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Introduction

The goal of this work is the production of Navier eigenvectors in cylindrical coordinates. It is well known [1] that in spherical geometry there exists a complete set of vector functions, the Navier eigenvectors, in the space of solutions of the time-independent reduced equation of dynamic elasticity. In many problems of elasticity having cylindrical structure a similar complete set of vector functions is needed in order to have a specific representation for every particular solution of equation of elasticity in form suitable to satisfy the boundary conditions of the problem.

The usefulness of this complete set emerges from the fact that it is much better from the application point of view to have a representation of a differential equation solution through a specific basis than to consider it as a formal function satisfying the equation. Especially this is true for boundary value problems of partial differential equations. The reason for this is that the basis representation transfers the problem of the determination of a function (the solution of the problem) to the problem of determination of its coefficients with respect to the specific basis. Consequently the differential equation problem is transformed to an algebraic problem and the satisfaction of boundary conditions forces these coefficients to satisfy some kind of linear non-homogeneous systems whose solution is much easier than other kind approach.

Especially for the equation of elasticity, even under the above mentioned representation schema there are two alternatives. The first one considers scalar basis functions and so requires vector coefficients to reproduce the vector elastic fields. The second one, which is proposed in our work, considers a basis of vector functions, the Navier eigenfunctions, and consequently the coefficients in this type of expansions require scalar coefficients, fact rendering the algebraic solution approach much more simple and efficient. The usefulness and application of Navier eigenvectors in boundary value problems in elasticity is clear in Ref. 2 and 3 where the dynamic characteristics of elastic structures of cylindrical symmetry are studied.

The construction of Navier eigenvectors is accomplished in two stages: first the application of Helmholtz decomposition expresses every elastic field as the superposition of a solenoidal transverse field and an irrotational longitudinal one. Following separation of variables techniques in cylindrical geometry we get the most general representations of these two fields mentioned above assuring completeness based on the fact that these fields satisfy Helmholtz equation with different though wavenumbers. Consequently every elastic field is represented through the constructed eigenvectors.

Construction of Navier Eigenvectors

We consider the time independent reduced equation of dynamic elasticity

$$\mu\nabla^2 \boldsymbol{u}(\boldsymbol{r}) + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}(\boldsymbol{r})) + \rho\omega^2\boldsymbol{u}(\boldsymbol{r}) = 0, \quad \boldsymbol{r}\in V, \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{r})$ is the displacement field characterising the harmonic motion of the elastic material defined completely by Lamè constants λ, μ and density ρ. The displacement field $\boldsymbol{u}(\boldsymbol{r})$ is decomposed as follows:

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{u}^p(\boldsymbol{r}) + \boldsymbol{u}^s(\boldsymbol{r}) \tag{2}$$

where

$$\nabla^2\boldsymbol{u}^p(\boldsymbol{r}) + k_p^2\boldsymbol{u}^p(\boldsymbol{r}) = 0, \quad \nabla\times\boldsymbol{u}^p(\boldsymbol{r}) = \boldsymbol{0} \tag{3}$$

$$\nabla^2\boldsymbol{u}^s(\boldsymbol{r}) + k_s^2\boldsymbol{u}^s(\boldsymbol{r}) = 0, \quad \nabla\cdot\boldsymbol{u}^s(\boldsymbol{r}) = \boldsymbol{0} \tag{4}$$

and

$$k_p = \frac{\Omega}{c_p}, \quad k_s = \frac{\Omega}{c_s}$$

where $c_p = \left(\dfrac{\lambda+2\mu}{\rho}\right)^{1/2}, \quad c_s = \left(\dfrac{\mu}{\rho}\right)^{1/2}.$

Decomposition (2) is known as Helmholtz decomposition and its verification is based on the remark that if

$$\boldsymbol{v}(\boldsymbol{r}) = -\frac{1}{4\pi}\int_V \frac{\boldsymbol{u}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d\sigma(\boldsymbol{r}') \tag{5}$$

then according to potential theory

$$\boldsymbol{u}(\boldsymbol{r}) = \nabla^2\boldsymbol{v}(\boldsymbol{r}). \tag{6}$$

