NP-completeness Results for some Problems on Subclasses of Bipartite and Chordal Graphs

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Abstract: Extending previous NP-completeness results for the harmonious coloring problem and the pair-complete coloring problem on trees, bipartite graphs and cographs, we prove that these problems are also NP-complete on connected permutation bipartite graphs. We also study the k-path partition problem and, motivated by a recent work of Steiner [23], where he left the problem open for the class of convex bipartite graphs, we prove that the k-path partition problem is NP-complete on convex bipartite graphs. Moreover, we study the complexity of these problems on two well-known subclasses of chordal graphs namely quasi-threshold and threshold graphs. Based on the work of Bodlaender [3], we show NP-completeness results for the pair-complete coloring and harmonious coloring problems on quasi-threshold graphs. Concerning the k-path partition problem, we prove that it is also NP-complete on this class of graphs. It is known that both the harmonious coloring problem and the k-path partition problem are polynomially solvable on threshold graphs. We show that the pair-complete coloring problem is also polynomially solvable on threshold graphs by describing a linear-time algorithm.

Keywords: Harmonious coloring, pair-complete coloring, k-path partition, permutation bipartite graphs, convex bipartite graphs, quasi-threshold graphs, threshold graphs, NP-completeness.

1 Introduction

A *harmonious coloring* of a simple graph G is a proper vertex coloring such that each pair of colors appears together on at most one edge, while a *pair-complete coloring* of G is a proper vertex coloring such that each pair of colors appears together on at least one edge; the *harmonious chromatic number* $h(G)$ of the graph G is the least integer k for which G admits a harmonious coloring with k colors and its *achromatic number* $\psi(G)$ is the largest integer k for which G admits a pair-complete coloring with k colors.

Harmonious coloring developed from the closely related concept of line-distinguishing coloring which was introduced independently by Frank et al. [10] and by Hopcroft and Krishnamoorthy [15] who showed that the harmonious coloring problem is NP-complete on general graphs. The achromatic number was introduced by Harary et al. [13, 14], while the pair-complete coloring problem was proved to be NP-hard on arbitrary graphs by Yannakakis and Gavril [26]. The complexity of both problems has been extensively studied on various classes of perfect graphs such as cographs, interval graphs,

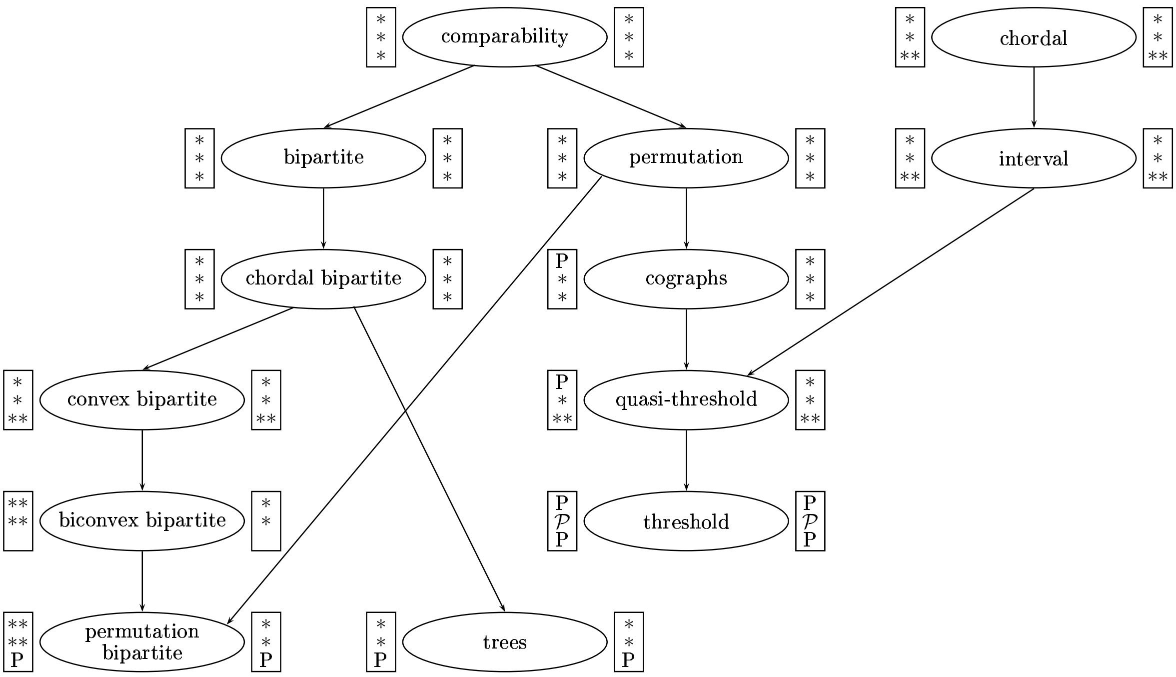

Figure 1: The complexity status of three problems for some graph subclasses of comparability and chordal graphs. $A \rightarrow B$ indicates that class A contains class B. The box to the left (resp. right) of each class contains the status of the harmonious coloring (top), pair-complete coloring (middle) and k-path partition (bottom) problems on connected (resp. disconnected) graphs. ($*$): NP-complete, previously known; ($**$): NP-complete, new result; (P): polynomial, previously known; ($\mathcal{P}$): polynomial, new result.

bipartite graphs and trees [2, 12]; see Fig. 1 for their complexity status[1]. Bodlaender [3] provides a proof for the NP-completeness of the pair-complete coloring problem for disconnected cographs and disconnected interval graphs, and extends his results for the connected cases. His proof also establishes the NP-hardness of the harmonious coloring problem for disconnected interval graphs and disconnected cographs. It is worth noting that the problem of determining the harmonious chromatic number of a connected cograph is trivial, since in such a graph each vertex must receive a distinct color as it is at distance at most 2 from all other vertices [4]. Bodlaender's results establish the NP-hardness of the pair-complete coloring problem for the class of permutation graphs and, also, the NP-hardness of the harmonious coloring problem when restricted to disconnected permutation graphs. Extending the above results, Asdre et al. [1] show that the harmonious coloring problem remains NP-complete on connected interval and permutation graphs.

