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Abstract — We classify the edges of a graph as either *free*, *semi-free* or *actual* and we define the class of *A-free* graphs as the class containing all the undirected graphs with no actual edges. We prove that the A-free graphs satisfy several important structural and algorithmic properties and are characterized by specific forbidden induced subgraphs. Based on these results, we show the relationship between A-free graphs and the classes of perfect graphs known as domination perfect, chordal (or triangulated), cographs, comparability, cocomparability, interval, permutation, ptolemaic, distance-hereditary, (*t, c, s*)-perfect, block, split and threshold. Moreover, we show structural and algorithmic properties of the normal product of two A-free graphs.

1. Introduction

An undirected graph $G = (V, E)$ is said to be *perfect* if it satisfies the following two properties: the χ-Perfect property: $\chi(G_A) = \omega(G_A)$ (for all $A \subseteq V$), and the α-Perfect property: $\alpha(G_A) = \varkappa(G_A)$ (for all $A \subseteq V$), where $\chi(G_A)$, $\omega(G_A)$, $\alpha(G_A)$ and $\varkappa(G_A)$ are the chromatic, clique, stability and clique-cover number of G_A, respectively, and G_A is an induced subgraph of G.

Our objective is to study structural and recognition properties for some important classes of perfect graphs known as domination perfect, chordal (or triangulated), cographs, comparability, cocomparability, interval, permutation, ptolemaic, distance-hereditary, (*t, c, s*)-perfect, block, split and threshold graphs. Many researchers have extensively studied these classes of perfect graphs and proposed algorithms for the recognition problem, as well as for many other problems such as colouring, minimal code-colouring, maximal matching, clique finding, constructing perfect elimination schemes, assigning transitive orientations, clustering, assigning transitive orientations, minimum weight domination, minimal path cover, isomorphism, etc (see, e.g., [9, 19]).

In this paper, we introduce an edge classification and show that it can be used as a constructive tool in proving recognition properties for the most important classes of perfect graphs. Based on this classification, we define the class of *A-free* graphs as the class which contains all the undirected graphs having no actual edges. We show structural properties and characterizations of the members of this class, which imply that A-free graphs form a subclass of chordal, cographs, ptolemaic, distance-hereditary, comparability, cocomparability, interval, permutation and (t, c, s)-perfect graphs. Moreover, we show recognition properties for block graphs, split graphs and threshold graphs, still using the proposed edge classification [12, 17].

Specifically, given an undirected graph, we partition the edges of the graph into three classes, called *free*, *semi-free* and *actual* edges, according to the relationship of the closed neighbourhoods of the endpoints (or end-vertices) of their edges. Moreover, we show that the vertex set V of an A-free graph, i.e., a graph which contains only free and semi-free edges, can be partitioned into $m \geq 2$ nonempty, disjoint vertex sets $V_1, V_2, ..., V_c, ..., V_k$ satisfying important algorithmic and structural properties. Furthermore, we show a close relationship between the structure of an A-free graph and the structure of the normal product of two A-free graphs. Consequently, we prove that any A-free graph possesses, among others, the following important properties: Chordality or property T; a graph satisfying T is said to be chordal or triangulated; Transitive orientation or property C; a graph satisfying C is said to be comparability; Transitive co-orientation or property C^c; a graph satisfying C^c is said to be cocomparability, i.e., its complement is a comparability graph; Clique-kernel intersection property or CK property [9, 4]. Moreover, based on the definition of the actual edges of a graph, we show that the A-free graphs are exactly the graphs not having a P_4 or a C_4 as an induced subgraph.

It is well-known that several classes of perfect graphs have already been characterized in terms of these properties, as well as in terms of forbidden induced subgraphs. For example, interval graphs satisfy properties T and C^c [10], permutation graphs satisfy properties C and C^c [18], cographs satisfy the CK property [4], cographs have no induced subgraphs isomorphic to P_4 [4], threshold graphs have no induced subgraph isomorphic to $2K_2$, P_4, or C_4 [5], t-perfect graphs have no induced subgraph isomorphic to P_4 or C_4 [2, 9], etc. Based on these properties and characterizations, we show that A-free graphs belong to the classes of domination perfect, chordal, cographs, comparability, cocomparability, interval, permutation, ptolemaic and distance-hereditary graphs. Moreover, we identify the precise structure possessed by certain subsets of vertices and/or edges of a graph in the case where it is a block, split or threshold graph.

We should point out that we can easily formulate a constant-time parallel algorithm for deciding whether or not an undirected graph contains actual edges, which can operate by examining specific relations of the closed neighbourhoods of the endpoints of each edge of the graph. This result, in turn, implies than all the above mentioned perfect graphs can be recognized in constant-time in the case where they contain no actual edges. Obviously, such an algorithm run on a Concurrent-Read, Concurrent-Write (CRCW) PRAM model of computation and use $O(m\,n)$ processors.

Throughout the paper we assume that all graphs are finite and that unless stated otherwise the term subgraph always refers to the notion of induced subgraph. Moreover, m denotes the number of edges and n denotes the number of vertices in a graph.

2. The Structure of A-free Graphs

Following the notation and terminology in [11, p.167], the neighbourhood of a vertex u is the set $N(u)$ consisting of all the vertices v which are adjacent with u. The closed neighbourhood is $N[u] = \{u\} \cup N(u)$. We call a graph trivial if it has only one vertex, and incomplete if it has at least two non-adjacent vertices. The subgraph of a graph G induced by a subset of vertices S will be denoted by G(S) or G_S, but sometimes also by S when there is no ambiguity.

Given a graph G = (V, E), we define three classes of edges in G, denoted by AE, FE and SE, according to relationship of the neighbourhood and closed neighbourhood of the endpoints of its edges [12, 17]. Let $x = (u, v)$ be an edge of G. Then,

$$(u, v) \in \text{FE} \quad \text{if} \quad N[u] = N[v]$$
$$(u, v) \in \text{SE} \quad \text{if} \quad N[u] \subset N[v]$$
$$(u, v) \in \text{AE} \quad \text{if} \quad N[u] - N[v] \neq \varnothing \text{ and } N[v] - N[u] \neq \varnothing$$

In words, edge (u, v) is a member of FE if its vertices u and v have the same closed neighbourhoods; it is a member of SE if the closed neighbourhood of one vertex u is a proper subset of the closed neighbourhood of the other vertex v; it is a member of AE if the closed neighbourhoods of vertices u and v, i.e., $N[u]$ and $N[v]$, are not comparable with respect to inclusion. An edge is said to be a *free*, *semi-free* and *actual* edge if it is a member of class FE, SE and AE, respectively. Obviously, E = FE + SE + AE. We illustrate with three graphs G, H and I shown in Figure 1. The edges in classes FE, SE and AE are denoted by *f*, *s* and *a*, respectively.

