WAVE PROPAGATION MODELING IN HUMAN LONG BONES

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SUMMARY

The dynamic behaviour of a dry long bone that has been modeled as a piezoelectric hollow cylinder of crystal class 6 is investigated. The solution for the wave propagation problem is expressed in terms of a potential function which satisfies an eight - order partial differential equation, whose solutions lead to the derivation of the explicit solution of the wave equation. The mechanical boundary conditions correspond to those of stress free lateral surfaces, while the electrical boundary conditions correspond to those of short circuit. The satisfaction of the boundary conditions lead to the dispersion relation which is solved numerically. The eigenfrequencies obtained are presented as a function of various parameters and they are compared well with other researchers' theoretical results.

1. INTRODUCTION

The discovery of the piezoelectric property in bone has opened a new field in biomedical engineering. The study of vibrations in piezoelectric materials will provide vital information which can be used in the monitoring of the rate of bone fracture healing.

The wave propagation in piezoelectric rods of hexagonal crystal symmetry was studied by Wilson and Morrison [1]. They presented an exact description for the vibration modes using an extension of Mirsky's technique [2]. Ambardar and Ferris [3] proposed a model for the wave propagation characteristics in long bones consisted of a two-layered cylindrical shell of crystal class 6mm. The bone behaves like crystal class 6. In the case of crystal class 6mm, there are three independent piezoelectric constants since e_{14}, is zero, but this constant for bone has the largest value [4]. However, neither Wilson and Morrison [1] nor Ambardar and Ferris [3] have presented numerical results for the characteristic frequency equation.

Güzelsu and Saha [5] studied electromagnetic wave propagation in the hexagonal crystal class piezoelectric hollow cylinder for dry bone shaft. In their work, a comparison was made between theoretical predictions and experimental data which were obtained for flexural waves by non-contacting device. However, they have not included the influence of the electric field in stresses.

The wave propagation in a piezoelectric bone of arbitrary cross section with a circular cylindrical cavity and in a piezoelectric bone with cylindrical cavity of arbitrary shape was studied by Paul and Vankatesan [6-7].

Ding and Chenbuo [8] obtained three general solutions for the coupled dynamic equations for a piezoelectric medium of crystal class 6 mm. These solutions are expressed in terms of two functions ψ and F, where ψ satisfies a second-degree partial differential equation and F a sixth-degree partial differential equation.

In this work, we study the wave propagation in an infinite piezoelectric hollow cylinder of crystal class 6. First, following the formalism of Ref. 8 for piezoelectric medium of crystal class 6 mm, we express the solutions for wave propagation problems in piezoelectric medium of crystal class 6 in terms of one function F which satisfies a quadruple-Helmholtz equation. Solving this equation for F, we derive explicit expressions for four solutions of the wave equation. The frequency equation for the system considered is obtained when the lateral surface is stress-free and coated with electrodes that are shorted. The dynamic characteristics are calculated in the case of a hollow cylinder which represents the cortical bone for various values of wave number as well as other parameters entering the system. We present also solutions for the simplified cases of non-piezoelectric and isotropic bones. Our results are presented for various parameters of the system.

2. PROBLEM FORMULATION

The system under consideration is shown in Fig. 1 and consists of a hollow piezoelectric circular cylinder of crystal class 6 with inner radius r_0 and the outer one r_1. The cylindrical polar system (r,θ,z') is introduced and the z-axis of the cylinder is assumed to be perpendicular to the isotropic plane of the medium.

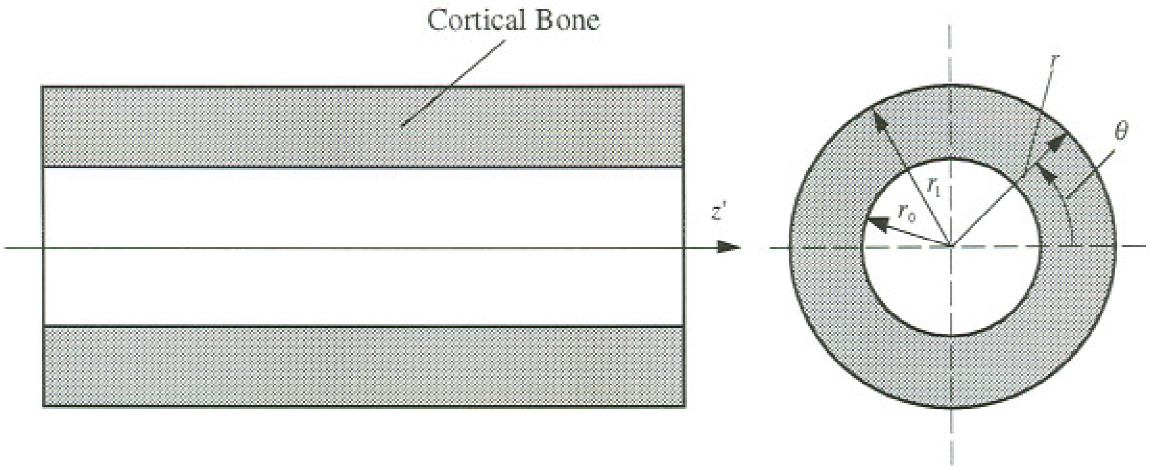


Figure 1: *Problem Geometry.*

For a piezoelectric material of crystal class 6 the equations of motion and the equation of Gauss in cylindrical coordinates are given as

$$c_{11}\left(u'_{r,rr} + r^{-1}u'_{r,r} - r^{-2}u'_r\right) + c_{66}r^{-2}u'_{r,\theta\theta} + c_{44}u'_{r,z'z'} + (c_{66}+c_{12})r^{-1}u'_{\theta,r\theta} - (c_{66}+c_{11})r^{-2}u'_{\theta,\theta}$$

$$+(c_{44}+c_{13})u'_{z,rz'} + (e_{15}+e_{31})V'_{,rz'} - e_{14}r^{-1}V'_{,\theta z'} = \rho_s \frac{\partial^2 u'_r}{\partial t'^2}, \qquad (1)$$

$$(c_{66}+c_{12})r^{-1}u'_{r,r\theta} + (c_{66}+c_{11})r^{-2}u'_{r,\theta} + c_{66}\left(u'_{\theta,rr} + r^{-1}u'_{\theta,r} - r^{-2}u'_\theta\right) + c_{11}r^{-2}u'_{\theta,\theta\theta}$$

$$+c_{44}u'_{\theta,z'z'} + (c_{44}+c_{13})r^{-1}u'_{z,\theta z'} + e_{14}V'_{,rz'} + (e_{15}+e_{31})r^{-1}V'_{,\theta z'} = \rho_s \frac{\partial^2 u'_\theta}{\partial t'^2}, \qquad (2)$$

$$(c_{44}+c_{13})\left(u'_{r,rz'} + r^{-1}u'_{r,z'} + r^{-1}u'_{\theta,\theta z'}\right) + c_{44}\left(u'_{z,rr} + r^{-1}u'_{z,r} + r^{-2}u'_{z,\theta\theta}\right) + c_{33}u'_{z,z'z'}$$

$$+e_{15}\left(V'_{,rr} + r^{-1}V'_{,r} + r^{-2}V'_{,\theta\theta}\right) + e_{33}V'_{,z'z'} = \rho_s \frac{\partial^2 u'_z}{\partial t'^2}, \qquad (3)$$

$$\epsilon_{11}\left(V'_{,rr} + r^{-1}V'_{,r} + r^{-2}V'_{,\theta\theta}\right) + \epsilon_{33}V'_{,z'z'} - (e_{15}+e_{31})\left(u'_{r,rz'} + r^{-1}u'_{r,z'} + r^{-1}u'_{\theta,\theta z'}\right)$$

$$+e_{14}\left(r^{-1}u'_{r,\theta z'}-u'_{\theta,rz'}-r^{-1}u'_{\theta,z'}\right)-e_{15}\left(u'_{z,rr}+r^{-1}u'_{z,r}+r^{-1}u'_{z,\theta\theta}\right)-e_{33}u'_{z,z'z'}=0. \quad (4)$$

where u'_r, u'_θ and u'_z are the elastic displacement components, V' is the electrostatic potential, c_{ij} are the elastic constants, e_{ij} are the piezoelectric constants, $\in_{ij}$ are the dielectric permittivities, ρ_s is the mass density and $u_{i,j}=\frac{\partial u_i}{\partial x_j}$.

The boundary conditions are:

$$T'_{rr}=T'_{r\theta}=T'_{rz}=0, \quad V'=0, \quad \text{at} \quad r=r_0,\ r_1, \quad (5)$$

where T'_{rr}, $T'_{r\theta}$ and T'_{rz} are components of the stress tensor which satisfy the constitutive relations:

$$T'_{rr}=c_{11}u'_{r,r}+c_{12}r^{-1}\left(u'_{\theta,\theta}+u'_r\right)+c_{13}u'_{z,z'}+e_{13}V'_{,z'}, \quad (6)$$

$$T'_{r\theta}=c_{66}\left[u'_{\theta,r}+r^{-1}u'_{r,\theta}-r^{-1}u'_\theta\right], \quad (7)$$

$$T'_{rz}=c_{44}\left[u'_{\theta,z'}+u'_{z,r}\right]+e_{15}V'_{,z'}-e_{14}r^{-1}V'_{,\theta}, \quad (8)$$

The boundary conditions (5) correspond to the situation where the inner and the outer surface of the cylinder are free of traction and coated with electrodes which are shorted.