Furthermore, using vector analysis arguments

$$\nabla^2\boldsymbol{v}(\boldsymbol{r}) = \nabla(\nabla\cdot\boldsymbol{v}(\boldsymbol{r})) + \nabla\times[-\nabla\times\boldsymbol{v}(\boldsymbol{r})]. \tag{7}$$

Combining equations (6) and (7) we obtain

$$\boldsymbol{u}(\boldsymbol{r}) = \nabla(\nabla \cdot \boldsymbol{v}(\boldsymbol{r})) + \nabla \times [-\nabla \times \boldsymbol{v}(\boldsymbol{r})]. \tag{8}$$

On the other hand applying on equation (1) the operators $\nabla(\nabla \cdot\ )$ and $\nabla \times (\nabla \times\ )$ we get

$$\nabla^2[\nabla(\nabla \cdot \boldsymbol{v}(\boldsymbol{r}))] + k_p^2 \nabla(\nabla \cdot \boldsymbol{v}(\boldsymbol{r})) = \boldsymbol{0} \tag{9}$$

and

$$\nabla^2[\nabla \times (\nabla \times \boldsymbol{v}(\boldsymbol{r}))] + k_s^2 [\nabla \times (\nabla \times \boldsymbol{v}(\boldsymbol{r}))] = \boldsymbol{0}. \tag{10}$$

Equations (8), (9) and (10) lead immediately to representation (2).

The irrotational character of $\boldsymbol{u}^p(\boldsymbol{r})$ imposes that there exists a scalar function $\Psi^p(\boldsymbol{r})$ such that $\boldsymbol{u}^p(\boldsymbol{r}) = \nabla \Psi^p(\boldsymbol{r})$. Introducing this expression in equation (3) we find that $\Psi^p(\boldsymbol{r})$ satisfies the scalar Helmholtz equation with wave number k_p.

The solenoidal character of $\boldsymbol{u}^s(\boldsymbol{r})$ imposes that two possible forms of this field exist:

$\nabla \Psi^s(\boldsymbol{r}) \times \underline{\hat{\alpha}}$ and $\nabla \times (\nabla \Psi^s(\boldsymbol{r}) \times \underline{\hat{\alpha}})$

where $\Psi^s(\boldsymbol{r})$ satisfies again the scalar Helmholtz equation with wave number k_s. The vector $\underline{\hat{\alpha}}$ is in general an arbitrary constant unit vector, though there are cases where it can be taken variant vector [4].

We have to determine the most general functions $\Psi^p(\boldsymbol{r})$ and $\Psi^s(\boldsymbol{r})$ introduced previously and replace them in the already mentioned expressions of $\boldsymbol{u}^p(\boldsymbol{r})$ and $\boldsymbol{u}^s(\boldsymbol{r})$ in order to get all possible forms of the displacement fields and to construct a complete set of vector eigenfunctions for the equation of elasticity.

The separation of variables technique for the cylindrical coordinate systems leads to the following representations of Ψ^t, $t = p,s$

$$\Psi_j^{m,l,t}(\boldsymbol{r};\lambda) = \Phi_{|m|}^l (x_t^j r) e^{im\varphi} Z_j(z;\lambda) \tag{11}$$

where $t = p,s$ stands for the longitudinal and transverse field, respectively, $m = 0, \pm 1, \pm 2, \ldots$,

$$\Phi_m^l(x) = \begin{cases} \begin{cases} J_m(x) & x \in \boldsymbol{R} \\ I_m\left(\dfrac{x}{i}\right) & x \in \boldsymbol{I} \end{cases} & \text{if } l = 1 \\ \begin{cases} Y_m(x) & x \in \boldsymbol{R} \\ K_m\left(\dfrac{x}{i}\right) & x \in \boldsymbol{I} \end{cases} & \text{if } l = 2 \end{cases},$$

$j = 1,2,3,4 \quad \lambda \in \boldsymbol{R}^+$ with

$$Z_1(z;\lambda)=\sin(\lambda z),\ \ Z_2(z;\lambda)=\sinh(\lambda z),$$
$$Z_3(z;\lambda)=\cos(\lambda z),\ \ Z_4(z;\lambda)=\cosh(\lambda z)'$$

and

$$x_t^j=\begin{cases}\sqrt{k_t^2-\lambda^2} & \text{if } j=1,2\\ \sqrt{k_t^2+\lambda^2} & \text{if } j=3,4\end{cases}.$$