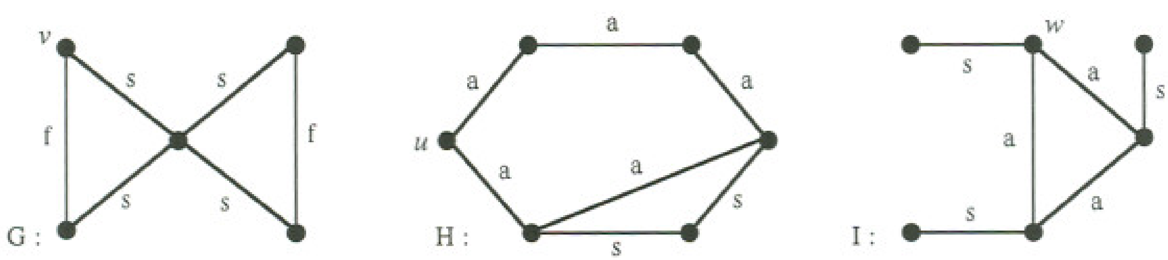

Figure 1. Three undirected graphs. Free, semi-free and actual edges are denoted by f, s and a, respectively.

Having classified the edges of a graph as either free, semi-free and actual, let us now define the class of *A-free graphs* as follows:

Definition 1. A undirected graph G = (V, E) is called *A-free* if every edge of G is either free or semi-free edge.

The graph G in Figure 1 is an A-free graph, while the graphs H and I in the same figure are not A-free graphs. A typical structure of an A-free graph is shown in Figure 2. The following results provide algorithmic and structural properties for the class of A-free graphs.

Theorem 1 (Nikolopoulos [1995]). The vertex set V of an A-free graph G = (V, E) can be partitioned into $k \geq 2$ nonempty, disjoint vertex sets $V_1, V_2, \ldots, V_c, \ldots, V_k$, i.e.,

$$V = V_1 + V_2 + \ldots + V_c + \ldots + V_k,$$

satisfying the following properties:

(P1) There exists a vertex set V_c such that $N[V_c] = V$, $1 \le c \le k$.

(P2) Every vertex set V_i induces a complete subgraph $G(V_i)$, i.e., V_i is a clique, $1 \le i \le k$.

(P3) Every vertex set $V_i \cup V_j$ induces either a complete graph $G(V_i \cup V_j)$ or a disconnected graph having two complete subgraphs $G(V_i)$ and $G(V_j)$, $1 \le i, j \le k$.

(P4) Edges with both endpoints in V_i are free edges, $1 \le i \le k$.

(P5) Edges with one endpoint in V_i and the other endpoint in V_j are semi-free edges, $1 \le i, j \le k$ and $i \ne j$.

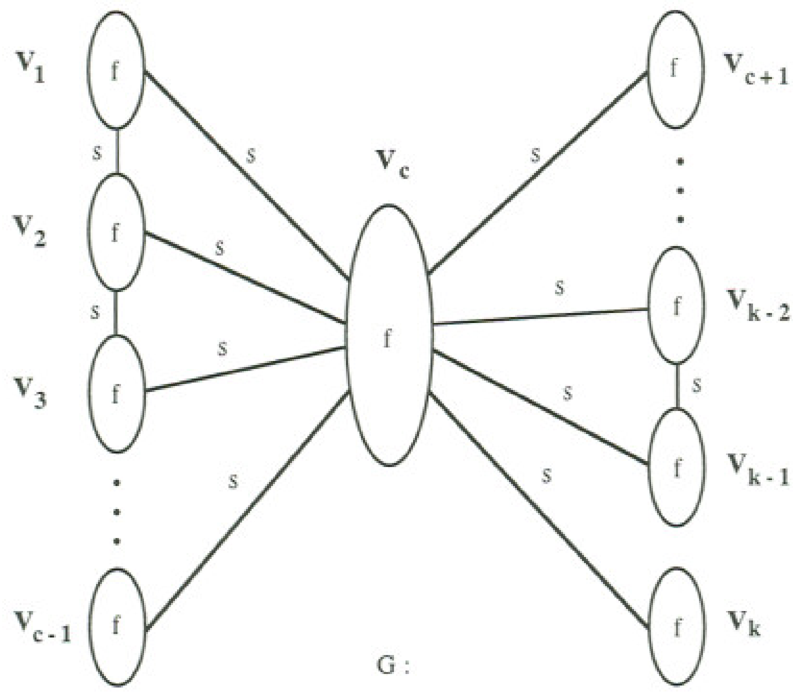

Figure 2. The typical structure of an A-free graph. A line between cells V_i and V_j indicates that each vertex in V_i is adjacent to each vertex of V_j. All edges in V_i are free edges; All edges between cells are semi-free edges.

If V_i and V_j are disjoint vertex sets of an A-free graph $G = (V, E)$, we say that V_i and V_j are *adjacent* and denote $V_i \sim V_j$ if there exists a semi-free edge (x, y) such that $x \in V_i$, $y \in V_j$ and $V_i \cup V_j$ is not a clique. If (x, y) is a semi-free edge for all $x \in V_i$ and $y \in V_j$ then we say that V_i and V_j are *clique-adjacent* and denote $V_i \approx V_j$. Obviously, if V_i and V_j are clique-adjacent then $G(V_i \cup V_j)$ is a complete graph. Throughout the paper $x \sim y$ means that (x, y) is an edge of G. Moreover, $i \ne j \ne k$ means that $i \ne j$, $i \ne k$ and $j \ne k$.

Let us now examine the effect of property P3 of Theorem 1 on the structure of an A-free graph. This property ensures that all the edges with both endpoints in a vertex set V_i are free edges, $1 \le i \le k$. A consequence of this property is that the vertex set $V_i \cup V_c$ is not always a maximal clique. We can easily see that $V_i \cup V_c$ is not a maximal clique if there exists a vertex set V_j such that $V_i \sim V_j$, $1 \le j \le k$ and $j \ne c$. Actually, the property P3 of Theorem 1 says that $V_i \approx V_j$ for every V_i, V_j such that $V_i \sim V_j$, $1 \le i, j \le k$.

Theorem 1 shows how to construct the vertex sets of the partition $V = V_1 + ... + V_c + ... + V_k$ of an A-free graph $G = (V, E)$. We consider now the case where the vertex set V of an A-free graph is partitioned into $k \geq 2$ nonempty disjoint vertex sets $V_1, ..., V_c, ..., V_k$, under the additional restriction that every vertex set $V_i \cup V_c$ is a maximal clique. In this case, the following results are obtained.

Theorem 2. Let $G = (V, E)$ be an A-free graph and let $V = V_1 + V_2 + ... + V_c + ... + V_k$ be a partition satisfying the following properties:

(i) There exists a vertex set V_c such that $N[V_c] = V$, $1 \leq c \leq k$.
(ii) Every vertex set V_i is a clique, $1 \leq i \leq k$.
(iii) Every vertex set $V_i \cup V_c$ is a maximal clique, $1 \leq i \leq k$ and $i \neq c$.