3. PROBLEM SOLUTION

We introduce the following dimensionless variables

$$x=\frac{r}{R}, \qquad z=\frac{z'}{R}, \qquad u_x=\frac{1}{R}u'_r, \qquad u_\theta=\frac{1}{R}u'_\theta, \qquad u_z=\frac{1}{R}u'_z$$

$$V=\frac{e_{33}}{Rc_{44}}V' \qquad \tilde{c}_{ij}=\frac{c_{ij}}{c_{44}}, \qquad \tilde{e}_{ij}=\frac{e_{ij}}{e_{33}}, \qquad \varepsilon_{i3}^2=\frac{e_{33}^2}{c_{44}\,\epsilon_{ii}}, \qquad t=\frac{1}{R}\sqrt{\frac{c_{44}}{\rho_s}}\ t',$$

where $R=r_1-r_0$.

To study the propagation of harmonic waves in the z' direction, we assume a solution of the form:

$$u_x=\left(G_{,x}+\frac{1}{x}\psi_{,\theta}\right)e^{i(\lambda z-\Omega t)}, \qquad u_\theta=\left(\frac{1}{x}G_{,\theta}-\psi_{,x}\right)e^{i(\lambda z-\Omega t)},$$

$$u_z=i\,w\,e^{i(\lambda z-\Omega t)}, \qquad V=i\,\phi\;e^{i(\lambda z-\Omega t)}, \tag{9}$$

where G, ψ, w and ϕ are functions of x and θ, $\Omega^2=\frac{(R\omega)^2\rho_s}{c_{44}}$, ω is the angular frequency, $\lambda=R\gamma$, and γ is the wave number.

Using (9) the system (1)-(4) can be simplified as follows:

$$\mathcal{D}\begin{bmatrix}G\\ \psi\\ w\\ \phi\end{bmatrix}\equiv\begin{bmatrix}\tilde{c}_{11}\nabla^2+\Omega^2-\lambda^2 & 0 & -\lambda(1+\tilde{c}_{13}) & -\lambda(\tilde{e}_{15}+\tilde{e}_{31})\\ 0 & \tilde{c}_{66}\nabla^2+\Omega^2-\lambda^2 & 0 & \lambda\tilde{e}_{14}\\ \lambda(1+\tilde{c}_{13})\nabla^2 & 0 & \nabla^2+\Omega^2-\tilde{c}_{33}\lambda^2 & \tilde{e}_{15}\nabla^2-\lambda^2\\ -\lambda(\tilde{e}_{15}+\tilde{e}_{31})\nabla^2 & \lambda\tilde{e}_{14}\nabla^2 & -(\tilde{e}_{15}\nabla^2-\lambda^2) & \varepsilon_{13}^{-2}\nabla^2-\varepsilon_{33}^{-2}\lambda^2\end{bmatrix}\begin{bmatrix}G\\ \psi\\ w\\ \phi\end{bmatrix}=\boldsymbol{0},$$

(10)

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial \theta^2}.$$

Following Ref. [8], we introduce a function $F = F(x,\theta)$ such that $\{\det \mathcal{D}\}F = 0$ or equivalently

$$\left[a\nabla^8 + b\nabla^6 + c\nabla^4 + d\nabla^2 + e\right]F = 0, \tag{11}$$

where the coefficients a, b, c, d and e are given in Appendix A. Calculating the algebraic components $\mathcal{D}_{pq}$ of the matrix $\mathcal{D}$ we can find four solutions of the system (10) given by

$$G = \mathcal{D}_{p1}F, \quad \psi = \mathcal{D}_{p2}F, \quad w = \mathcal{D}_{p3}F, \quad \phi = \mathcal{D}_{p4}F, \qquad p = 1,2,3,4 \tag{12}$$

where

$$\mathcal{D}_{pq} = d_1^{pq}\nabla^6 + d_2^{pq}\nabla^4 + d_3^{pq}\nabla^2 + d_4^{pq}, \tag{13}$$

and d_s^{pq}, $p,q,s = 1,2,3,4$ are given in Appendix B.

Using (12) the proposed solution (9) takes the following form:

$$u_x = \left[\frac{\partial}{\partial x}\left(\mathcal{D}_{p1}F\right) + \frac{1}{x}\frac{\partial}{\partial \theta}\left(\mathcal{D}_{p2}F\right)\right]e^{i(\lambda z - \Omega t)},$$

$$u_\theta = \left[\frac{1}{x} \frac{\partial}{\partial \theta} \left(\mathcal{D}_{p1} F \right) - \frac{\partial}{\partial x} \left(\mathcal{D}_{p2} F \right) \right] e^{i(\lambda z - \Omega t)},$$

$$u_z = i \left(\mathcal{D}_{p3} F \right) e^{i(\lambda z - \Omega t)}, \tag{14}$$

$$V = i \left(\mathcal{D}_{p4} F \right) e^{i(\lambda z - \Omega t)},$$

where the function F satisfies equation (11).

In order to solve equation (11) we recognize that it can be factored as

$$\left(\nabla^2 + k_1^2 \right) \left(\nabla^2 + k_2^2 \right) \left(\nabla^2 + k_3^2 \right) \left(\nabla^2 + k_4^2 \right) F = 0, \tag{15}$$

where the k_j^2 's, $j = 1,2,3,4$ are the roots of the equation:

$$a k^8 - b k^6 + c k^4 - d k^2 + e = 0. \tag{16}$$

There are four roots k_j^2 for equation (16). Since the coefficients of (16) are real, the roots k_j^2 are either real or occur in conjugate pairs. The case of the repeated roots require different consideration and is not considered here because for the properties of bone used this case does not appear.

In the case of distinguished roots, if $F_j(x, \theta)$, $j = 1,2,3,4$ are four functions which satisfy

$$\left(\nabla^2 + k_j^2\right)F_j = 0, \qquad j = 1,2,3,4, \tag{17}$$

respectively, then the function

$$F = \sum_{j=1}^{4} F_j \tag{18}$$

is a solution of equation (11) since the order of the operators in (15) can be interchanged. But for each j, equation (17) is a Helmholtz equation which admits a solution of the form:

$$F_j = \sum_{l=1}^{2} \left\{a_j^{m,l}\cos(m\theta) + \beta_j^{m,l}\sin(m\theta)\right\}\zeta^{m,l}\left(k_j x\right), \qquad m \in \mathbf{N} \tag{19}$$

where $a_j^{m,l}$ and $\beta_j^{m,l}$ are arbitrary constants,

$$\zeta^{m,l}\left(k_j x\right) = \begin{cases} \begin{cases} J^m\left(k_j x\right), & l = 1, \quad (Bessel\ of\ 1st\ kind) \\ Y^m\left(k_j x\right), & l = 2, \quad (Bessel\ of\ 2nd\ kind) \end{cases} & if\ k_j^2 > 0, \\ \begin{cases} I^m\left(k_j x\right), & l = 1, \quad (mod. Bessel\ of\ 1nd\ kind) \\ K^m\left(k_j x\right), & l = 2, \quad (mod. Bessel\ of\ 2nd\ kind) \end{cases} & if\ k_j^2 < 0, \end{cases}$$

and $k_j = \left|k_j^2\right|^{1/2}$ when k_j^2 is real and $\zeta^{m,l}(k_j x)$ is the Bessel function with complex argument when k_j^2 is complex.

Using (14), (18) and (19)we find that the elastic displacements and the electric potential can be written as

$$u_x = \sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}\delta_j^{p1}\frac{\partial}{\partial x}\zeta^{m,l}\left(k_j x\right)+\beta_j^{m,l}\delta_j^{p2}\frac{m}{x}\zeta^{m,l}\left(k_j x\right)\right]\cos(m\theta)\right.$$

$$\left.+\left[-a_j^{m,l}\delta_j^{p2}\frac{m}{x}\zeta^{m,l}\left(k_j x\right)+\beta_j^{m,l}\delta_j^{p1}\frac{\partial}{\partial x}\zeta^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{20}$$

$$u_\theta = \sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[-a_j^{m,l}\delta_j^{p2}\frac{\partial}{\partial x}\zeta^{m,l}\left(k_j x\right)+\beta_j^{m,l}\delta_j^{p1}\frac{m}{x}\zeta^{m,l}\left(k_j x\right)\right]\cos(m\theta)\right.$$

$$\left.-\left[a_j^{m,l}\delta_j^{p1}\frac{m}{x}\zeta^{m,l}\left(k_j x\right)+\beta_j^{m,l}\delta_j^{p2}\frac{\partial}{\partial x}\zeta^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{21}$$

$$u_z = i\sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}\delta_j^{p3}\zeta^{m,l}\left(k_j x\right)\right]\cos(m\theta)+\left[\beta_j^{m,l}\delta_j^{p3}\zeta^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{22}$$

$$V = i\sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}\delta_j^{p4}\zeta^{m,l}\left(k_j x\right)\right]\cos(m\theta)+\left[\beta_j^{m,l}\delta_j^{p4}\zeta^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{23}$$

where

$$\delta_j^{pq} = -d_1^{pq}k_j^6 + d_2^{pq}k_j^4 - d_3^{pq}k_j^2 + d_4^{pq}, \qquad p,q,j = 1,2,3,4.$$

The stresses given by the constitutive equations (6-8) are expressed as:

$$T_{xx} = \sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}P_{p,j}^{m,l}\left(k_j x\right)+\beta_j^{m,l}Q_{p,j}^{m,l}\left(k_j x\right)\right]\cos(m\theta)\right.$$

$$\left.+\left[-a_j^{m,l}Q_{p,j}^{m,l}\left(k_j x\right)+\beta_j^{m,l}P_{p,j}^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{24}$$

$$T_{x\theta} = \bar{c}_{66}\sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}R_{p,j}^{m,l}\left(k_j x\right)+\beta_j^{m,l}S_{p,j}^{m,l}\left(k_j x\right)\right]\cos(m\theta)\right.$$
$$\left.+\left[-a_j^{m,l}S_{p,j}^{m,l}\left(k_j x\right)+\beta_j^{m,l}R_{p,j}^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{25}$$

$$T_{xz} = i\sum_{j=1}^{4}\sum_{l=1}^{2}\left\{\left[a_j^{m,l}T_{p,j}^{m,l}\left(k_j x\right)+\beta_j^{m,l}U_{p,j}^{m,l}\left(k_j x\right)\right]\cos(m\theta)\right.$$
$$\left.+\left[-a_j^{m,l}U_{p,j}^{m,l}\left(k_j x\right)+\beta_j^{m,l}T_{p,j}^{m,l}\left(k_j x\right)\right]\sin(m\theta)\right\}e^{i(\lambda z-\Omega t)}, \tag{26}$$

where the quantities $P_{p,j}^{m,l}$, $Q_{p,j}^{m,l}$, $R_{p,j}^{m,l}$, $S_{p,j}^{m,l}$, $T_{p,j}^{m,l}$ and $U_{p,j}^{m,l}$ are given in Appendix C.

