determination of the scattered field in the exterior space of the stratified spheroidal body. This field - in its time-independent form after assuming time harmonic dependence - satisfies a specific boundary value problem for it satisfies the Helmholtz equation, the Sommerfeld radiation condition at infinity (Morse and Feshbach, 1953) and suitable boundary conditions on scatterer surface, where it is related to the incident field as well the penetrating the scatterer surface field. As a matter of fact, all the boundary conditions on the interfaces of the stratified structure involve implicitely all the exterior and interior waves of the system. The scattered field as well as the interior fields are regular functions wherever are defined, in contrast to the incident spherical field, which disposes a singularity at the point source location point. We expand all waves of the problem in terms of the spheroidal wave functions, incorporating asymptotic properties and then we force them to satisfy the several boundary conditions of the problem. Exploiting suitably the properties of the special functions involved in these expansions, we determine, after extended analytical procedures, the coefficients in the above mentioned expansions.

Consequently we determine the scattered field as well, which incorporates in parametric form all the information about the physical and geometrical properties of the scatterer. The undecoding of this information from the knowledge of the scattered field is the inverse scattering problem and merits special interest in medical applications similar to that motivating our research. Of course, the alternatives we have in the position and distance of the receiver, measuring the scattered field, relative to the position of the point source and the scatterer is a very significant factor for the solution of the inverse scattering problem and are going to be examined under the guideness of the direct problem analysis and the already existed monitoring and measurement techniques in the accompanying application paper.

## 2. Spheroidal Geometry - Spheroidal wave functions

Before stating the problem under consideration, it is better to prepare the necessary framework concerning the geometrical and physical background of the problem. The underlying geometry, as it has already been stated, is the spheroidal one and the fields describing the harmonic response of the system satisfy the Helmholtz equation. Let us present briefly how Helmholtz equation behaves in the spheroidal coordinate system, following mainly the arguments given in Morse and Fesbach (1953)..

The connection between cartesian and spheroidal coordinates as well as the scalar factors are given by the relations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\alpha \sinh\mu \sin\theta \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$z = \frac{1}{2}\alpha \cosh\mu \cos\theta$$

$$h_\mu = h_\theta = \frac{1}{2}\alpha\sqrt{\cosh\mu^2 - \cos^2\theta}$$

$$h_\varphi = \frac{1}{2}\alpha \sinh\mu \sin\theta$$

where the spheroidal coordinates range over the intervals

$$\mu \geq 0, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi .$$

The case $\mu = 0$ corresponds to the line interval connecting the two focii of the spheroidal system located at the points $z = -\frac{1}{2}\alpha$ end $z = \frac{1}{2}\alpha$ .

The Laplace operator in spheroidal coordinates takes the form:

$$\Delta\psi = \frac{\frac{4}{a^2}}{\cosh\mu^2 - \cos^2\theta}\left[\begin{array}{l} \frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}\left(\sinh\mu\frac{\partial\psi}{\partial\mu}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) \\ + \frac{\sinh\mu^2 + \sin^2\theta}{\sinh\mu^2 \sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} \end{array}\right] . \qquad (1)$$

Let us consider the Helmholtz equation

$$\Delta\psi + \mathrm{k}^2\psi = 0 \qquad (2)$$

Applying separation of variables techniques we conclude that

$$\psi = \begin{pmatrix} \cos \\ \sin \end{pmatrix}(\mathrm{m}\varphi) R(\xi) S(\eta), \;\; \eta = \cos\vartheta, \; \xi = \cosh\mu \qquad (3)$$

where the functions $R$, $S$ satisfy the equations

$$\left.\begin{aligned}\frac{d}{d\xi}\left[(\xi^2-1)\frac{dR}{d\xi}\right]-\left[\lambda_{mn}-c^2\xi^2+\frac{m^2}{\xi^2-1}\right]R=0\\ \frac{d}{dn}\left[(n^2-1)\frac{dS}{dn}\right]-\left[\lambda_{mn}-c^2n^2+\frac{m^2}{n^2-1}\right]S=0\end{aligned}\right\}\quad c=\frac{1}{2}\alpha k\,, \tag{4}$$

and $\lambda_{mn}$ stand for separation of variable constants.