The following properties are hold:

(P1) Edges with both endpoints in V_c are free edges, $1 \leq c \leq k$, while edges with both endpoints in V_i are free or semi-free edges, $1 \leq i \leq k$ and $i \neq c$.
(P2) Edges with one endpoint in V_i and the other endpoint in V_j are semi-free edges, $1 \leq i, j \leq k$ and $i \neq j$.
(P3) Every vertex set $V_i \cup V_j$ induces either an incomplete graph $G(V_i \cup V_j)$ or a disconnected graph having two complete subgraphs $G(V_i)$ and $G(V_j)$, $1 \leq i, j \leq k$ and $i \neq j \neq c$.

Proof. (P1) Theorem 1 implies that edges with both endpoints in V_i are free edges, $1 \leq i \leq k$; see property P4. Since any partition of the set $V\text{-}V_c$ does not affect the structure of the vertex set V_c, it follows that all the edges with both endpoints in V_c are free edges. Let V_i be a vertex set, other than V_c, $1 \leq i \leq k$. We distinguish two cases. Case I: There exists no vertex set V_j such that $V_i \sim V_j$, $j \neq c$. Since $V_j \approx V_c$, it implies that all the edges with both endpoints in V_j are free edges. Case II: There exists a vertex set V_j such that $V_i \sim V_j$, $i \neq j$ and $j \neq c$. Then, there are vertices $x \in V_i$ and $y \in V_j$ such that $(x, y) \in E$. Since $V_j \approx V_c$, there exists vertex $z \in V_i$ such that $(z, x) \in E$ and $(z, y) \notin E$. Thus, (z, x) is a semi-free edge. Obviously, every edge (z, z') having both endpoints in V_i is a free edge in the case where $N(\{z, z'\}) \cap V_j = \varnothing$.
(P2) Let (x, y) be an edge such that $x \in V_i$ and $y \in V_j$. Suppose that (x, y) is a free edge. Then, $(z, y) \in E$ for every vertex $z \in V_i$. Thus, $V_i \cup V_c$ is not a maximal clique which is absurd.
(P3) It follows directly from the property P2 and the fact that $V_i \sim V_j$ for every $i \neq j$, where $i \neq c$ and $j \neq c$. □

Corollary 1. For every pair of vertex sets V_i and V_j there exist vertices $x \in V_i$ and $y \in V_j$ such that $(x, y) \notin E$, where $1 \leq i, j \leq k$ and $i \neq j \neq c$.

The properties provided by Theorem 1 and Theorem 2 say that the structure of an A-free graph $G = (V, E)$ is entirely determined by the structures of the vertex sets of the partition $V = V_1 + V_2 + ... + V_c + ... + V_k$. More precisely, we have shown that the properties of Theorem 1 ensure that all the edges with both endpoints in a vertex set V_i are free edges, while the properties of Theorem 2 ensure that every vertex set $V_i \cup V_c$ is a maximal clique, $1 \leq i \leq k$. Thus, we obtain two deferent structures of an A-free graph. We shall refer to the structure which meets the properties of Theorem 1 as *A-free-I* and the structure which meets the properties of Theorem 2 as *A-free-II*. The structure A-free-II of an A-free graph is shown in Figure 3.

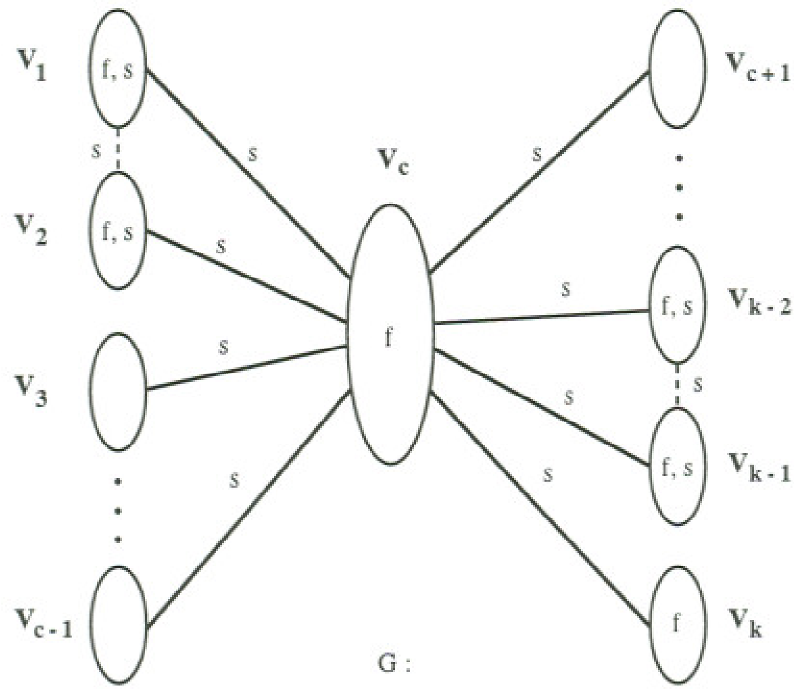

Figure 3. The structure of an A-free graph which is derived from Theorem 2. A line between cells V_i and V_c indicates that $V_i \approx V_c$, while a dashed line between cells V_i and V_j indicates that $V_i \sim V_j$. All edges in V_c are free edges, while all edges in V_i are free or semi-free edges; All edges between cells are semi-free edges.

Let G = (V, E) be an A-free graph and let $V = V_1 + V_2 + \ldots + V_c + \ldots + V_k$ be a partition of the vertex set V of G such that $V_i \cup V_c$ is a maximal clique. i.e., the partition which satisfy the properties of the structure A-free-II, $1 \le i \le k$. Then, it is obvious that the following partition

$$V = A_1 + A_2 + \ldots + A_{c-1} + A_{c+1} + \ldots + A_k$$

where $A_i = V_i \cup V_c$, $i = 1, 2, \ldots, c-1, c+1, \ldots, k$, is a *clique cover* of size k-1. Based on Theorem 2, we can easily prove that k-1 is the size of a smallest possible clique cover of G. Thus, k-1 equals the clique-cover number of G, i.e., $\varkappa(G) = k-1$. Moreover, it is well-known that a stable (independent) set is a subset X of vertices no two of which are adjacent. We can find a vertex x_i in A_i such that $X = \{x_1, x_2, \ldots, x_{c-1}, x_{c+1}, \ldots, x_k\}$ is a stable set. Since A_i is a clique, the stable set X is of maximum cardinality. Thus, k-1 equals the stability number of G, i.e., $\alpha(G) = k-1$. Therefore, we have the following result.

Lemma 1. Let $V = V_1 + V_2 + \ldots + V_c + \ldots + V_k$ be a partition of the vertex set V of an A-free graph G = (V, E) such that it determines the structure A-free-II. Then,

(i) k-1 equals the clique-cover number of G, i.e., $\varkappa(G) = k-1$.
(ii) k-1 equals the stability number of G, i.e., $\alpha(G) = k-1$.