4. NUMERICAL SOLUTION

Replacing the expressions (23)-(26) into the boundary conditions (5) and using orthogonality arguments we infer that for every specific pair (m,p) we obtain an algebraic system with 16 unknowns, which can be written as

$$\boldsymbol{Ax} = \boldsymbol{0}, \tag{27}$$

where $\boldsymbol{x} = \left[\alpha_j^{m,1}, \beta_j^{m,1}, \alpha_j^{m,2}, \beta_j^{m,2}\right]^T$, $j = 1,2,3,4$.

In order for the system (27) to have a nontrivial solution, the determinant of matrix $\boldsymbol{A}$ must vanish, that is

$$\det\left(A_{rs}\right) = 0, \qquad r,s = 1,2,\ldots 16. \tag{28}$$

This condition provides the frequency equation, the roots of which are the eigenfrequency coefficients $\Omega_{m,p}(\lambda)$, $m = 1,2,\ldots;\ p = 1,2,3,4$ of the system under discussion. The elements of the matrix $\boldsymbol{A}$ are

$$A_{1,j} = P_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{1,j+4} = P_{p,j}^{m,2}\left(k_j\tilde{r}_0\right), \quad A_{1,j+8} = Q_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{1,j+12} = Q_{p,j}^{m,2}\left(k_j\tilde{r}_0\right),$$

$$A_{2,j} = -A_{1,j+8}, \quad A_{2,j+4} = -A_{1,j+12}, \quad A_{2,j+8} = A_{1,j}, \quad A_{2,j+12} = A_{1,j+4},$$

$$A_{3,j} = R_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{3,j+4} = R_{p,j}^{m,2}\left(k_j\tilde{r}_0\right), \quad A_{3,j+8} = S_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{3,j+12} = S_{p,j}^{m,2}\left(k_j\tilde{r}_0\right),$$

$$A_{4,j} = -A_{3,j+8}, \quad A_{4,j+4} = -A_{3,j+12}, \quad A_{4,j+8} = A_{3,j}, \quad A_{4,j+12} = A_{3,j+4},$$

$$A_{5,j} = T_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{5,j+4} = T_{p,j}^{m,2}\left(k_j\tilde{r}_0\right), \quad A_{5,j+8} = U_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{5,j+12} = U_{p,j}^{m,2}\left(k_j\tilde{r}_0\right),$$

$$A_{6,j} = -A_{5,j+8}, \quad A_{6,j+4} = -A_{5,j+12}, \quad A_{6,j+8} = A_{5,j}, \quad A_{6,j+12} = A_{5,j+4},$$

$$A_{7,j} = W_{p,j}^{m,1}\left(k_j\tilde{r}_0\right), \quad A_{7,j+4} = W_{p,j}^{m,2}\left(k_j\tilde{r}_0\right), \quad A_{7,j+8} = 0, \quad A_{7,j+12} = 0,$$

$$A_{8,j} = 0, \quad A_{8,j+4} = 0, \quad A_{8,j+8} = A_{7,j}, \quad A_{8,j+12} = A_{7,j+4},$$

for $j=1,2,3,4$. The remaining eight rows can be obtained from the above relations by replacing $\tilde{r}_0=\frac{r_0}{R}$ by $\tilde{r}_1=\frac{r_1}{R}$.

As we have mentioned previously the roots of the equation (16) are real, occur in complex conjugate pairs or both. In order to simplify our numerical calculations we consider the following cases:

1st case: All the roots k_j^2 are real.

In this case all terms A_{pq} are real and the frequency equation is given by equation (28).

2nd case: $k_2^2=\overline{k_1^2}$, $k_4^2=\overline{k_3^2}$, where the bar denotes the complex conjugate.

In this case the terms A_{pq} satisfy the relations:

$A_{rt}=\overline{A_{r,t-1}}, \qquad r=1,2,\ldots,16, \quad t=2,4,\ldots,16.$

Using elementary determinant properties the frequency equation (28) can be simplified as follows:

$$\det(B_{rs})=0, \qquad r,s=1,2,\ldots,16 \tag{29}$$

where

$$\begin{aligned} B_{rt}&=\mathrm{Re}[A_{rt}], \\ B_{r,t+1}&=\mathrm{Im}[A_{rt}], \end{aligned} \qquad t=1,3,5,\ldots,15 \quad r=1,2,\ldots,16$$

3rd case: k_2^2, k_1^2 are real and $k_4^2=\overline{k_3^2}$.

In this case the frequency equation (28) becomes:

$$\det(C_{rs})=0, \qquad r,s=1,2,\ldots,16, \tag{31}$$

where

$$C_{rs} = \mathrm{Re}[A_{rs}], \qquad s \neq 4,\ 8,\ 12,\ 16,$$
$$C_{rt} = \mathrm{Im}[A_{r,t-1}], \qquad t = 4,\ 8,\ 12,\ 16.$$

For the system under discussion the stiffness matrix at constant electric fields is

$$\boldsymbol{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

with $c_{66} = \frac{1}{2}(c_{11} - c_{12})$.

The piezoelectric stress matrix is

$$\boldsymbol{e} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & -e_{14} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}.$$

The dielectric matrix at constant strain for hexagonal crystal is

$$\epsilon = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}.$$

The values used in our computations for the above quantities are given in Table 1.

Table 1: Material Coefficients in S.I. Units.

Elastic Coefficients [9] $\left(N/m^2\right)$	Piezoelectric Coefficients [4] $\left(C/m^2\right)$	Dielectric Coefficients [10] (F/m)
$c_{11} = 2.12\times10^{10}$	$e_{31} = 1.50765\times10^{-3}$	$\in_{11} = 88.54\times10^{-12}$
$c_{12} = 0.95\times10^{10}$	$e_{33} = 1.87209\times10^{-3}$	$\in_{33} = 106.248\times10^{-12}$
$c_{13} = 1.02\times10^{10}$	$e_{14} = 17.88215\times10^{-3}$	
$c_{33} = 3.76\times10^{10}$	$e_{15} = 3.57643\times10^{-3}$	
$c_{44} = 0.75\times10^{10}$		

The frequency equation is solved numerically and for this purpose a matrix determinant computation routine was used for different Ω and λ, along with a root finding method to refine steps close to its roots. For each pair (Ω,λ) the equation (16) is solved first using Laguerre's method. The roots obtained indicate the case which must be treated in order to obtain the roots of the frequency equation.

Results for the non-piezoelectric and isotropic cylinder can be obtained following the procedure shown in Appendix D.

The frequency spectra obtained for the case of isotropic bone are shown in Table 2 for various wavelengths. It is obvious that the first frequency decreases as wavelength increases.

Table 2: Frequency Spectra for Isotropic Hollow Bone as a Function of Wavelength.

No.	$\alpha = 0.01m$	$\alpha = 0.03m$	$\alpha = 0.05m$	$\alpha = 0.08m$	$\alpha = 0.10m$	$\alpha = 0.15m$	$\alpha = 0.20m$
1	3.7700	0.9094	0.5174	0.2918	0.2162	0.1190	0.0746
2		0.9290	0.5586	0.4022	0.3194	0.2514	0.1574
3		1.0618	0.6540	0.4104	0.3770	0.2522	0.1886
4		1.1034	0.6796	0.4540	0.3780	0.2552	0.1894
5		1.1150	0.7534	0.4664	0.5864	0.2888	0.1912
6		1.1728	0.7540	0.4714	0.6206	0.4472	0.2544
7		1.2566	0.7554	0.4722	0.6358	0.4480	0.2964

The same behavior is observed for the case of non-piezoelectric bone as it is shown in Table 3 for the first seven eigenfrequencies.

Table 3: Frequency Spectra for non-piezoelectric Hollow Bone as a Function of Wavelength.

No.	$\alpha = 0.01m$	$\alpha = 0.03m$	$\alpha = 0.05m$	$\alpha = 0.08m$	$\alpha = 0.10m$	$\alpha = 0.15m$	$\alpha = 0.20m$
1	3.7016	0.8008	0.6454	0.3562	0.2606	0.1362	0.0764
2	3.7694	0.9248	0.6820	0.4096	0.3216	0.2084	0.1542
3	3.7700	0.9726	0.7540	0.4714	0.3772	0.2514	0.1886
4		1.0750	0.7836	0.5930	0.5344	0.3674	0.2778
5		1.1606	0.8204	0.6402	0.6818	0.4526	0.3678
6		1.1806	0.9798	0.6990	0.7008	0.4770	0.4220
7		1.2564	0.9882	0.7656	0.7490	0.5630	0.5538

The results for a Crystal class 6 bone are shown in Table 4. It is observed that the same behaviour occurs for the variation of the first eigenfrequency with respect to the wavelength. However, the computed eigenfrequencies are smaller than the observed in the case of non-piezoelectric cylinder. The comparison of the first eigenfrequency obtained for each of those cases is shown in Figure 2.

Table 4: Frequency Spectra for Crystal Class 6 Hollow Bone as a Function of Wavelength.