It is proved [12], that the functions $R$, $S$ are given by the relations

$$S_{mn}(n;c)=\sum_{k=0,1}^{\infty}{}' d_k^{mn}(c)P_{m+k}^{m}(n)=\begin{cases}\sum_{k=0}^{\infty} d_{2k}^{mn}(c)P_{m+2k}^{m}(n), & n=m,m+2,\ldots\\ \sum_{k=0}^{\infty} d_{2k+1}^{mn}(c)P_{m+2k+1}^{m}(n), & n=m+1,m+3,\ldots\end{cases} \tag{5}$$

$$R_{mn}^{(p)}(\xi;c)=\left[\sum_{k=0,1}^{\infty}{}'\frac{(2m+k)!}{k!}d_k^{mn}(c)\right]^{-1}\left(1-\frac{1}{\xi^2}\right)^{m/2}\sum_{k=0,1}^{\infty}{}' i^{k+m-n}\frac{(2m+k)!}{k!}d_k^{mn}(c)Z_{m+k}^{(p)}(c\xi) \tag{6}$$

where there exist four alternatives for the spherical Bessel functions $Z_{m+k}^{(p)}$

$$\begin{aligned}Z_n^{(1)}(z)&=j_n(z)\\ Z_n^{(2)}(z)&=y_n(z)\\ Z_n^{(3)}(z)&=h_n^{(1)}(z)=\left(j_n(z)+iy_n(z)\right)\\ Z_n^{(4)}(z)&=h_n^{(2)}(z)=\left(j_n(z)-iy_n(z)\right)\end{aligned}$$

while $P_n^m(n)$ denote the Legendre functions. In addition, the symbol $\sum_{k=0,1}^{\infty}{}'$, as it is clear from Eq. (5), indicates summation over even or odd indices, depending on the starting index.

The crucial point is the determination of the coefficients $d_k^{mn}(c)$.

Inserting Eqs. (5), (6) in Eq. (2) and exploiting recurrence relations for Legendre functions we conclude to the following recursive scheme:

$$\frac{(2m+k+2)(2m+k+1)}{(2m+2k+5)(2m+2k+3)}c^2 d_{k+2}^{mn}(c) + \left[\begin{array}{l}(m+k)(m+k+1)-\lambda_{mn}(c)+ \\ \dfrac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}c^2\end{array}\right] d_k^{mn}(c)$$
$$+\frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}c^2 d_{k-2}^{mn}(c)=0 \qquad (7)$$

Setting for simplicity

$$\alpha_k = \frac{(2m+k+2)(2m+k+1)}{(2m+2k+5)(2m+2k+3)}c^2$$
$$\beta_k = (m+k)(m+k+1) + c^2\,\frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}$$
$$\gamma_k = \frac{k(k-1)}{(2m+2k-3)(2m+2k-1)}c^2$$

we obtain the homogeneous system

$$\begin{array}{l}\alpha_0 d_2^{mn}(c) + [\beta_0 - \lambda_{mn}(c)]d_0^{mn}(c) = 0 \\ \alpha_1 d_3^{mn}(c) + [\beta_1 - \lambda_{mn}(c)]d_1^{mn}(c) = 0 \\ \ldots \\ \alpha_k d_{k+2}^{mn}(c) + [\beta_k - \lambda_{mn}(c)]d_k^{mn}(c) + \gamma_k d_{k-2}^{mn}(c) = 0\end{array} \qquad (8)$$

We obtain then, separately, the following eigenvalue problems

$$\begin{bmatrix}\beta_0 & \alpha_0 & 0 & 0 & \ldots & \ldots \\ \gamma_2 & \beta_2 & \alpha_2 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & .. & \gamma_{2k} & \beta_{2k} & \alpha_{2k} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\end{bmatrix}\begin{bmatrix}d_0^{mn}(c) \\ d_2^{mn}(c) \\ d_4^{mn}(c) \\ \ldots \\ d_{2k}^{mn}(c) \\ \ldots\end{bmatrix} = \lambda_{mn}(c)\begin{bmatrix}d_0^{mn}(c) \\ d_2^{mn}(c) \\ d_4^{mn}(c) \\ \ldots \\ d_{2k}^{mn}(c) \\ \ldots\end{bmatrix} \qquad (9a)$$

for $(n-m)$ even.

$$
\begin{bmatrix}
\beta_1 & \alpha_1 & 0 & 0 & \cdots & \cdots \\
\gamma_3 & \beta_3 & \alpha_3 & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & .. & \gamma_{2k+1} & \beta_{2k+1} & \alpha_{2k+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\begin{bmatrix}
d_1^{mn}(c) \\ d_3^{mn}(c) \\ d_5^{mn}(c) \\ \cdots \\ d_{2k+1}^{mn}(c) \\ \cdots
\end{bmatrix}
= \lambda_{mn}(c)
\begin{bmatrix}
d_1^{mn}(c) \\ d_3^{mn}(c) \\ d_5^{mn}(c) \\ \cdots \\ d_{2k+1}^{mn}(c) \\ \cdots
\end{bmatrix}
\tag{9b}
$$

for $(n-m)$ odd.