Based on the properties of Theorem 1 and Theorem 2, as well as on the fact that every induced subgraph of an A-free graph contains no actual edges, we obtain the following result.

Lemma 2. Every induced subgraph H of an A-free graph G is also an A-free graph having $\alpha(H) = \varkappa(H)$.

We should point out that, the structure A-free-II of an A-free graph G also provides algorithmic properties of major importance since they can be used, among others, for the computation of all the maximal cliques or maximal independent sets of G.

3. Normal Product of A-free Graphs

Let $G_1 = (X, E_1)$ and $G_2 = (Y, E_2)$ be two undirected graphs. Their *normal product* is the graph $G = G_1 \wedge G_2 = (V, E)$, where $V = X \times Y$ and $(x, y) \sim (x', y')$ if and only if $x \sim x'$ and $y = y'$, or $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y \sim y'$.

From now on and until the end of this section, we reserve the letter G for the normal product of two A-free graphs G_1 and G_2, and the letters X and Y for the vertex sets of G_1 and G_2, respectively. The following theorem provides algorithmic and structural properties for the normal product of two A-free graphs.

Theorem 3. Let $G_1 = (X, E_1)$ and $G_2 = (Y, E_2)$ be two A-free graphs and let $G = (V, E)$ be their normal product. Let

$$X = X_1 + X_2 + \dots + X_c + \dots + X_k$$

and

$$Y = Y_1 + Y_2 + \dots + Y_{c'} + \dots + Y_{k'}$$

be the vertex partitions of X and Y, respectively, which determine the structures A-free-II. The vertex set V of the normal product $G = (V, E)$ can be partitioned into $kk' \geq 2$ nonempty, disjoint vertex sets $V_{11}, V_{12}, \dots, V_{cc'}, \dots, V_{kk'}$, i.e.,

$$V = V_{11} + V_{12} + \dots + V_{21} + V_{22} + \dots + V_{cc'} + \dots + V_{kk'},$$

satisfying the following properties:

(P1) There exists a vertex set $V_{cc'}$ such that $N[V_{cc'}] = V$, $1 \leq c \leq k$ and $1 \leq c' \leq k'$.

(P2) Every vertex set V_{ij} is a clique, $1 \leq i \leq k$ and $1 \leq j \leq k'$.

(P3) Every vertex set $V_{ij} \cup V_{cc'}$ is a clique, $1 \leq i, c \leq k$ and $1 \leq j, c' \leq k'$.

(P4) Every vertex set $V_{ij} \cup V_{cc'} \cup V_{ic'} \cup V_{cj}$ is a maximal clique, $1 \leq i, c \leq k$ and $1 \leq j, c' \leq k'$.

(P5) Edges with both endpoints in $V_{cc'}$ are free edges, $1 \leq c \leq k$ and $1 \leq c' \leq k'$.

(P6) Edges with one endpoint in $V_{cc'}$ and the other endpoint in V_{ij} are semi-free edges, $1 \leq i, c \leq k$ and $1 \leq j, c' \leq k'$.

(P7) Edges with one endpoint in $V_{ic'}$ and the other endpoint in V_{cj} are actual edges, $1 \leq i, c \leq k$, $1 \leq j, c' \leq k'$, $i \neq c$ and $j \neq c'$.

(P8) Every vertex set $V_{ij} \cup V_{pq}$ induces either an incomplete graph $G(V_{ij} \cup V_{pq})$ or a disconnected graph having two complete subgraphs $G(V_{ij})$ and $G(V_{pq})$, where $1 \leq i, p \leq k$, $1 \leq j, q \leq k'$ and either ($i \neq c$, $p \neq c$ and $j = q = c'$) or ($j \neq c'$, $q \neq c'$ and $i = p = c$) or ($i \neq c$, $p \neq c$, $j \neq c'$ and $q \neq c'$).

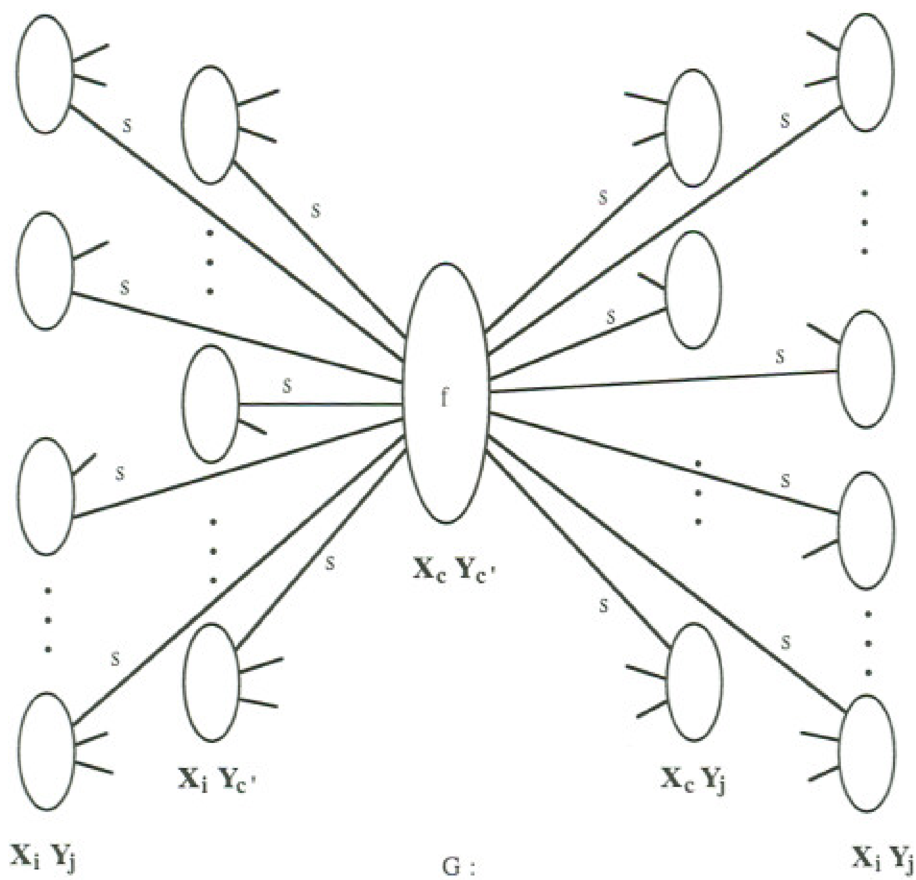

Figure 4. The adjacency relationship between $X_cY_{c'}$ and the other vertex sets of a partition of the normal product of two A-free graphs. A line between cells indicates that each vertex in one cell is adjacent to each vertex of the other set. The vertex set X_iY_j is the Cartesian product $X_i \times Y_j$.