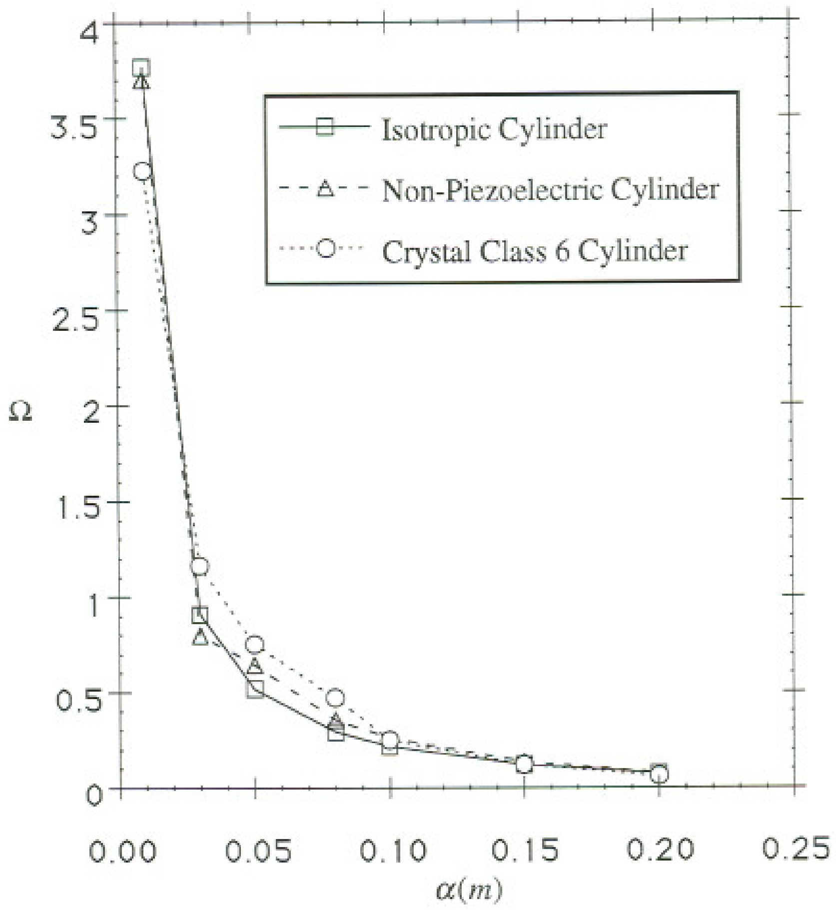
No.	$\alpha = 0.01m$	$\alpha = 0.03m$	$\alpha = 0.05m$	$\alpha = 0.08m$	$\alpha = 0.10m$	$\alpha = 0.15m$	$\alpha = 0.20m$
1	3.2260	1.1640	0.7540	0.4712	0.2502	0.1208	0.0584
2	3.7700	1.2566	1.0472	0.6544	0.3770	0.2514	0.1886
3	5.0766	1.5278	1.1688	0.7308	0.5236	0.3490	0.2618
4		1.7452	1.3300	0.7372	0.5896	0.3898	0.2924
5			1.6752	0.8312	0.6650	0.3930	0.2948
6			1.8150	0.9730	0.8376	0.4434	0.3326
7				1.0552		0.5585	0.4188

The eigenfrequecies of the system decrease with increasing ratio of inner to outer radius as it is shown in Table 5. This behavior has been observed to appear in the same manner in the treatment of the isotropic hollow cylinder of finite length in Ref. [11].

Table 5: Frequency Spectra for Crystal Class 6 Hollow Bone as a Function of the Ratio r_0/r_1 for $\alpha = 0.10m$.

No.	*Solid Cylinder*	$r_0/r_1 = 0.21$	$r_0/r_1 = 0.43$	$r_0/r_1 = 0.57$	$r_0/r_1 = 0.71$	$r_0/r_1 = 0.86$
1	0.4736	0.3906	0.3130	0.2502	0.1760	0.0920
2	0.8790	0.6912	0.5026	0.3770	0.2514	0.1256
3	1.2208	0.9598	0.6982	0.5236	0.3490	0.1746
4	1.3644	1.0810	0.7796	0.5896	0.3898	0.1950
5	1.3750	1.2192	0.8866	0.6650	0.3930	0.2216
6	1.5506	1.5356	1.1168	0.8376	0.4434	0.2676
7	1.9530	1.8532	1.8448		0.5584	0.2792

Figure 2: Comparison of the First Eigenfrequency for the Different Models of Hollow Cylinder (First Flexural Mode).



5. CONCLUDING REMARKS

In this work, we have studied the wave propagation in an infinite piezoelectric hollow cylinder of crystal class 6. We adopted the analysis of Ref. 8 and the solution of the problem was expressed in terms of a potential function. The resulting dispersion relation has been solved numerically. We considered also two simpler cases, the case of the isotropic cylinder and the non-piezoelectric cylinder which are simplifications of the general problem but require special treatment. The analysis focused on the eigenfrequencies of the system and we have shown that the first eigenfrequency of the system varies in a similar qualitative manner for the three models. This can be further used to obtain qualitative characteristics for cases which are difficult to be modeled (e.g. osteogenesis, bone healing, etc.). The obtained results compare well with analogous of other researchers as it is shown in Figures 3 and 4 for the cases of non-piezoelectric and crystal class 6 cylinder, respectively.

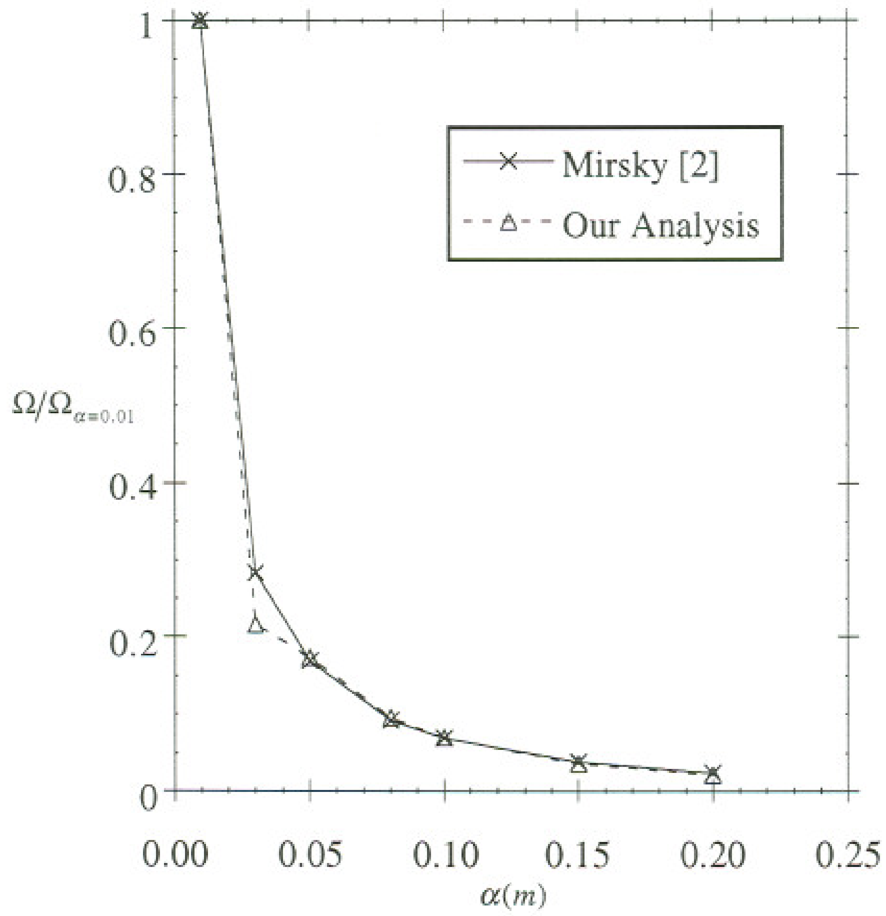


Figure 3: Comparison of normalized eigenfrequencies of our analysis for the non-piezoelectric hollow cylinder with the results obtained by Mirsky [2].

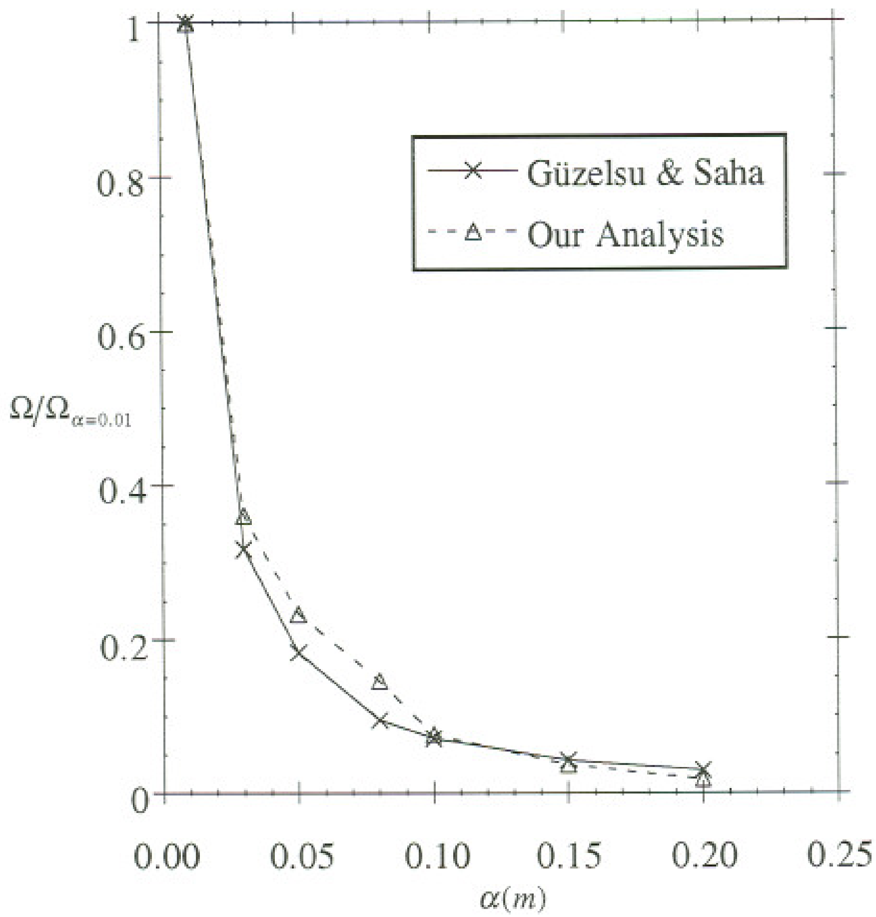


Figure 4: *Comparison of normalized eigenfrequencies of our analysis for crystal class 6-hollow cylinder with the results obtained by Güzelsu and Saha [5].*