Eq. (9α) provides with the eigenvalues $\lambda_{m,m}(c), \lambda_{m,m+2}(c), \lambda_{m,m+4}(c),\ldots$ and Eq. (9β) provides with the eigenvalues $\lambda_{m,m+1}(c), \lambda_{m,m+3}(c), \lambda_{m,m+5}(c),\ldots$

For every $\lambda_{mn}(c)$ determined above, the coefficients $d_k^{mn}(c)$ are determined modulo a multiplicative constant. These coefficients are fully determined when a normalization condition is imposed.

In Ref. 12, we find

$$
\sum_k \frac{(k+2m)!}{k!} d_k^{mn}(c) = \frac{(n+m)!}{(n-m)!} \tag{10}
$$

Under the above condition, Eq. (6) which expresses the "radial" functions becomes simpler and takes the form

$$
R_{mn}^{(P)}(\xi;c) = \frac{(n-m)!}{(n+m)!}\left(1-\frac{1}{\xi^2}\right)^{m/2} \sum_{k=0,1}^{\infty}{}' i^{k+m-n} \frac{(2m+k)!}{k!} d_k^{mn}(c) Z_{m+k}^{(p)}(c\xi) . \tag{11}
$$

## 3. Statement of the problem

A multi-layer isotropic and homogeneous stratified spheroidal body occupying region $V$, centered at the coordinate system origin, is reached by an acoustic spherical wave emanated by a point source located at point M having position vector $\mathbf{r}' = (\mu', \theta', \varphi')$. The core spheroidal region–surrounded by surface $S_0$- is inpenetrable and rigid. The exterior surface $S_I$, as well the interfaces separating the interior spheroidals shells, occupying regions $V_i$, are penetrable and must compensate pressures and normal velocities on their sides. The several layers are characterized physically by their densities $\rho_i$ and the velocity of sound propagation $c_i$ in every particular component. The surrounding space has density $\rho$ and the sound propagates in this region with speed $c_{ext}$. The geometry of the problem is given in Fig. 1.

The result of interference of the incident spherical field with the scatterer is the creation of the scattered wave propagating outwards the scatterer as well as the creation of penetrating acoustic fields enclosed in the several spheroidal layers. The direct scattering problem consists in the determination of the scattered field, given the geometrical and physical characteristics of the scatterer as well as the parameters describing completely the excitation (i.e. position of the point source and particular form of the incident field).

The scattered field incorporates all the information about the scatterer and the undecoding of this information constitutes the inverse scattering problem, whose solution is based-as in all problems with simple geometry-to the suitable exploitation of the solution of the direct problem.