Before giving the proof of Theorem 3, we need to present some technical lemmas about the structure of the disjoint vertex sets V_{11}, V_{12}, ..., $V_{cc'}$, ..., $V_{kk'}$ of a partition of the normal product of two A-free graphs.

Let $G_1 = (X, E_1)$ and $G_2 = (Y, E_2)$ be two A-free graphs, and let $G = (V, E)$ be their normal product. By definition, the vertex set V contains $|X||Y|$ vertices (x, y) such that $x \in X$ and $y \in Y$. We define the set V_{ij} to be the subset of the vertex set V which contains all the vertices (x, y) such that $x \in X_i$ and $y \in Y_j$, $1 \le i \le k$ and $1 \le j \le k'$. Hereafter, the vertices of G will be denoted by xy and the vertex set V_{ij} by X_iY_j.

Lemma 3. Let $G_1 = (X, E_1)$ and $G_2 = (Y, E_2)$ be A-free graphs and let $G = (V, E)$ be their normal product. Let $V_{11} + V_{12} + ... + V_{cc'} + ... + V_{kk'}$ be a partition of the vertex set V into nonempty disjoint subsets such that $xy \in V_{ij}$ if and only if $x \in X_i$ and $y \in Y_j$, $1 \le i \le k$ and $1 \le j \le k'$. Then the following statements hold.

(i) V_{ij} is a clique.

(ii) $V_{ij} \cup V_{cc'}$ is a clique.

(iii) $V_{ic'} \cup V_{cj}$ is a clique.

(iv) $V_{ij} \cup V_{cc'} \cup V_{ic'} \cup V_{cj}$ is a maximal clique.

Proof. (i) By definition, the set V_{ij} contains all the vertices xy of G such that $x \in X_i$ and $y \in Y_j$,

$1 \le i \le k$ and $1 \le j \le k'$. Since X_i and Y_j are cliques, any two vertices xy and $x'y'$ in V_{ij} are adjacent. Thus, V_{ij} is a clique. In this case, it is easy to see that if $xy \sim x'y'$ then either $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$ or $x \sim x'$ and $y \sim y'$.

(ii) Let xy and $x'y'$ be vertices in V_{ij} and $V_{cc'}$, respectively. By definition, $x \in X_i$, $x' \in X_c$ and $y \in Y_j$, $y' \in Y_{c'}$. Since $X_i \approx X_c$ and $Y_j \approx Y_{c'}$, we have that $x \sim x'$ and $y \sim y'$. Thus, $V_{ij} \cup V_{cc'}$ is a clique.

(iii) The proof is similar to (ii).

(iv) It is easy to prove that the vertex sets $V_{ij} \cup V_{ic'}$ and $V_{ij} \cup V_{cj}$ are cliques. Since $V_{cc'} \cup V_{ij}$ is a clique for every i, j, where $1 \le i \le k$ and $1 \le j \le k'$, the vertex set $V_{ijcc'} = V_{ij} \cup V_{cc'} \cup V_{ic'} \cup V_{cj}$ is a clique. Suppose that the set $V_{ijcc'}$ is not a maximal clique; then there exist a vertex set V_{pq}, $p \neq q$, and a vertex $xy \in V_{pq}$ such that $\{xy\} \cup V_{ijcc'}$ is a clique, where $p \neq i \neq c$ and $q \neq j \neq c'$. Then $\{xy\} \cup V_{ij}$ is a clique, and in consequence the vertex $x \in X_p$ is adjacent with every vertex $x' \in X_i$, a contradiction. (Notice that $X_p \cup X_i$ is not a clique for every p, i such that $p \neq i \neq c$; see property P3 of Theorem 2.) □

The properties established so far suffice to prove some of the statements of Theorem 3. For the proof of Theorem 3, we need some additional results concerning the adjacency relationship between the vertex sets $V_{ic'}$, V_{cj} and V_{ij} of the partition of the vertex set of the normal product of two A-free graphs G_1 and G_2, where $i \neq c$ and $j \neq c'$. We therefor classify the vertex sets V_{11}, V_{12}, ..., $V_{cc'}$, ..., $V_{kk'}$ into four classes, according to the properties provided by Lemma 3. The first class contains the vertex set $V_{cc'}$ and the other three, called A, B and C, contain the following vertex sets:

$A = \{V_{ic'} : 1 \le i \le k, i \neq c\}$,
$B = \{V_{cj} : 1 \le j \le k', j \neq c'\}$, and
$C = \{V_{ij} : 1 \le i \le k, 1 \le j \le k', i \neq c \text{ and } j \neq c'\}$.

It is not difficult to show that the cardinalities of the sets A, B and C are as follows:

$|A| = |B| = k + k' - 2$ and
$|C| = (k - 1)(k' - 1)$,

where k and k' are the sizes of the partitions of the vertex sets of G_1 and G_2, respectively.

Figure 5 depicts the classes of A, B and C, as well as the adjacency relationship between the vertex sets of the partition of the set V of the graph G = (V, E).

Lemma 4. Let $G_1 = (X, E_1)$ and $G_2 = (Y, E_2)$ be A-free graphs and let G = (V, E) be their normal product. Let $V_{11} + V_{12} + ... + V_{cc'} + ... + V_{kk'}$ be the partition of the vertex set V of G. Then the following properties hold.

(i) For any pair of vertex sets $a \in A$ and $b \in B$, $a \cup b$ is a clique.

(ii) Edges with one endpoint in A and the other endpoint in B are actual edges.

(iii) For every pair of vertex sets $a, a' \in Z$, where Z is either the set A, B or C, the set $a \cup a'$ induces either an incomplete graph $G(a \cup a')$ or a disconnected graph having two complete subgraphs $G(a)$ and $G(a')$.

Proof. Statement (i) follows directly from the statement (iii) of Lemma 3, while (iii) is implied by property P3 of Theorem 2; $a = X_iY_{c'}$, $b = X_cY_j$, and $V_{ic'} \cup V_{cj}$ is a clique. (ii) Let $a, a' \in A$ and

b, $b' \in$ B, and let $(x_a y_a, x_b y_b)$ be an edge with one endpoint in set a and the other endpoint in set b. Statement (iii) implies that there exist vertices $x_{a'} y_{a'} \in a'$ and $x_{b'} y_{b'} \in b'$ such that $(x_a y_a, x_{a'} y_{a'}) \notin$ E and $(x_b y_b, x_{b'} y_{b'}) \notin$ E. Since $a \cup b$ and $a' \cup b'$ are cliques, it follows that $(x_a y_a, x_{b'} y_{b'}) \in$ E and $(x_b y_b, x_{a'} y_{a'}) \in$ E. Thus, $(x_a y_a, x_b y_b)$ is an actual edge. □