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APPENDIX A

$$a=\tilde{c}_{11}\tilde{c}_{66}\left(\tilde{e}_{15}^{2}+\varepsilon_{13}^{-2}\right),$$

$$b=-\left\{\begin{array}{l}\tilde{c}_{66}\left[\tilde{e}_{31}^{2}-\tilde{e}_{15}^{2}-2\tilde{c}_{13}\tilde{e}_{15}(\tilde{e}_{15}+\tilde{e}_{31})-\varepsilon_{13}^{-2}(1+\tilde{c}_{13})^{2}\right]\\+\tilde{c}_{11}\tilde{e}_{14}^{2}+\tilde{c}_{11}\tilde{c}_{66}\left[2\tilde{e}_{15}+\tilde{c}_{33}\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}\right]+(\tilde{c}_{11}+\tilde{c}_{66})\left(\tilde{e}_{15}^{2}+\varepsilon_{13}^{-2}\right)\end{array}\right\}\lambda^{2}$$
$$+\left\{(\tilde{c}_{11}+\tilde{c}_{66})\left(\tilde{e}_{15}^{2}+\varepsilon_{13}^{-2}\right)+\tilde{c}_{11}\tilde{c}_{66}\varepsilon_{13}^{-2}\right\}\Omega^{2}$$

$$c=\left\{(\tilde{e}_{15}+\tilde{e}_{31})^{2}(1+\tilde{c}_{33}\tilde{c}_{66})-(1+\tilde{c}_{13})^{2}\left(\tilde{e}_{14}^{2}+\varepsilon_{13}^{-2}+\tilde{c}_{66}\varepsilon_{33}^{-2}\right)\right.$$
$$-2(\tilde{e}_{15}+\tilde{e}_{31})(1+\tilde{c}_{13})(\tilde{c}_{66}+\tilde{e}_{15})+\tilde{e}_{14}^{2}(1+\tilde{c}_{11}\tilde{c}_{33})$$
$$\left.+(\tilde{c}_{11}+\tilde{c}_{66})\left(\tilde{c}_{33}\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}+2\tilde{e}_{15}\right)+\tilde{e}_{15}^{2}+\varepsilon_{13}^{-2}+\tilde{c}_{11}\tilde{c}_{66}\left(1+\tilde{c}_{33}\varepsilon_{33}^{-2}\right)\right\}\lambda^{4}$$
$$-\left\{\begin{array}{l}(\tilde{e}_{15}+\tilde{e}_{31})^{2}(1+\tilde{c}_{66})+\tilde{e}_{14}^{2}(1+\tilde{c}_{11})+(\tilde{c}_{11}+\tilde{c}_{66})\left(\tilde{c}_{33}\varepsilon_{13}^{-2}+\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}+2\tilde{e}_{15}\right)\\-(1+\tilde{c}_{13})\left[2(\tilde{e}_{15}+\tilde{e}_{31})\tilde{e}_{15}+(1+\tilde{c}_{13})\varepsilon_{13}^{-2}\right]+2\tilde{e}_{15}^{2}+2\varepsilon_{13}^{-2}+\tilde{c}_{11}\tilde{c}_{66}\varepsilon_{33}^{-2}\end{array}\right\}\lambda^{2}\Omega^{2}$$
$$+\left\{\tilde{e}_{15}^{2}+\varepsilon_{13}^{-2}(1+\tilde{c}_{11}+\tilde{c}_{66})\right\}\Omega^{4}$$

$$d=\left(\Omega^{2}-\lambda^{2}\right)\left\{\left[-(\tilde{e}_{15}+\tilde{e}_{31})\left[2(1+\tilde{c}_{13})-(\tilde{e}_{15}+\tilde{e}_{31})\tilde{c}_{33}\right]+(\tilde{c}_{11}+\tilde{c}_{66})\left(1+\tilde{c}_{33}\varepsilon_{33}^{-2}\right)\right.\right.$$
$$\left.+\tilde{c}_{33}\left(\tilde{e}_{14}^{2}+\varepsilon_{13}^{-2}\right)+2\tilde{e}_{15}-\tilde{c}_{13}\varepsilon_{33}^{-2}(2+\tilde{c}_{13})\right]\lambda^{4}$$
$$\left.-\left[(\tilde{e}_{15}+\tilde{e}_{31})^{2}+\tilde{e}_{14}^{2}+2\tilde{e}_{15}+\varepsilon_{13}^{-2}(1+\tilde{c}_{33})+\varepsilon_{33}^{-2}(1+\tilde{c}_{11})+\tilde{c}_{66}\varepsilon_{33}^{-2}\right]\lambda^{2}\Omega^{2}+\varepsilon_{13}^{-2}\Omega^{4}\right\},$$

$$e=\left\{1+\tilde{c}_{33}\varepsilon_{33}^{-2}\right\}\lambda^{8}-\left\{2+\varepsilon_{33}^{-2}(1+2\tilde{c}_{33})\right\}\lambda^{6}\Omega^{2}+\left\{1+\varepsilon_{33}^{-2}(2+\tilde{c}_{33})\right\}\lambda^{4}\Omega^{4}-\varepsilon_{33}^{-2}\lambda^{2}\Omega^{6}.$$

APPENDIX B

$$d_1^{11} = \tilde{c}_{66}\left(\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\right),$$

$$d_2^{11} = -\left\{\tilde{e}_{14}^2 + \tilde{e}_{15}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\left[2\tilde{e}_{15} + \tilde{c}_{33}\varepsilon_{13}^{-2} + \varepsilon_{33}^{-2}\right]\right\}\lambda^2 + \left\{\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\left[1 + \tilde{c}_{66}\right]\right\}\Omega^2,$$

$$\begin{aligned} d_3^{11} = &\left\{\tilde{c}_{66} + 2\tilde{e}_{15} + \varepsilon_{33}^{-2} + \tilde{c}_{33}\left(\tilde{e}_{14}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\varepsilon_{33}^{-2}\right)\right\}\lambda^4 \\ &-\left\{\left(\tilde{e}_{14}^2 + 2\tilde{e}_{15}\right) + \varepsilon_{13}^{-2}\left(1 + \tilde{c}_{33}\right) + \varepsilon_{33}^{-2}\left(1 + \tilde{c}_{66}\right)\right\}\lambda^2\Omega^2 + \varepsilon_{13}^{-2}\Omega^4, \end{aligned}$$

$$d_4^{11} = -\left\{1 + \tilde{c}_{33}\varepsilon_{33}^{-2}\right\}\lambda^6 + \left\{1 + \varepsilon_{33}^{-2}\left(1 + \tilde{c}_{33}\right)\right\}\lambda^4\Omega^2 - \varepsilon_{33}^{-2}\lambda^2\Omega^4,$$

$$d_1^{12} = 0,$$

$$d_2^{12} = -\tilde{e}_{14}\left\{\tilde{e}_{31} - \tilde{c}_{13}\tilde{e}_{15}\right\}\lambda^2,$$

$$d_3^{12} = -\tilde{e}_{14}\left\{\left[\left(1 + \tilde{c}_{13}\right) - \tilde{c}_{33}\left(\tilde{e}_{15} + \tilde{e}_{31}\right)\right]\lambda^4 + \left(\tilde{e}_{15} + \tilde{e}_{31}\right)\lambda^2\Omega^2\right\},$$

$$d_4^{12} = 0,$$

$$d_1^{13} = -\tilde{c}_{66}\left\{\tilde{e}_{15}\left(\tilde{e}_{15} + \tilde{e}_{31}\right) + \varepsilon_{13}^{-2}\left(1 + \tilde{c}_{13}\right)\right\}\lambda,$$

$$d_2^{13} = \left\{(1+\tilde{c}_{13})\left(\tilde{e}_{14}^2 + \varepsilon_{13}^{-2} + \tilde{c}_{66}\varepsilon_{33}^{-2}\right) + (\tilde{c}_{66} + \tilde{e}_{15})(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^3$$
$$-\left\{\tilde{e}_{15}(\tilde{e}_{15} + \tilde{e}_{31}) + \varepsilon_{13}^{-2}(1+\tilde{c}_{13})\right\}\lambda\Omega^2,$$

$$d_3^{13} = \left\{\varepsilon_{33}^{-2}(1+\tilde{c}_{13}) + (\tilde{e}_{15} + \tilde{e}_{31})\right\}\left(\Omega^2 - \lambda^2\right)\lambda^3,$$

$$d_4^{13} = 0,$$

$$d_1^{14} = -\tilde{c}_{66}(\tilde{c}_{13}\tilde{e}_{15} - \tilde{e}_{31})\lambda,$$

$$d_2^{14} = \left\{\tilde{c}_{66}(1+\tilde{c}_{13}) + (\tilde{c}_{13}\tilde{e}_{15} - \tilde{e}_{31}) - \tilde{c}_{66}\tilde{c}_{33}(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^3$$
$$-\left\{(\tilde{c}_{13}\tilde{e}_{15} - \tilde{e}_{31}) - \tilde{c}_{66}(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda\Omega^2,$$