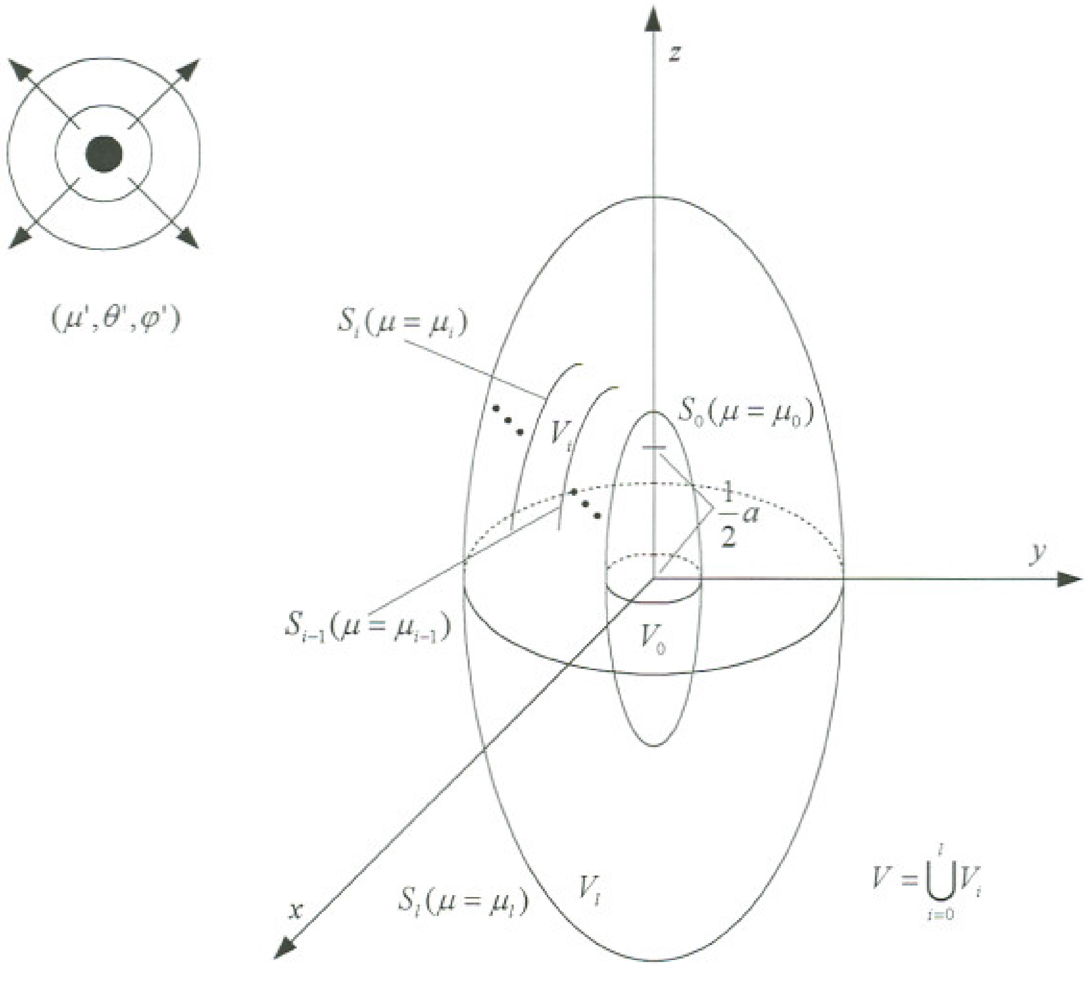


*Fig. 1: Problem Geometry.*

The mathematical formulation of the problem under consideration is the following.

The point source located at point $\mathrm{M}(\mu',\theta',\varphi')$, emits a harmonic spherical wave of the form

$$u^{in}(\mathbf{r},t;\mathbf{r}') = u^{in}(\mathbf{r};\mathbf{r}')\, e^{-i\omega t} \tag{12}$$

where

$$u^{in}(\mathbf{r};\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{e^{ik\mathbf{R}}}{\mathbf{R}} \tag{13}$$

stands for the time-independent spherical wave. The harmonic time-dependence is induced through the factor $e^{-i\omega t}$, fact rendering all the fields appearing in the problem dependent on time through the same multiplicative factor. Consequently, we restrict ourselves to study the reduced equations and waves after supressing the time harmonic dependence. In addition, the implicit dependence of all the functions on the point source position $\mathbf{r}'$ is omitted in the sequel, for simplicity.

In Eq. (13), $k = \dfrac{\omega}{c_{ext}}$ stands for the wave number of the propagation process in the exterior space.

The scattered field satisfies the Helmholtz equation,

$$\Delta u^{scat}(\mathbf{r}) + k^2 u^{scat}(\mathbf{r}) = 0, \quad \mathbf{r} \in R^3 \setminus V \tag{14}$$

as well as the radiation condition of Sommerfeld,

$$\frac{\partial}{\partial r} u^{scat}(\mathbf{r}) - iku^{scat}(\mathbf{r}) \to 0, \quad \text{as } r \to \infty \tag{15}$$

The interior fields $u^i(\mathbf{r})$ satisfies also Helmholtz equation

$$\Delta u^i(\mathbf{r}) + k_i^2 u^i(\mathbf{r}) = 0, \quad \mathbf{r} \in V_i \tag{16}$$

(where $k_i = \dfrac{\omega}{c_i}$ stands for the wave number in medium $V_i$).

All the fields are connected through the boundary conditions, which after some possible field renormalizations (Colton and Kress, 1983), take the form

Surface $S_l$: $\rho\left(u^{scat} + u^{in}\right) = \rho_l\, u^l$ (17a)

$$\frac{\partial}{\partial n}\left(u^{scat} + u^{in}\right) = \frac{\partial}{\partial n}\, u^l \qquad (17b)$$

Surface $S_i$ $(1 \le i \le n-1)$: $\rho_i u^i = \rho_{i+1} u^{i+1}$ (18a)

$$\frac{\partial}{\partial n} u^i = \frac{\partial}{\partial n} u^{i+1} \qquad (18b)$$

Surface $S_0$: $\frac{\partial}{\partial n} u^1 = 0$. (19)

It is proved that the boundary value problem constituted by Eqs. (14-19) is a well posed mathematical problem.