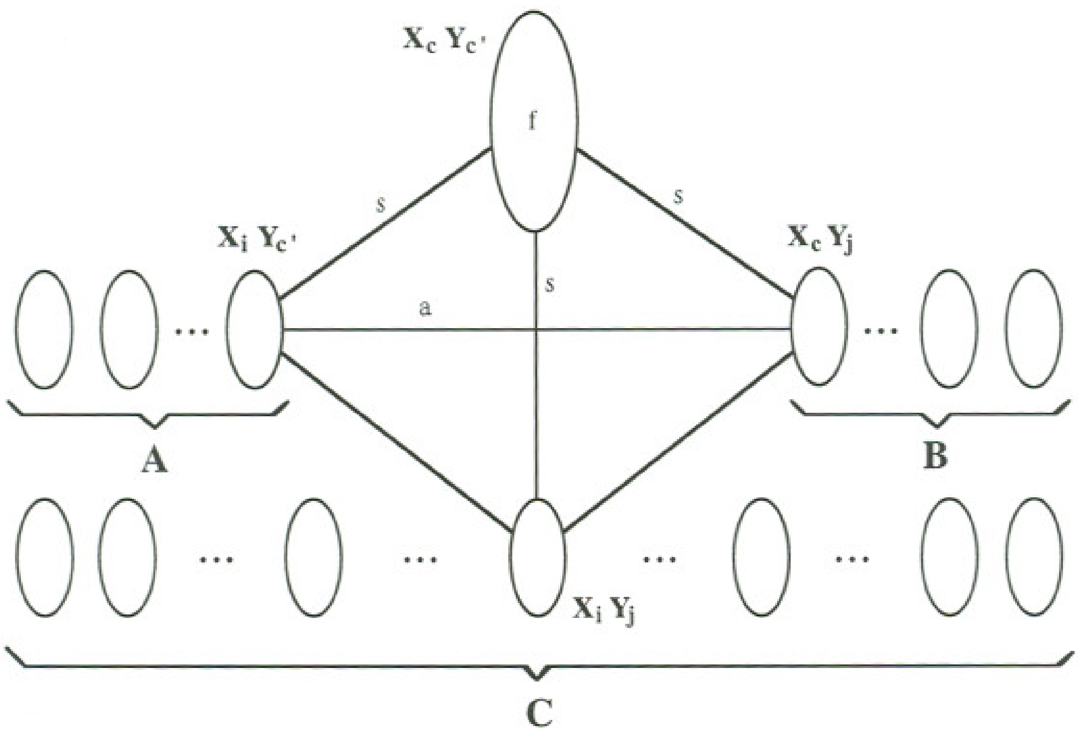

Figure 5. The relationship between the disjoint vertex sets of a partition of the normal product of two A-free graphs.

The following are immediate results of the preceding Lemma 3 and Lemma 4.

Corollary 2. Let G_1 be an incomplete A-free graph and let G_2 be a complete graph. Then their normal product $G = G_1 \wedge G_2$ is an A-free graph.

Corollary 3. For every pair of vertex sets a, $a' \in$ Z, where Z is either the set A, or B or C, there exist vertices $xy \in a$ and $x'y' \in a'$ such that $(xy, x'y') \notin$ E.

Corollary 4. The normal product of two incomplete A-free graphs is not an A-free graph.

Based on the results presented in Lemmas 3 and 4, we can easily prove the main theorem of this section.

Proof of Theorem 3. Properties (P1) through (P4) follow from Lemma 3, while (P7) and (P8) are implied from Lemma 4. Property (P3) tells us that $V_{ij} \cup V_{cc'}$ is a clique for every $1 \le i \le k$ and $1 \le j \le k'$. Thus, it directly implies (P5) and (P6). □

Note. Although we have assumed that both the A-free graphs G_1 and G_2 are incomplete graphs, Theorem 3 is also true in the case where one of the A-free graphs is either trivial or complete. Obviously, we have nothing to prove in the case where both A-free graphs are either trivial or complete.

4. Relationship between A-free and Perfect Graphs

In this section we prove that the A-free graphs satisfy important properties which are later used as a base for showing the relationship between the class of A-free graphs and many other classes of perfect graphs. Moreover, based on the structure A-free-II we show important properties of the normal product of two A-free graphs.

A graph is a *diagonal* graph or *D*-graph if for every path in G with edges (v_1, v_2), (v_2, v_3), (v_3, v_4), the graph also contains the edges (v_1, v_3) or (v_2, v_4). It is important to point out that Wolk [23] showed that the *D*-graphs are precisely the comparability graphs of rooted trees. This result was later quoted incorrectly as "A graph without induced subgraph isomorphic to P_4, i.e., a cograph, is the comparability graph of rooted trees". The graph C_4 is a counter-example to this statement. By definition, it is easy to see that *D*-graphs contain no actual edges. Therefore, we are in a position to state our first result.

Theorem 4. Diagonal graphs (or *D*-graphs) are precisely the undirected graphs with no actual edges, i.e., the A-free graphs.

Based on the definition of the actual edges of a graph, we can easily show that the A-free graphs are exactly the graphs not having a P_4 or a C_4 as an induced subgraph. Thus, the following theorem holds.

Theorem 5. A graph G is an A-free graph if and only if it contains no induced subgraph isomorphic to P_4 or C_4.

An immediate consequence of Theorem 5 is that A-free graphs are exactly the *chordal cographs* [6, 9, 15]. Moreover, cographs form a subclass of the class of *distance-hereditary* graphs [13, 14] (each connected induced subgraph preserves distances), and therefore, they form a subclass of the class of *parity* graphs [1, 4, 16]. An important class of perfect graphs, known as *ptolemaic* graphs, forms a subclass of the distance-hereditary graphs. Actually, a graph G is a ptolemaic graph if and only if it is chordal and distance-hereditary graph. Thus, we can present the following theorem and its corollary.

Theorem 6. Let G be an A-free graph. Then G is a ptolemaic graph.

Corollary 5. A-free graphs form a subclass of distance-hereditary and parity graphs.

We have shown that an A-free graph is a cograph. Since cographs are a subclass of *permutation* graphs [9, 18], cographs are *comparability* and *cocomparability* graphs [9, 10]. Moreover, it is well-known that permutation graphs are exactly those graphs which are comparability graphs and cocomparability graphs. It is also known that a comparability graph is a *superperfect* graph [9] and an *interval* graph is chordal and cocomparability [8, 9, 10]. Therefore, the following theorem and its corollary hold.

Theorem 7. Let G = (V, E) be an A-free graph. Then G is cograph, comparability graph and cocomparability graph.

Corollary 6. An A-free graph is a permutation graph, interval graph and superperfect graph.

A *sun* of order p, or p-*sun* ($p \geq 3$) is a chordal graph on vertex set $\{x_1, x_2, ..., x_p, y_1, y_2, ..., y_p\}$, where $\{y_1, y_2, ..., y_p\}$ is an independent set, $(x_1, x_2, ..., x_p)$ is a cycle, and each vertex y_i has exactly two neighbours, x_{i-1} and x_i. By definition, every p-*sun* ($p \geq 3$) contains an actual edge. So, we obtain the following results.