$$d_3^{14} = -\left\{(1+\tilde{c}_{13}) - \tilde{c}_{33}(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^5$$
$$-\left\{-(1+\tilde{c}_{13}) + (1+\tilde{c}_{33})(\tilde{e}_{15} + \tilde{e}_{31})\right\}\lambda^3\Omega^2 + (\tilde{e}_{15} + \tilde{e}_{31})\lambda\Omega^4,$$

$$d_4^{14} = 0,$$

$$d_1^{21} = 0,$$

$$d_2^{21} = -\tilde{e}_{14}(\tilde{e}_{31} - \tilde{c}_{13}\tilde{e}_{15})\lambda^2,$$

$$d_3^{21} = -\tilde{e}_{14}\left\{\left[(1+\tilde{c}_{13}) - \tilde{c}_{33}(\tilde{e}_{15} + \tilde{e}_{31})\right]\lambda^4 + (\tilde{e}_{15} + \tilde{e}_{31})\lambda^2\Omega^2\right\},$$

$$d_4^{21} = 0,$$

$$d_1^{22} = \tilde{c}_{11}\left(\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}\right),$$

$$d_2^{22} = \Big\{-(\tilde{e}_{15}+\tilde{e}_{31})^2 + 2(1+\tilde{c}_{13})(\tilde{e}_{15}+\tilde{e}_{31})\overline{e}_{15} - \tilde{e}_{15}(2\tilde{c}_{11}+\tilde{e}_{15}) + \varepsilon_{13}^{-2}\tilde{c}_{13}(2+\tilde{c}_{13}) - \tilde{c}_{11}\left(\tilde{c}_{33}\varepsilon_{13}^{-2}+\varepsilon_{33}^{-2}\right)\Big\}\lambda^2 + \left\{\tilde{e}_{15}^2 + \varepsilon_{13}^{-2}(1+\tilde{c}_{11})\right\}\Omega^2,$$

$$d_3^{22} = \begin{Bmatrix} -2(1+\tilde{c}_{13})(\tilde{e}_{15}+\tilde{e}_{31}) + \tilde{c}_{11}(1+\tilde{c}_{33}\varepsilon_{33}^{-2}) + \tilde{c}_{33}(\tilde{e}_{15}+\tilde{e}_{31})^2 + \\ 2\tilde{e}_{15} + \tilde{c}_{33}\varepsilon_{13}^{-2} - \tilde{c}_{13}\varepsilon_{33}^{-2}(2+\tilde{c}_{13}) \end{Bmatrix}\lambda^4 + \left\{-(\tilde{e}_{15}+\tilde{e}_{31})^2 - 2\tilde{e}_{15} - \varepsilon_{13}^{-2}(1+\tilde{c}_{33}) - \varepsilon_{33}^{-2}(1+\tilde{c}_{11})\right\}\lambda^2\Omega^2 + \varepsilon_{13}^{-2}\Omega^4,$$

$$d_4^{22} = -\left\{1+\tilde{c}_{33}\varepsilon_{33}^{-2}\right\}\lambda^6 + \left\{1+(1+\tilde{c}_{33})\varepsilon_{33}^{-2}\right\}\lambda^4\Omega^2 - \varepsilon_{33}^{-2}\lambda^2\Omega^4$$

$$d_1^{23} = \tilde{c}_{11}\tilde{e}_{14}\tilde{e}_{15}\lambda,$$

$$d_2^{23} = -\tilde{e}_{14}\left\{\left[\tilde{c}_{11} - \tilde{e}_{31} - \tilde{c}_{13}(\tilde{e}_{15}+\tilde{e}_{31})\right]\lambda^3 - \tilde{e}_{15}\lambda\Omega^2\right\},$$

$$d_3^{23} = -\tilde{e}_{14}\left\{-\lambda^5 + \lambda^3\Omega^2\right\},$$

$$d_4^{23} = 0,$$

$$d_1^{24} = -\tilde{e}_{14}\tilde{c}_{11}\lambda,$$

$$d_2^{24} = \tilde{e}_{14}\left\{\left[-\tilde{c}_{13}(2+\tilde{c}_{13}) + \tilde{c}_{11}\tilde{c}_{33}\right]\lambda^3 - (1+\tilde{c}_{11})\lambda\Omega^2\right\},$$

$$d_3^{24} = \tilde{e}_{14}\left\{-\tilde{c}_{33}\lambda^5 + (1+\tilde{c}_{33})\lambda^3\Omega^2 - \lambda\Omega^4\right\},$$

$$d_4^{24} = 0,$$

$$d_1^{31} = 0,$$

$$d_2^{31} = \tilde{c}_{66}\left\{\tilde{e}_{15}(\tilde{e}_{15}+\tilde{e}_{31}) + \varepsilon_{13}^{-2}(1+\tilde{c}_{13})\right\}\lambda,$$

$$d_3^{31} = \left\{-\tilde{e}_{14}^2(1+\tilde{c}_{13}) - (\tilde{e}_{15}+\tilde{e}_{31})(\tilde{c}_{66}+\tilde{e}_{15}) - (1+\tilde{c}_{13})\left(\varepsilon_{13}^{-2} + \tilde{c}_{66}\varepsilon_{33}^{-2}\right)\right\}\lambda^3$$
$$+\left\{\tilde{e}_{15}(\tilde{e}_{15}+\tilde{e}_{31}) + \varepsilon_{13}^{-2}(1+\tilde{c}_{13})\right\}\lambda\Omega^2,$$

$$d_4^{31} = \left\{(\tilde{e}_{15}+\tilde{e}_{31}) + \varepsilon_{33}^{-2}(1+\tilde{c}_{13})\right\}\left(\lambda^5 - \lambda^3\Omega^2\right),$$

$$d_1^{32} = 0,$$

$$d_2^{32} = -\tilde{e}_{14}\tilde{c}_{11}\tilde{e}_{15}\lambda,$$

$$d_3^{32} = -\tilde{e}_{14}\left[\left\{-\tilde{c}_{11} + \tilde{c}_{13}(\tilde{e}_{15}+\tilde{e}_{31}) + \tilde{e}_{31}\right\}\lambda^3 + \tilde{e}_{15}\lambda\Omega^2\right],$$

$$d_4^{32} = -\tilde{e}_{14}\left\{\lambda^5 - \lambda^3\Omega^2\right\},$$

$$d_1^{33} = \tilde{c}_{11}\tilde{c}_{66}\varepsilon_{13}^{-2},$$

$$d_2^{33} = -\left\{\tilde{c}_{11}\tilde{e}_{14}^2 + \tilde{c}_{66}(\tilde{e}_{15} + \tilde{e}_{31})^2 + \varepsilon_{13}^{-2}(\tilde{c}_{11} + \tilde{c}_{66}) + \tilde{c}_{11}\tilde{c}_{66}\varepsilon_{33}^{-2}\right\}\lambda^2 + \varepsilon_{13}^{-2}(\tilde{c}_{11} + \tilde{c}_{66})\Omega^2,$$

$$d_3^{33} = \left\{\tilde{e}_{14}^2 + \left(\tilde{e}_{15} + \tilde{e}_{31}\right)^2 + \varepsilon_{13}^{-2} + \varepsilon_{33}^{-2}\left(\tilde{c}_{11} + \tilde{c}_{66}\right)\right\}\lambda^4 - \left\{\tilde{e}_{14}^2 + \left(\tilde{e}_{15} + \tilde{e}_{31}\right)^2 + 2\varepsilon_{13}^{-2} + \varepsilon_{33}^{-2}\left(\tilde{c}_{11} + \tilde{c}_{66}\right)\right\}\lambda^2\Omega^2 + \varepsilon_{13}^{-2}\Omega^4,$$

$$d_4^{33} = -\varepsilon_{33}^{-2}\left\{\lambda^6 - 2\lambda^4\Omega^2 + \lambda^2\Omega^4\right\},$$

$$d_1^{34} = \tilde{c}_{11}\tilde{c}_{66}\tilde{e}_{15},$$

$$d_2^{34} = -\left\{\tilde{c}_{11}\tilde{c}_{66} + \tilde{c}_{11}\tilde{e}_{15} - \tilde{c}_{66}\tilde{e}_{31} - \tilde{c}_{13}\tilde{c}_{66}\left(\tilde{e}_{15} + \tilde{e}_{31}\right)\right\}\lambda^2 + \tilde{e}_{15}\left(\tilde{c}_{11} + \tilde{c}_{66}\right)\Omega^2,$$

$$d_3^{34} = -\left\{-\tilde{c}_{11} - \tilde{c}_{66} + \tilde{c}_{13}\tilde{e}_{15} + \tilde{e}_{31}\left(1 + \tilde{c}_{13}\right)\right\}\lambda^4 - \left\{\tilde{c}_{11} + \tilde{c}_{66} + \tilde{e}_{15}\left(1 - \tilde{c}_{13}\right) - \tilde{e}_{31}\left(1 + \tilde{c}_{13}\right)\right\}\lambda^2\Omega^2 + \tilde{e}_{15}\Omega^4,$$

$$d_4^{34} = -\left\{\lambda^6 - 2\lambda^4\Omega^2 + \lambda^2\Omega^4\right\},$$

$$d_1^{41} = 0,$$

$$d_2^{41} = -\tilde{c}_{66}\left\{\tilde{c}_{13}\tilde{e}_{15} - \tilde{e}_{31}\right\}\lambda,$$