## 4. The Solution of the Direct Scattering Problem.

In order to solve the boundary value problem under consideration, we expand first all the fields satisfying Helmholtz equation in terms of the spheroidal wave functions as follows:

$$u^i(r) = \sum_{mn} \frac{1}{\Lambda_{mn}\left(c^{(i)}\right)} \left\{ \begin{array}{l} a^i_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) je_{mn}\left(\cosh\mu;\, c^{(i)}\right) + \\ \beta^i_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) je_{mn}\left(\cosh\mu;\, c^{(i)}\right) + \\ \gamma^i_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) ye_{mn}\left(\cosh\mu;\, c^{(i)}\right) + \\ \delta^i_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) ye_{mn}\left(\cosh\mu;\, c^{(i)}\right) \end{array} \right\}, \quad r \in V_i. \qquad (20)$$

(for $1 \le i \le l$)

and

$$u^{scat}(r)=\sum_{mn}\frac{1}{\Lambda_{mn}(c)}\left\{\begin{array}{l}a^{ext}_{mn}\cos(m\varphi)\,S_{mn}(\cos\theta;\,c)he_{mn}(\cosh\mu;\,c)+\\ \beta^{ext}_{mn}\sin(m\varphi)\,S_{mn}(\cos\theta;\,c)he_{mn}(\cosh\mu;\,c)\end{array}\right\},\qquad r\in R^3\setminus V \quad (21)$$

where $je_{mn}=R^{(1)}_{mn}$, $\mathrm{ye}_{mn}=\mathrm{R}^{(2)}_{mn}$ and $he_{mn}=R^{(3)}_{mn}$.

(We incorporate the asymptotic condition of Sommerfeld (Kolton and Kress, 1983) by choosing the radial Hankel function $he_{mn}(\cosh\mu;c)$ to express the scattered field.)

In addition, $c^{(i)}=\frac{1}{2}k_i a$ and $c=\frac{1}{2}ka$. Finally, summation $\sum_{mn}$ denotes $\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}$, while the constants $\Lambda_{mn}\left(c^{(i)}\right)$ relate to the normalization constants of the «angle» functions $S_{mn}\left(cos\vartheta;\,c^{(i)}\right)$ as follows:

$$\Lambda_{mn}\left(c^{(i)}\right)=\int_{-1}^{1}\left|S_{mn}\left(\eta\,;\,c^{(i)}\right)\right|^2 d\eta=\sum_k\left|d_k^{mn}\right|^2\left(\frac{2}{2k+2m+1}\right)\frac{(k+2m)!}{(k)!} \qquad (22)$$

The problem of determination of the several waves of the problem has been transferred - using the expansion in spheroidal wave functions - to the determination of the coefficients entering the expansions.

Similarly to the secondary fields, the incident field has to be expanded in the same function basis. In order to expand a spherical wave in spheroidal coordinates we use an appropriate addition theorem (Morse and Fesbach, 1953) to obtain

$$\begin{aligned}\frac{e^{ikR}}{R}&=2ik\sum_{m,n}\frac{e_m}{\Lambda_{mn}}S_{mn}(\cos\theta';c)S_{mn}(\cos\theta';c)\cos[m(\varphi-\varphi')]\\&\times\begin{cases}je_{mn}(\cosh\mu';c)he_{mn}(\cosh\mu;c); & \mu>\mu'\\ je_{mn}(\cosh\mu;c)he_{mn}(\cosh\mu';c); & \mu<\mu'\end{cases}\end{aligned} \qquad (23)$$

where

$$e_m = \begin{cases} 2 & m \neq 0 \\ 1 & m = 0 \end{cases}.$$

Expressions (21), (22) and (23) are the suitable forms that must be inserted in the boundary conditions in order for the expansion coefficients to be determined.

The normal derivative operator appeared in boundary conditions has the following form in spheroidal coordinate system

$$\frac{\partial}{\partial n} = \frac{1}{h_\mu}\frac{\partial}{\partial \mu} = \frac{1}{\left(\frac{1}{2}\alpha\right)\sqrt{\cosh \mu^2 - \cos^2\theta}}\frac{\partial}{\partial \mu}. \tag{24}$$

<u>*Treatment of Boundary Conditions*</u>

i) Surface $S_0$

Equation (19) is written as:

$$\sum_{mn}\frac{1}{\Lambda_{mn}(c^{(1)})}\left\{\begin{array}{l} a^1_{mn}\cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(1)}\right) je'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) + \\ \beta^1_{mn}\sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(1)}\right) je'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) + \\ \gamma^1_{mn}\cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(1)}\right) ye'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) + \\ \delta^1_{mn}\sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(1)}\right) ye'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) \end{array}\right\} = 0 \tag{25}$$

for $0 \le \theta \le \pi, 0 \le \varphi \le 2\pi$.