Applying the procedure described previously, after selecting $\underline{\hat{\alpha}}\equiv\underline{\hat{z}}$, as it is induced by the cylindrical geometry of the problem, we get the following expansions for the eigenvectors under examination,

$$\begin{aligned}\boldsymbol{L}_j^{m,l}(\boldsymbol{r};\lambda)&=\frac{1}{x_p^j}\nabla\Psi_j^{m,l,p}(\boldsymbol{r};\lambda)\\&=\dot{\Phi}_m^l(x_p^jr)\boldsymbol{P}_j^m(\varphi,z;\lambda)+\frac{\Phi_m^l(x_p^jr)}{x_p^jr}\left(im\boldsymbol{B}_j^m(\varphi,z;\lambda)+r\boldsymbol{C}_j^m(\varphi,z;\lambda)\right)\end{aligned}\tag{12}$$

$$\begin{aligned}\boldsymbol{M}_j^{m,l}(\boldsymbol{r};\lambda)&=\frac{1}{x_s^j}\nabla\Psi_j^{m,l,s}(\boldsymbol{r};\lambda)\times\underline{\hat{z}}\\&=\frac{\Phi_m^l(x_s^jr)}{x_s^jr}im\boldsymbol{P}_j^m(\varphi,z;\lambda)-\dot{\Phi}_m^l(x_s^jr)\boldsymbol{B}_j^m(\varphi,z;\lambda)\end{aligned}\tag{13}$$

$$\begin{aligned}\boldsymbol{N}_j^{m,l}(\boldsymbol{r};\lambda)&=\frac{1}{x_s^j}\frac{\partial}{\partial z}\nabla\times\boldsymbol{M}_j^{m,l}(\boldsymbol{r};\lambda)\\&=(-1)^j\lambda^2\dot{\Phi}_m^l(x_s^jr)\boldsymbol{P}_j^m(\varphi,z;\lambda)+(-1)^jim\lambda^2\frac{\Phi_m^l(x_s^jr)}{x_s^jr}\boldsymbol{B}_j^m(\varphi,z;\lambda)\\&+x_s^j\Phi_m^l(x_s^jr)\boldsymbol{C}_j^m(\varphi,z;\lambda)\end{aligned}\tag{14}$$

where

$$\begin{aligned}\boldsymbol{P}_j^m(\varphi,z;\lambda)&=e^{im\varphi}Z_j(z;\lambda)\underline{\hat{r}}\\\boldsymbol{B}_j^m(\varphi,z;\lambda)&=e^{im\varphi}Z_j(z;\lambda)\underline{\hat{\varphi}}\quad,\\\boldsymbol{C}_j^m(\varphi,z;\lambda)&=e^{im\varphi}\frac{\partial}{\partial z}Z_j(z;\lambda)\underline{\hat{z}}\end{aligned}$$

and $\dot{\Phi}_m^l(x)$ stands for the derivative with respect to its argument.

Notice that instead of using $\frac{1}{x_s^j}\nabla\times \boldsymbol{M}_j^{m,l}(\boldsymbol{r};\lambda)$ for $\boldsymbol{N}_j^{m,l}(\boldsymbol{r};\lambda)$ we have used $\frac{\partial}{\partial z}\left(\frac{1}{x_s^j}\nabla\times \boldsymbol{M}_j^{m,p}(\boldsymbol{r};\lambda)\right)$ which satisfies also the vector Helmholtz equation (the operator $\frac{\partial}{\partial z}$ commutes with ∇^2) and constitutes a vector function independent of $\boldsymbol{M}_j^{m,l}(\boldsymbol{r};\lambda)$.

This choice is based on the fact that the finally chosen $\boldsymbol{N}_j^{m,l}(\boldsymbol{r};\lambda)$ is expressible in terms of vector functions $\boldsymbol{P}_j^m$, $\boldsymbol{B}_j^m$, $\boldsymbol{C}_j^m$ instead of $\frac{1}{x_s^j}\nabla\times \boldsymbol{M}_j^{m,l}(\boldsymbol{r};\lambda)$, which is not.

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