$$d_3^{41} = -\left\{-\tilde{c}_{66}(1+\tilde{c}_{13}) - \tilde{e}_{15}(\tilde{c}_{13} - \tilde{c}_{33}\tilde{c}_{66}) + \tilde{e}_{31}(1+\tilde{c}_{33}\tilde{c}_{66})\right\}\lambda^3$$
$$-\left\{\tilde{e}_{15}(\tilde{c}_{13} - \tilde{c}_{66}) - \tilde{e}_{31}(1+\tilde{c}_{66})\right\}\lambda\Omega^2,$$

$$d_4^{41} = -\left\{(1+\tilde{c}_{13}) - \tilde{c}_{33}(\tilde{e}_{15}+\tilde{e}_{31})\right\}\lambda^5 + \left\{(1+\tilde{c}_{13}) - (\tilde{e}_{15}+\tilde{e}_{31})(1+\tilde{c}_{33})\right\}\lambda^3\Omega^2 + (\tilde{e}_{15}+\tilde{e}_{31})\lambda\Omega^4,$$

$$d_1^{42} = 0,$$

$$d_2^{42} = -\tilde{e}_{14}\tilde{c}_{11}\lambda,$$

$$d_3^{42} = -\tilde{e}_{14}\left\{\left[\tilde{c}_{13}(2+\tilde{c}_{13}) - \tilde{c}_{11}\tilde{c}_{33}\right]\lambda^3 + (1+\tilde{c}_{11})\lambda\Omega^2\right\},$$

$$d_4^{42} = \tilde{e}_{14}\left\{-\tilde{c}_{33}\lambda^5 + (1+\tilde{c}_{33})\lambda^3\Omega^2 - \lambda\Omega^4\right\},$$

$$d_1^{43} = -\tilde{c}_{11}\tilde{c}_{66}\tilde{e}_{15},$$

$$d_2^{43} = \left\{\tilde{c}_{66}\left[\tilde{e}_{15} + \tilde{c}_{11} - (1+\tilde{c}_{13})(\tilde{e}_{15}+\tilde{e}_{31})\right] + \tilde{c}_{11}\tilde{e}_{15}\right\}\lambda^2 - \tilde{e}_{15}(\tilde{c}_{11}+\tilde{c}_{66})\Omega^2,$$

$$d_3^{43} = -\left\{\tilde{c}_{66} + \tilde{e}_{15} + \tilde{c}_{11} - (1+\tilde{c}_{13})(\tilde{e}_{15}+\tilde{e}_{31})\right\}\lambda^4$$
$$+\left\{\tilde{c}_{66} + 2\tilde{e}_{15} + \tilde{c}_{11} - (1+\tilde{c}_{13})(\tilde{e}_{15}+\tilde{e}_{31})\right\}\lambda^2\Omega^2 - \tilde{e}_{15}\Omega^4,$$

$$d_4^{43} = \lambda^6 - 2\lambda^4\Omega^2 + \Omega^4\lambda^2,$$

$$d_1^{44} = \tilde{c}_{11}\tilde{c}_{66},$$

$$d_2^{44} = \left\{-\tilde{c}_{11}\left(1+\tilde{c}_{33}\tilde{c}_{66}\right)+\tilde{c}_{13}\tilde{c}_{66}\left(2+\tilde{c}_{13}\right)\right\}\lambda^2 + \left\{\tilde{c}_{11}+\tilde{c}_{66}+\tilde{c}_{11}\tilde{c}_{66}\right\}\Omega^2,$$

$$d_3^{44} = \left\{-\tilde{c}_{13}\left(2+\tilde{c}_{13}\right)+\tilde{c}_{33}\left(\tilde{c}_{66}+\tilde{c}_{11}\right)\right\}\lambda^4 + \left\{-1+\tilde{c}_{13}\left(2+\tilde{c}_{13}\right)-\left(1+\tilde{c}_{33}\right)\left(\tilde{c}_{66}+\tilde{c}_{11}\right)\right\}\Omega^2\lambda^2 + \left(1+\tilde{c}_{66}+\tilde{c}_{11}\right)\Omega^4,$$

$$d_4^{44} = -\tilde{c}_{33}\lambda^6 + (1+2\tilde{c}_{33})\Omega^2\lambda^4 - (2+\tilde{c}_{33})\Omega^4\lambda^2 + \Omega^6$$

APPENDIX C

$$P_{p,j}^{m,l}\left(k_j x\right) = \delta_j^{p1}\left\{\tilde{c}_{11}\frac{\partial^2 \zeta^{m,l}\left(k_j x\right)}{\partial x^2} + \tilde{c}_{12}\left(\frac{1}{x}\frac{\partial \zeta^{m,l}\left(k_j x\right)}{\partial x} - \frac{m^2}{x^2}\zeta^{m,l}\left(k_j x\right)\right)\right\}$$

$$-\lambda\left[\tilde{c}_{13}\delta_j^{p3} + \tilde{e}_{31}\delta_j^{p4}\right]\zeta^{m,l}\left(k_j x\right)$$

$$Q_{p,j}^{m,l}\left(k_j x\right) = 2\tilde{c}_{66}\delta_j^{p2}\frac{m}{x}\left[\frac{\partial \zeta^{m,l}\left(k_j x\right)}{\partial x} - \frac{1}{x}\zeta^{m,l}\left(k_j x\right)\right],$$

$$R_{p,j}^{m,l}(k_j x) = -\delta_j^{p2}\left[\frac{\partial^2 \zeta^{m,l}(k_j x)}{\partial x^2} - \frac{1}{x}\frac{\partial \zeta^{m,l}(k_j x)}{\partial x} + \frac{m^2}{x^2}\zeta^{m,l}(k_j x)\right],$$

$$S_{p,j}^{m,l}(k_j x) = \delta_j^{p1}\frac{2m}{x}\left[\frac{\partial \zeta^{m,l}(k_j x)}{\partial x} - \frac{1}{x}\zeta^{m,l}(k_j x)\right],$$

$$T_{p,j}^{m,l}(k_j x) = \left(\lambda\delta_j^{p1} + \delta_j^{p3} + \tilde{e}_{15}\delta_j^{p4}\right)\frac{\partial \zeta^{m,l}(k_j x)}{\partial x},$$

$$U_{p,j}^{m,l}(k_j x) = \frac{m}{x}\left(\lambda\delta_j^{p2} - \tilde{e}_{14}\delta_j^{p4}\right)\zeta^{m,l}(k_j x),$$

$$W_{p,j}^{m,l}\left(k_j x\right) = \delta_j^{p4}\zeta^{m,l}\left(k_j x\right).$$

APPENDIX D

The case of non - piezoelectric cylinder.

The simplifications of the proposed model include that $e_{ij} = 0$ and $\varepsilon_{ij} = 0$ and the system of equations (1) - (4) is simplified as

$$\begin{bmatrix} \tilde{c}_{11}\nabla^2 + \Omega^2 - \lambda^2 & 0 & -\lambda(1+\tilde{c}_{13}) \\ 0 & \tilde{c}_{66}\nabla^2 + \Omega^2 - \lambda^2 & 0 \\ \lambda(1+\tilde{c}_{13})\nabla^2 & 0 & \nabla^2 + \Omega^2 - \tilde{c}_{33}\lambda^2 \end{bmatrix} \begin{bmatrix} G \\ \psi \\ w \end{bmatrix} = \boldsymbol{0} \tag{D.1}$$

or equivalently

$$\left(\tilde{c}_{66}\nabla^2 + \Omega^2 - \lambda^2\right)\psi = 0, \tag{D.2a}$$

$$\begin{bmatrix} \tilde{c}_{11}\nabla^2 + \Omega^2 - \lambda^2 & -\lambda(1+\tilde{c}_{13}) \\ \lambda(1+\tilde{c}_{13})\nabla^2 & \nabla^2 + \Omega^2 - \tilde{c}_{33}\lambda^2 \end{bmatrix} \begin{bmatrix} G \\ w \end{bmatrix} = \mathcal{D} \begin{bmatrix} G \\ w \end{bmatrix} = \boldsymbol{0}. \tag{D.2b}$$

Following the methodology proposed in section 2 of this work we introduce a function $F = F(x,\theta)$ such that $\{\det \mathcal{D}\}F = 0$ or equivalently

$$\begin{gathered} \left[c\nabla^4 + d\nabla^2 + e\right]F = 0 \quad \Leftrightarrow \\ \left(\nabla^2 + k_1^2\right)\left(\nabla^2 + k_2^2\right)F = 0 \end{gathered}, \tag{D.3}$$

where k_j^2 are the roots of the equation $ck_j^4 - dk_j^2 + e = 0$, where $c = \tilde{c}_{11}$, $d = (1+\tilde{c}_{11})\Omega^2 - (1+\tilde{c}_{11}\tilde{c}_{33})\lambda^2 + (1+\tilde{c}_{13})^2\lambda^2$ and $e = \Omega^4 - (1+\tilde{c}_{33})\lambda^2\Omega^2 + \tilde{c}_{33}\lambda^4$.

The solution of (D.3) is given by the relation (19) and the solution of (D.2b) can be expressed as

$$G = \mathcal{D}'_{p1} F, \quad w = \mathcal{D}'_{p3} F, \quad \mathrm{p} = 1,2, \tag{D.4}$$

where

$$\mathcal{D}'_{pq} = d_3^{pq}\nabla^2 + d_4^{pq}, \ \ p = 1,2; \ \ q = 1,3,$$

and

$$\begin{aligned} d_3^{11} &= 1 & d_4^{11} &= \Omega^2 - \tilde{c}_{33}\lambda^2 \\ d_3^{21} &= 0 & d_4^{21} &= \lambda(1+\tilde{c}_{13}) \\ d_3^{13} &= -\lambda(1+\tilde{c}_{13}) & d_4^{13} &= 0 \\ d_3^{23} &= \tilde{c}_{11} & d_4^{23} &= \Omega^2 - \lambda^2. \end{aligned}$$

Finally from (D.3) and (D.4) we obtain

$$G = \sum_{j=1}^{2}\sum_{l=1}^{2}\left\{\alpha_j^{m,l}\delta_j^{p1}\cos(m\theta) + \beta_j^{m,l}\delta_j^{p1}\sin(m\theta)\right\}\zeta^{m,l}(k_j x),$$

$$w = \sum_{j=1}^{2}\sum_{l=1}^{2}\left\{\alpha_j^{m,l}\delta_j^{p3}\cos(m\theta) + \beta_j^{m,l}\delta_j^{p3}\sin(m\theta)\right\}\zeta^{m,l}(k_j x),$$

where $\delta_j^{pq} = -d_3^{pq}k_j^2 + d_4^{pq}, \ \ p = 1,2; \ \ q = 1,3$.