Given that $\cos(m\varphi)$, $\sin(m\varphi)$ are orthogonal functions and that $\mathrm{S}_{mn}\left(\cos\theta; \mathrm{c}^{(1)}\right)$ are orthogonal for constant $m$, as $n$ varies we obtain:

$$a^1_{mn}\, je'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) + \gamma^1_{mn}\, ye'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) = 0, \quad m \ge 0,\ n = m, m+1, m+2, \ldots \tag{26a}$$

$$\beta^1_{mn}\, je'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) + \delta^1_{mn}\, ye'_{mn}\left(\cosh\mu_0;\, c^{(1)}\right) = 0, \quad m \ge 0,\ n = m, m+1, m+2, \ldots \tag{26b}$$

ii) Surface $S_i$ $1 \le i \le l-1$

Equation (18a) takes the form

$$\sum_{mn} \frac{\rho_i}{\Lambda_{mn}\left(c^{(i)}\right)} \left\{ \begin{array}{l} a^{i}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) je_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) + \\ \beta^{i}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) je_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) + \\ \gamma^{i}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) ye_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) + \\ \delta^{i}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) ye_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) \end{array} \right\} =$$

$$\sum_{mn} \frac{\rho_{i+1}}{\Lambda_{mn}\left(c^{(i+1)}\right)} \left\{ \begin{array}{l} a^{i+1}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) je_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) + \\ \beta^{i+1}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) je_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) + \\ \gamma^{i+1}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) ye_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) + \\ \delta^{i+1}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) ye_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) \end{array} \right\}. \qquad (27)$$

Equation (18b), in a similar way, takes the form

$$\sum_{mn} \frac{1}{\Lambda_{mn}(c^{(i)})} \left\{ \begin{array}{l} a^{i}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) je'_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) + \\ \beta^{i}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) je'_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) + \\ \gamma^{i}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) ye'_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) + \\ \delta^{i}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i)}\right) ye'_{mn}\left(\cosh\mu_i;\, c^{(i)}\right) \end{array} \right\} =$$

$$\sum_{mn} \frac{1}{\Lambda_{mn}(c^{(i+1)})} \left\{ \begin{array}{l} a^{i+1}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) je'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) + \\ \beta^{i+1}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) je'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) + \\ \gamma^{i+1}_{mn} \cos(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) ye'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) + \\ \delta^{i+1}_{mn} \sin(m\varphi)\, S_{mn}\left(\cos\theta;\, c^{(i+1)}\right) ye'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right) \end{array} \right\}. \qquad (28)$$

The orthogonality of trigonometric functions $cos(m\varphi)$ and $sin(m\varphi)$ guarantees that for every specific $m$ the corresponding terms of the above double sums are equal to each other. If we project then these equations containing the simple sums $\sum_n$ to the complete and orthogonal set of functions $S_{mn}\left(\cos\theta;\, c^{(i+1)}\right)$, $n=m,m+1,m+2,\ldots$, we obtain the following relations:

$$\rho_{i+1}\left[\alpha_{mn}^{i+1}\, je_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)+\gamma_{mn}^{i+1}\, ye_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)\right]=$$

$$\rho_i \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(i)})}\left[\alpha_{mn'}^{i}\, je_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)+\gamma_{mn'}^{i}\, ye_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)\right]\left\langle S_{mn'}^{i},\, S_{mn}^{i+1}\right\rangle, \quad (29a)$$

$$\rho_{i+1}\left[\beta_{mn}^{i+1}\, je_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)+\delta_{mn}^{i+1}\, ye_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)\right]=$$

$$\rho_i \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(i)})}\left[\beta_{mn'}^{i}\, je_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)+\delta_{mn'}^{i}\, ye_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)\right]\left\langle S_{mn'}^{i},\, S_{mn}^{i+1}\right\rangle. \quad (29b)$$

and

$$\left[\alpha_{mn}^{i+1}\, je'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)+\gamma_{mn}^{i+1}\, ye'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)\right]=$$

$$\sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(i)})}\left[\alpha_{mn'}^{i}\, je'_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)+\gamma_{mn'}^{i}\, ye'_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)\right]\left\langle S_{mn'}^{i},\, S_{mn}^{i+1}\right\rangle, \quad (30a)$$

$$\left[\beta_{mn}^{i+1}\, je'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)+\delta_{mn}^{i+1}\, ye'_{mn}\left(\cosh\mu_i;\, c^{(i+1)}\right)\right]=$$

$$\sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(i)})}\left[\beta_{mn'}^{i}\, je'_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)+\delta_{mn'}^{i}\, ye'_{mn'}\left(\cosh\mu_i;\, c^{(i)}\right)\right]\left\langle S_{mn'}^{i},\, S_{mn}^{i+1}\right\rangle. \quad (30b)$$