Theorem 8. Let G be an A-free graph. Then G contains no induced subgraph isomorphic to a p-sun ($p \geq 3$).

For a graph G the k-th power G^k of G is the graph with the same vertex set as G where two vertices are adjacent if and only if their distance is at most k in G. The *clique graph* K(G) of G is the graph whose vertices are the maximal cliques $K^1, K^2, ..., K^p$ of G, in which (K^i, K^j) is an edge if and only if $N(K^i) \cap K^j \neq \emptyset$, where $i \neq j$. The following theorem clarify the relationship between G^2 and K(G) of an A-free graph G.

Theorem 9. Let G be an A-free graph. Then both G^2 and K(G) are complete graphs.

A graph G is called *strongly chordal* if G is chordal and G contains no sun, G is called *balanced chordal* if G is chordal and G contains no sun of odd order, and G is called *compact* if G contains no sun of order 3. We have showed that an A-free graph is a chordal graph (Theorem 6) and it contains no induced subgraph isomorphic to a p-sun, $p \geq 3$ (Theorem 8). These prove the following result.

Theorem 10. Let G be an A-free graph. Then G is a strongly chordal graph, a balanced chordal graph and a compact graph.

We know the following three statements are equivalent for a chordal graph G: (i) G^2 is chordal; (ii) K(G) is chordal; (iii) every sun of G of order greater than 3 is suspended [22]. If G is an A-free graph, then G^2 and K(G) are complete graphs and, therefore, chordal graphs. Thus, we have the following result.

Theorem 11. Let G be an A-free graph. Then both G^2 and K(G) are chordal graphs and every sun of G of order greater than 3 is suspended.

Let γ(G) and ι(G) be the domination number and independent domination number of a graph G, respectively. A graph G is called a *domination perfect* graph if $\gamma(H) = \iota(H)$, for every induced subgraph H of G. The domination number γ(G) is the minimum cardinality taken over all dominating sets of G, and the independent domination number ι(G) is the minimum cardinality taken over all maximal independent sets of vertices of G. Based on the properties (P1) and (P2) of Theorem 1, we can prove that $\gamma(H) = \iota(H) = 1$ for every induced subgraph H of an A-free graph G. Thus, we obtain the following theorem.

Theorem 12. Let G be an A-free graph. Then G is a domination perfect graph.

Let G = (V, E) be a graph. We define **C**(G) to be the set of all maximal cliques of G and similarly, we define **S**(G) to be the set of all independent sets of G. Let $F = (V_i)_{i \in I}$ be a family of subsets of the set V. Following the definition in [2], we call a *transversal* of F a subset T of V such that T intersects the sets V_i for all $i \in I$; if all these intersections consist of exactly one vertex, we call T a *perfect transversal*. A perfect transversal of **C**(G) (**S**(G), respectively) will be called a *stable* (*complete*, respectively) *transversal* of G, since a transversal of **C**(G) (**S**(G), respectively) is perfect if and only if it is a maximal stable set (maximal clique, respectively) of G.

A graph is called *c-perfect* (*s-perfect*, respectively) if all its induced subgraphs have a stable (complete, respectively) transversal. Based on the structure A-free-II, we can easily show that every A-free graph G = (V, E) is both *c*-perfect and *s*-perfect graph. Let $V = V_1 + ... + V_c + ... + V_k$ be a partition of its vertex set V which determines the structure A-free-II. Corollary 1 implies that there exists a stable set $S = \{v_1, ..., v_{c-1}, v_{c+1}, ..., v_k\}$ of k-1 cardinality such that $v_i \in V_i$, where $1 \leq i \leq k$ and $i \neq k$. By Lemma 1, S is a maximal stable set. Moreover, the vertex set $C = V_i \cup V_c$ is a maximal clique, $1 \leq i \leq k$ and $i \neq k$ (Note that k-1 equals the number of maximal cliques in G.) It can be easily proved that S is a stable transversal and C is a complete transversal of G. Moreover, a graph is called *t-perfect* if for every induced subgraph H of G, $\alpha(H)$ equals the number of maximal cliques contained in H. Lemma 2 tell us that every incudes subgraph H of an A-free graph G is also an A-free graph having $\alpha(H) = \varkappa(H)$. Thus, we obtain the following results.

Theorem 13. *t*-perfect graphs are precisely the undirected graphs with no actual edges, that is, the A-free graphs.

Corollary 7. A graph is *t*-perfect if and only if it contains no induced subgraph isomorphic to P_4 or C_4 (see also [9]).

Corollary 8. Every A-free graph is *c*-perfect and *s*-perfect graph.

We should point out that it is well-known that *t*-perfectness implies *c*-perfectness and *s*-perfectness but generally, the converse is false. We also point out that the properties of Theorem 2 show us the algorithmic way to compute stable and complete transversals of an A-free graph.

Based on the properties and the structure of the normal product of two A-free graphs, we can prove the main result of [2] with less effort. Specifically, we can prove that the normal product of two *t*-perfect graphs is *c*-perfect. Let G = (V, E) be the normal product of two A-free graphs and $W \subseteq V$. We shall sketch the construction of a stable transversal for the graph G(W)). Here, for simplicity, we consider the case where both A-free graphs have no pair of vertex sets V_i, V_j such that $V_i \sim V_j$, i.e., $G(V_i \cup V_j)$ is an incomplete graph; see property P3 of Theorem 2. The results can be easily extended to prove the general case. Let $V = V_{11} + ... + V_{cc'} + ... + V_{kk'}$ be the partition of the vertex set V of G which satisfy the properties of Theorem 3. We construct a stable set $S \subseteq V$ by selecting one vertex of each of the kk'-2 vertex sets of the class C. Recall that the class C contains all the vertex sets V_{ij} of the partition V such that $i \neq c$ and $j \neq c'$. We can easily show that S is a maximal stable set. By Theorem 3, all maximal cliques of G have the form $V_{ij} \cup V_{cc'} \cup V_{ic'} \cup V_{cj}$, where $1 \leq i, c \leq k$, $1 \leq j, c' \leq k'$. Thus, S is a stable transversal. We can show that if $W = V - \{xy\}$ for every $xy \in a$, where $a = V_{cc'}$, or a is a member of either the class A or B, then S is a maximal stable set of G(W). In the case where $W = V - V_{ij}$, where V_{ij} is a member of the class C, we select a vertex xy from either the set $a \in$ A or $a \in$ B such that $N(a) \cap C \neq \emptyset$, and we add it in S. The previous discussion and Theorem 10 imply the following result.

Theorem 14. The normal product of two t-perfect graphs is a c-perfect graph.

5. Other Properties of A-free Graphs

We have shown that many classes of perfect graphs properly contain the class of A-free graphs. Moreover, we have shown that important vertex sets of the normal product of two A-free graphs can be easily constructed. Next, we identify the precise structure possessed by certain subsets of the vertices and/or edges of an A-free graph in the case where it is a block, split or threshold graph.