The solution of (D.2a) is given by

$$\psi = \sum_{l=1}^{2}\left\{\alpha_3^{m,l}\cos(m\theta) + \beta_3^{m,l}\sin(m\theta)\right\}j^{m,l}(k_3 x),$$

where $k_3^2 = 1/\tilde{c}_{66}\left(\Omega^2 - \lambda^2\right)$.

The proposed solution takes the form

$$\begin{aligned} u_x &= \left(\frac{\partial G}{\partial x} + \frac{1}{x}\frac{\partial \psi}{\partial \theta}\right)e^{i(\lambda z - \Omega t)} \\ u_\theta &= \left(\frac{1}{x}\frac{\partial G}{\partial \theta} - \frac{\partial \psi}{\partial x}\right)e^{i(\lambda z - \Omega t)} \\ u_z &= iwe^{i(\lambda z - \Omega t)}. \end{aligned}$$

Following the same methodology with the described in section 2 we obtain an algebraic equation with 12 unknowns, which can be written as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}, \tag{D.6}$$

where $\boldsymbol{x} = \left[\alpha_1^{m,1}, \alpha_1^{m,2}, \alpha_2^{m,1}, \alpha_2^{m,2}, \alpha_3^{m,1}, \alpha_3^{m,2}, \beta_1^{m,1}, \beta_1^{m,2}, \beta_2^{m,1}, \beta_2^{m,2}, \beta_3^{m,1}, \beta_3^{m,2}\right]^T$, and the elements of $\boldsymbol{A}$ are:

$$A_{1,1} = P_{p,1}^{m,1}(k_1\tilde{r}_0),\ A_{1,2} = P_{p,2}^{m,1}(k_2\tilde{r}_0),\ A_{1,3} = 0,\ A_{1,4} = P_{p,1}^{m,2}(k_1\tilde{r}_0),\ A_{1,5} = P_{p,2}^{m,2}(k_2\tilde{r}_0),\ A_{1,6} = 0,$$
$$A_{1,7} = 0,\ A_{1,8} = 0,\ A_{1,9} = Q_{p,3}^{m,1}(k_3\tilde{r}_0),\ A_{1,10} = 0,\ A_{1,11} = 0,\ A_{1,12} = Q_{p,3}^{m,2}(k_3\tilde{r}_0),$$
$$A_{2,1} = 0,\ A_{2,2} = 0,\ A_{2,3} = -Q_{p,3}^{m,1}(k_3\tilde{r}_0),\ A_{2,4} = 0,\ A_{2,5} = 0,\ A_{2,6} = -Q_{p,3}^{m,2}(k_3\tilde{r}_0),$$
$$A_{2,7} = P_{p,1}^{m,1}(k_1\tilde{r}_0),\ A_{2,8} = P_{p,2}^{m,1}(k_2\tilde{r}_0),\ A_{2,9} = 0,\ A_{2,10} = P_{p,1}^{m,2}(k_1\tilde{r}_0),\ A_{2,11} = P_{p,2}^{m,2}(k_2\tilde{r}_0),\ A_{2,12} = 0,$$
$$A_{3,1} = 0,\ A_{3,2} = 0,\ A_{3,3} = R_{p,3}^{m,1}(k_3\tilde{r}_0),\ A_{3,4} = 0,\ A_{3,5} = 0,\ A_{3,6} = R_{p,3}^{m,2}(k_3\tilde{r}_0),$$
$$A_{3,7} = S_{p,1}^{m,1}(k_1\tilde{r}_0),\ A_{3,8} = S_{p,2}^{m,1}(k_2\tilde{r}_0),\ A_{3,9} = 0,\ A_{3,10} = S_{p,1}^{m,2}(k_1\tilde{r}_0),\ A_{3,11} = S_{p,2}^{m,2}(k_2\tilde{r}_0),\ A_{3,12} = 0,$$
$$A_{4,1} = -S_{p,1}^{m,1}(k_1\tilde{r}_0),\ A_{4,2} = -S_{p,2}^{m,1}(k_2\tilde{r}_0),\ A_{4,3} = 0,\ A_{4,4} = -S_{p,1}^{m,2}(k_1\tilde{r}_0),\ A_{4,5} = -S_{p,2}^{m,2}(k_2\tilde{r}_0),\ A_{4,6} = 0,$$
$$A_{4,7} = 0,\ A_{4,8} = 0,\ A_{4,9} = R_{p,3}^{m,1}(k_3\tilde{r}_0),\ A_{4,10} = 0,\ A_{4,11} = 0,\ A_{4,12} = R_{p,3}^{m,2}(k_3\tilde{r}_0),$$
$$A_{5,1} = T_{p,1}^{m,1}(k_1\tilde{r}_0),\ A_{5,2} = T_{p,2}^{m,1}(k_2\tilde{r}_0),\ A_{5,3} = 0,\ A_{5,4} = T_{p,1}^{m,2}(k_1\tilde{r}_0),\ A_{5,5} = T_{p,2}^{m,2}(k_2\tilde{r}_0),\ A_{5,6} = 0,$$
$$A_{5,7} = 0,\ A_{5,8} = 0,\ A_{5,9} = U_{p,3}^{m,1}(k_3\tilde{r}_0),\ A_{5,10} = 0,\ A_{5,11} = 0,\ A_{5,12} = U_{p,3}^{m,2}(k_3\tilde{r}_0),$$
$$A_{6,1} = 0,\ A_{6,2} = 0,\ A_{6,3} = -U_{p,3}^{m,1}(k_3\tilde{r}_0),\ A_{6,4} = 0,\ A_{6,5} = 0,\ A_{6,6} = -U_{p,3}^{m,2}(k_3\tilde{r}_0),$$
$$A_{6,7} = T_{p,1}^{m,1}(k_1\tilde{r}_0),\ A_{6,8} = T_{p,2}^{m,1}(k_2\tilde{r}_0),\ A_{6,9} = 0,\ A_{6,10} = T_{p,1}^{m,2}(k_1\tilde{r}_0),\ A_{6,11} = T_{p,2}^{m,2}(k_2\tilde{r}_0),\ A_{6,12} = 0.$$

The remaining six rows can be obtained from the above relations replacing $\tilde{r}_0$ with $\tilde{r}_1$.

The quantities $P_{p,j}^{m,l}(k_jx)$, $S_{p,j}^{m,l}(k_jx)$, $T_{p,j}^{m,l}(k_jx)$, $U_{p,3}^{m,l}(k_3x)$, $Q_{p,3}^{m,l}(k_3x)$, $R_{p,3}^{m,l}(k_3x)$ are

$$P_{p,j}^{m,l}(k_jx) = \delta_j^{p1}\left\{\tilde{c}_{11}\frac{\partial^2\zeta^{m,l}(k_jx)}{\partial x^2} + \tilde{c}_{12}\left(\frac{1}{x}\frac{\partial\zeta^{m,l}(k_jx)}{\partial x} - \frac{m^2}{x^2}\zeta^{m,l}(k_jx)\right)\right\} - \lambda\tilde{c}_{13}\delta_j^{p3}\zeta^{m,l}(k_jx),$$

$$S_{p,j}^{m,l}(k_jx) = \delta_j^{p1}\frac{2m}{x}\left[\frac{\partial\zeta^{m,l}(k_jx)}{\partial x} - \frac{1}{x}\zeta^{m,l}(k_jx)\right],$$

$$T_{p,j}^{m,l}(k_jx) = \left(\lambda\delta_j^{p1} + \delta_j^{p3}\right)\frac{\partial\zeta^{m,l}(k_jx)}{\partial x},$$

$$U_{p,3}^{m,l}(k_3x) = \lambda \frac{m}{x} \zeta^{m,l}(k_3x),$$

$$Q_{p,3}^{m,l}(k_3x) = 2\tilde{c}_{66} \frac{m}{x} \left[\frac{\partial \zeta^{m,l}(k_3x)}{\partial x} - \frac{1}{x} \zeta^{m,l}(k_3x) \right],$$

$$R_{p,3}^{m,l}(k_jx) = -\left[\frac{\partial^2 \zeta^{m,l}(k_3x)}{\partial x^2} - \frac{1}{x} \frac{\partial \zeta^{m,l}(k_3x)}{\partial x} + \frac{m^2}{x^2} \zeta^{m,l}(k_3x) \right].$$

For the system (D.6) different cases, which depend on the nature of roots k_1, k_2, can be treated as it is explained in section 4 of our work.

The case of a piezoelectric cylinder can be further reduced to isotropic hollow cylinder if we set

$$\tilde{c}_{12} = \tilde{c}_{13}, \qquad \tilde{c}_{44} = \tilde{c}_{66} = 1, \qquad \tilde{c}_{11} = \tilde{c}_{33},$$
$$\tilde{c}_{12} = \frac{\lambda}{\mu}, \qquad \frac{1}{2}(\tilde{c}_{11} - \tilde{c}_{12}) = \mu,$$

where λ, μ are the Lamè's constants, $\lambda = E\nu/((1+\nu)(1-2\nu))$, $\mu = E/(2(1+\nu))$, E, ν are the Young's modulus and Poisson's ratio, respectively.