$n=m,m+1,\ldots$

where

$$\left\langle S_{mn'}^{i},\, S_{mn}^{i+1}\right\rangle = \int_{-1}^{1} S_{mn'}^{i}\left(\eta;\, c^{(i)}\right) S_{mn}^{i+1}\left(\eta;\, c^{(i+1)}\right) d\eta =$$

$$\sum_{k}{}^{'}\sum_{k'}{}^{'} d_{k'}^{mn'}\left(c^{(i)}\right) d_{k}^{mn}\left(c^{(i+1)}\right)\int_{-1}^{1} P_{m+k'}^{m}(\eta) P_{m+k}^{m}(\eta)\, d\eta =$$

$$\sum_{k}{}^{'} d_{k}^{mn'}\left(c^{(i)}\right) d_{k}^{mn}\left(c^{(i+1)}\right)\frac{2}{2k+2m+1}\frac{(k+2m)!}{k!} \quad (31)$$

are the mixed inner products of "radial" functions, which would be diagonal only in the very specific case $c^{(i)} = c^{(i+1)}$.

iii) Surface $S_I$

We handle Eqs. (17) following the same steps, i.e exploiting orthogonality of azimuthal and radial functions, obtaining the following results

$$\rho\left[a_{mn}^{ext}\, he_{mn}(\cosh\mu_I;\, c) + 2ike_m S_{mn}(\cos\theta';\, c)\cos(m\varphi')\, je_{mn}(\cosh\mu_I;\, c) he_{mn}(\cosh\mu';\, c)\right]$$

$$= \rho_I \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(I)})}\left[a_{mn'}^{I}\, je_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right) + \gamma_{mn'}^{I}\, ye_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right)\right]\left\langle S_{mn'}^{I}, S_{mn}\right\rangle \quad (32a)$$

$$\rho\left[\beta_{mn}^{ext}\, he_{mn}(\cosh\mu_I;\, c) + 2ike_m S_{mn}(\cos\theta';\, c)\sin(m\varphi')\, je_{mn}(\cosh\mu_I;\, c) he_{mn}(\cosh\mu';\, c)\right]$$

$$= \rho_I \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(I)})}\left[\beta_{mn'}^{I}\, je_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right) + \delta_{mn'}^{I}\, ye_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right)\right]\left\langle S_{mn'}^{I}, S_{mn}\right\rangle \quad (32b)$$

and

$$\left[a_{mn}^{ext}\, he'_{mn}(\cosh\mu_I;\, c) + 2ike_m S_{mn}(\cos\theta';\, c)\cos(m\varphi')\, je'_{mn}(\cosh\mu_I;\, c) he_{mn}(\cosh\mu';\, c)\right]$$

$$= \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(I)})}\left[\alpha_{mn'}^{I}\, je'_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right) + \gamma_{mn'}^{I}\, ye'_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right)\right]\left\langle S_{mn'}^{I}, S_{mn}\right\rangle \quad (33a)$$

$$\left[\beta_{mn}^{ext}\, he'_{mn}(\cosh\mu_I;\, c) + 2ike_m S_{mn}(\cos\vartheta';\, c)\sin(m\varphi')\, je'_{mn}(\cosh\mu_I;\, c) he_{mn}(\cosh\mu';\, c)\right.$$

$$= \sum_{n'} \frac{1}{\Lambda_{mn'}(c^{(I)})}\left[\beta_{mn'}^{I}\, je'_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right) + \delta_{mn'}^{I}\, ye'_{mn'}\left(\cosh\mu_I;\, c^{(I)}\right)\right]\left\langle S_{mn'}^{I}, S_{mn}\right\rangle. \quad (33b)$$

Let us define the matrices

$$\mathbf{\Delta}_{i+1}^{m,n} = \begin{bmatrix} \rho_{i+1} je_{mn}(\cosh\mu_i; c^{(i+1)}) & \rho_{i+1} ye_{mn}(\cosh\mu_i; c^{(i+1)}) \\ je'_{mn}(\cosh\mu_i; c^{(i+1)}) & ye'_{mn}(\cosh\mu_i; c^{(i+1)}) \end{bmatrix},\ i = 0,1,2,\ldots,I-1,$$

$$\mathbf{B}_{i+1}^{m,n,n'} = \frac{< S_{mn'}^{i+1}, S_{mn}^{i+2} >}{\Lambda_{mn}(c^{(i+1)})} \mathbf{\Delta}_{i+1}^{m,n}, \quad i = 0,1,2,\dots,l-1 \text{ where } S_{mn}^{l+1} = S_{mn},$$

$$\mathbf{y}_{m,n} = \left[\rho he_{mn}(\cosh\mu_l; c), he'_{mn}(\cosh\mu_l; c)\right]^T$$
$$\boldsymbol{\omega}_{m,n} = \left[je'_{mn}(\cosh\mu_0; c^{(1)}), ye'_{mn}(\cosh\mu_0; c^{(1)})\right]$$

and the supermatrices ( $(2l+1)\times(2l+1)$ )