5.1. Block Graphs

A graph G is called *block* graph if it is connected and every block (i.e., maximal 2-connected subgraph) is complete [3]. Howorka [14] offered the following purely metric characterization: a connected graph is a block graph if and only if its distance function d satisfies the four-point condition, i.e., for any four vertices u, v, x, y, the larger two of the distance sum

$$d(u, v) + d(x, y), \quad d(u, x) + d(v, y), \quad d(u, y) + d(v, x)$$

are equal.

Unfortunately, all the A-free graphs do not satisfy the above four-point condition. For example, the A-free graph $K_2 + 2K_1$ give distance sums 2, 2 and 3. The next theorem provide us with another type of metric characterization, namely, via forbidden isometric subgraphs.

Theorem 15 (Bandelt and Mulder [1986]). Let G be a connected graph with distance function d. Then, the following statements are equivalent:

(i) G is a block graph;
(ii) d satisfies the four-point condition;
(iii) neither K_4 minus an edge nor C_n with $n \geq 4$ is an isometric subgraph of G;

We focus on statements (i) and (iii) of Theorem 15. By definition, an A-free graph does not contain subgraphs isomorphic to C_n with $n \geq 4$, and therefore, it does not contain C_n ($n \geq 4$) as an isomorphic subgraph. It is easy to see that a graph is a block graph if and only if it is chordal and each edge appears only in one clique.

Theorem 16. Let G = (V, E) be an A-free graph and let $|V_c| = 1$, where V_c is a clique satisfying the properties of Theorem 1 (or 2). Then G is a block graph if and only if there exists no semi-free edge (x, y) in G such that $x, y \notin V_k$.

Proof. (⇒) Let u be the vertex of set V_c. Suppose that there exists a semi-free edge (x, y) in G such that such that $x, y \notin V_c$. This implies that $x \in V_i$ and $y \in V_j$ where $i \neq j$. Since (x, y) is a semi-free edge, there exists a vertex $z \in V_p$, where $p \neq i$ and $p \neq j$, having the property $(z, x) \in E$ (or $(z, y) \in E$). Obviously, (x, u) appears in more than one clique; an absurd. (⇐) It is easy to see that $N(z) = V_i \cup \{u\}$ for every $z \in V_i$ and $i \neq c$, where $u \in V_c$. Since u is a cutpoint and G is a chordal graph, there follows that G is a block graph. □

Theorem 17. Let G = (V, E) be an A-free graph and let $|V_c| > 1$, where V_c is a clique satisfying the properties of Theorem 1 (or 2). Then G is a block graph if and only if G is a complete graph.

Proof. (⇒) Let (u, v) be an A-free edge such that $u, v \in V_c$. Suppose that G is not a complete graph, and let x, y be two vertices such that $(x, y) \notin E$. Then, it is easy to see that G contains an induced subgraph $K_2 + 2K_1$ (a K_4 minus an edge), i.e., $G(\{u, v, x, y\})$. Thus, edge (u, v) appears in more than one clique, and therefore, G is not a block graph; an absurd. (⇐) Obviously, G is a block graph. □

5.2. Split Graphs

An undirected graph G = (V, E) is defined to be *split* if there is a partition V = K + S of its vertex set V into a complete set K and a stable set S.

It is well know that split graphs are characterized in terms of the properties T and T^c, i.e., split graphs ≡ T + T^c (see Foldes and Hammer [7]). That is, a graph G is a split graph if and only if G and its complement G^c are chordal graphs.

Theorem 18 (Foldes and Hammer [1977]). Let G be a undirected graph. The following conditions are equivalent:

(i) G is a split graph;
(ii) G and G^c are chordal graphs;
(iii) G contains no induced subgraph isomorphic to $2K_2$, C_4, or C_5;

Unfortunately, A-free graphs do not satisfy the property T^c since the complement of a split graph is not always a chordal graph. For example, the complement of the graph $2K_2$, which is the graph C_4, obviously is not a chordal graph. Therefore, in the context of this work, statements (i) and (ii) seems not to give us any useful information. On the other hand, statements (i) and (iii) provide us with a characterization of split graphs in terms of forbidden induced subgraphs. It is easy to see that an A-free graph contains no induced subgraph isomorphic to C_4 or C_5; see the structure A-free-I or A-free-II. Thus, we obtain the following result on split graphs.

Theorem 19. Let G be an A-free graph. Then G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2$.

5.3. Threshold Graphs

The class of *threshold graphs*, a well-known class of perfect graphs, is defined to contain those graphs where stable subsets of their vertex sets can be distinguished by using a single linear inequality. Equivalently, a graph G = (V, E) is threshold if there exists a threshold assignment

$[\alpha, t]$ consisting of a labelling α of the vertices by non-negative integers and an integer threshold t such that: S is a stable set iff $\alpha(v_1) + \alpha(v_2) + ... + \alpha(v_p) \leq t$, where $v_i \in$ S, $1 \leq i \leq p$ and S $\subseteq$ V.

We have seen that most of the classes of perfect graphs we consider are characterized by forbidden (isometric in some cases) subgraphs. Chvátal and Hammer [5] have characterized the threshold graphs as the graphs which contain no induced subgraphs isomorphic to $2K_2$, P_4 or C_4.

Theorem 20 (Chvátal and Hammer [1973]). Let G be a undirected graph. Then the following statements are equivalent:

(i) G is a threshold graph;

(ii) G has no induced subgraph isomorphic to $2K_2$, P_4, or C_4;

We have proved that an A-free graph contains no induced subgraph isomorphic to P_4 or C_4; see Theorem 5. By combining these results with the results of Theorem 20, we obtain the following theorem.

Theorem 21. Let G be an A-free graph. Then G is a threshold graph if and only if G contains no induced subgraph isomorphic to $2K_2$.

6. Conclusions

In this paper we classified the edges of a graph as either free, semi-free or actual, we defined the class of A-free graphs, i.e., the class of all the graphs with no actual edges, and we proved that the members of this class possess several important structural and algorithmic properties. Moreover, we showed important structural properties for the normal product of two A-free graphs. Based on the fact that many classes of perfect graphs are characterized in terms of similar properties and forbidden induced subgraphs, we proved that A-free graphs belong to the class of domination perfect, chordal (or triangulated), cographs (or complement reducible), ptolemaic, distance-hereditary, comparability, cocomparability, interval, permutation and (t, c, s)-perfect graphs. Furthermore, recognition properties for block, split and threshold graphs containing no actual edges have been also shown, leading to a constant-time parallel recognition algorithm. The recognition of an A-free graph can be easily done in constant-time by using a powerful parallel model of computation.

We are currently studying other recognition properties and characterizations of A-free graphs in order to extend classes of perfect and/or non perfect graphs in which they might belong. We hope our study will also enable us to further extend classes of perfect graphs whose members can be recognized in parallel constant-time.

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