$$\mathbf{B}_{n,n}^{m} = \begin{bmatrix} \boldsymbol{\omega}_{m,n} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & 0 \\ -\mathbf{B}_1^{m,n,n} & \mathbf{\Delta}_2^{m,n} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \overline{\mathbf{0}} & -\mathbf{B}_2^{m,n,n} & \mathbf{\Delta}_3^{m,n} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & -\mathbf{B}_3^{m,n,n} & \mathbf{\Delta}_4^{m,n} & \dots & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & -\mathbf{B}_{l-1}^{m,n,n} & \mathbf{\Delta}_l^{m,n} & \mathbf{0}^T \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & -\mathbf{B}_l^{m,n,n} & \mathbf{y}_{m,n} \end{bmatrix}$$

$$\mathbf{B}_{n,n'}^{m} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & 0 \\ -\mathbf{B}_1^{m,n,n'} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \overline{\mathbf{0}} & -\mathbf{B}_2^{m,n,n'} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & -\mathbf{B}_3^{m,n,n'} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & -\mathbf{B}_{l-1}^{m,n,n'} & \overline{\mathbf{0}} & \mathbf{0}^T \\ \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \overline{\mathbf{0}} & \dots & \overline{\mathbf{0}} & -\mathbf{B}_l^{m,n,n'} & \mathbf{0}^T \end{bmatrix}, \quad n \neq n'$$

where $\mathbf{0} = [0,0]$ and $\overline{\mathbf{0}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Then, the system of coefficients satisfies the non – homogeneous linear system

$$\mathbf{D}^{(m)}\mathbf{x}^{(m)} = \mathbf{b}^{(m)} \tag{34}$$

where

$$\mathbf{x}^{(m)} = [\mathbf{x}_m, \mathbf{x}_{m+1}, \mathbf{x}_{m+2}, \ldots, \mathbf{x}_n, \ldots]^T \tag{35}$$

with

$$\mathbf{x}_n = \begin{bmatrix} a_{mn}^1 & \gamma_{mn}^1 & a_{mn}^2 & \gamma_{mn}^2 & \ldots & a_{mn}^l & \gamma_{mn}^l & a_{mn}^{ext} \end{bmatrix} \tag{36}$$

(or $\mathbf{x}_n = \begin{bmatrix} \beta_{mn}^1 & \delta_{mn}^1 & \beta_{mn}^2 & \delta_{mn}^2 & \ldots & \beta_{mn}^l & \delta_{mn}^l & \beta_{mn}^{ext} \end{bmatrix}$)

$$\mathbf{D}^{(m)} = \begin{bmatrix} \mathbf{B}_{m,m}^m & \mathbf{B}_{m,m+1}^m & \mathbf{B}_{m,m+2}^m & \ldots & \ldots \\ \mathbf{B}_{m+1,m}^m & \mathbf{B}_{m+1,m+1}^m & \mathbf{B}_{m+1,m+2}^m & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{B}_{m+\nu,m}^m & \mathbf{B}_{m+\nu,m+1}^m & \ldots & \mathbf{B}_{m+\nu,m+\nu}^m & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

(common for $(a,\gamma)$ and $(\beta,\delta)$)

$$\mathbf{b}^{(m)} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_n, \ldots]^T, \tag{37}$$

and

$$\mathbf{b}_n = \begin{bmatrix} 0,0,\ldots,0,-\rho 2ik\varepsilon_m S_{mn}(\cos\theta';c)\cos(m\varphi')\, je_{mn}(\cosh\mu_l;c)he_{mn}(\cosh\mu';c), \\ -2ik\varepsilon_m S_{mn}(\cos\theta';c)\cos(m\varphi)\, je'_{mn}(\cosh\mu_l;c)he_{mn}(\cosh\mu';c) \end{bmatrix}$$

is of dimension $(2l+1)$.

(for the case of $\beta, \delta, \beta^{ext}$ $\cos(m\varphi')$ is replaced by $\sin(m\varphi')$).

The solution of the system (34) is possible when the system is truncated to a finite value of $N$. The same procedure must be repeated for every value of azimuthal number $m$. This leads to the determination of the scattered field expansion coefficients to the desired accuracy. This is examined for the case of the two layer spheroidal human kidney in Ref. 9.

## 5. Concluding Remarks

In this work we give the most general approach to the point source excitation scattering problem by stratified spheroidal structures without restrictions to frequency region, geometrical, physical or measurement characteristics. We exploited the powerful background of spheroidal wave functions and expanded all the fields in terms of them. This approach gives a series of infinite linear systems (because of the spheroidal geometry – in the case of spherical geometry those systems are degenerated to finite ones) satisfied by the expansion scattered field coefficients which can be determined by truncation. This method finally constitutes the direct scattering problem solution. The inverse scattering problem solution is under investigation for the specific application problem (Charalambopoulos et al. 1999a) and is based on the suitable use of the direct problem solution.

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