Concerning the class of bipartite graphs and subclasses of this class (see Fig. 1), Farber et al. [9] show that the harmonious coloring problem and the pair-compete coloring problem are NP-complete for the class of bipartite graphs. In addition, Edwards et al. [7, 8] show that these problems are NP-complete for trees. Their results also establish the NP-completeness of these problems for the classes of convex bipartite graphs and disconnected permutation bipartite graphs. However, the complexity of these problems for connected permutation bipartite graphs and biconvex bipartite graphs is not straightforward.

[1]Figure 1 shows a diagram of class inclusions for a number of graph classes, subclasses of comparability and chordal graphs, and the current complexity status for the harmonious coloring problem, the pair-complete coloring problem, and the k-path partition problem on these classes; for definitions of the classes shown, see [2, 12].

Motivated by this issue we prove that the harmonious coloring problem and the pair-complete coloring problem is NP-complete for connected permutation bipartite graphs, and thus, the same holds for the class of biconvex bipartite graphs. Moreover, based on Bodlaender's results [3], we show that the pair-complete coloring problem is NP-complete for quasi-threshold graphs and that the harmonious coloring problem is NP-complete for disconnected quasi-threshold graphs. It has been shown that the harmonious coloring problem is polynomially solvable on threshold graphs. In this paper we show that the pair-complete coloring problem is also polynomially solvable on this class by proposing a simple linear-time algorithm.

We also study the k-path partition problem, a generalization of the path partition problem [11]; the *path partition problem* is to determine the minimum number of paths in a path partition of a simple graph G, while a path partition of G is a collection of vertex disjoint paths $P_1, P_2, \dots, P_r$ in G whose union is $V(G)$. A path partition is called a *k-path partition* if none of the paths has length more than k, for a given positive integer k. The *k-path partition problem* is to determine the minimum number of paths in a k-path partition of a graph G. It is a natural graph problem with applications in broadcasting in computer and communications networks [23, 25] and it is NP-complete for general graphs [11]. Yan et al. [25] gave a polynomial time algorithm for finding the minimum number of paths in a k-path partition of a tree, while Steiner [24] showed that the problem is NP-complete even for cographs if k is considered to be part of the input, but it is polynomially solvable if k is fixed; he also presented a linear-time solution for the problem, with any k, for threshold graphs. Quite recently, Steiner [23] showed that the k-path partition problem remains NP-complete on the class of chordal bipartite graphs if k is part of the input and on the class of comparability graphs even for $k = 3$. Furthermore, he presented a polynomial time solution for the problem, with any k, on permutation bipartite graphs and left the problem open for the class of convex bipartite graphs.

Motivated by Steiner's work [23], we prove that the k-path partition problem is NP-complete on convex bipartite graphs. Furthermore, we show that this problem is NP-complete for quasi-threshold graphs, and thus, it is also NP-complete for interval and chordal graphs. For some graph classes, the complexity status of the k-path partition problem is illustrated in Fig. 1.

Our work is organized as follows. In Section 2 we show that the harmonious coloring problem and the pair-complete coloring problem are NP-complete on permutation bipartite graphs, and in Section 3 we show that the k-path partition problem is NP-complete on convex bipartite graphs, a superclass of permutation bipartite graphs. In Section 4 we present structural properties of the class of quasi-threshold graphs and NP-completeness results on this class, while in Section 5 we describe a simple linear-time algorithm for the pair-complete coloring problem on threshold graphs. Finally, Section 8 concludes the paper and discusses open problems.

2 Permutation Bipartite Graphs

The formulations of the harmonious coloring problem and the pair-complete coloring problem in [4] are equivalent to the following formulations.

Harmonious Coloring Problem
Instance: Graph $G = (V, E)$, positive integer $K \leq |V|$.
Question: Is there a positive integer $k \leq K$ and a proper coloring using k colors such that each pair of colors appears together on at most one edge?

Pair-complete Coloring Problem
Instance: Graph $G = (V, E)$, positive integer $K \leq |V|$.
Question: Is there a positive integer $k \geq K$ and a proper coloring using k colors such that each pair of colors appears together on at least one edge?

We next prove our main result, that is, the harmonious coloring problem is NP-complete for connected permutation bipartite graphs. A bipartite graph $G = (X, Y; E)$ is a *permutation bipartite graph* if and only if it has a strong ordering of its vertices [22]; a *strong ordering* of the vertices of $G = (X, Y; E)$ is an ordering $\{x_1, x_2, \ldots, x_r\}$ of the vertices in X and an ordering $\{y_1, y_2, \ldots, y_r\}$ of the vertices in Y such that whenever $x_i y_\ell, x_j y_m \in E$ with $i < j$ and $\ell > m$ then we also have $x_i y_m, x_j y_\ell \in E$ [22].

Theorem 2.1. *The harmonious coloring problem is NP-complete when restricted to connected permutation bipartite graphs.*

Proof. Harmonious coloring is obviously in NP. In order to prove NP-hardness, we use a transformation from 3-PARTITION.

Let a set $A = \{a_1, \ldots, a_{3m}\}$ of $3m$ elements, a positive integer b and let positive integer sizes $s(a_i)$ for each $a_i \in A$ be given, such that $\frac{1}{4}b < s(a_i) < \frac{1}{2}b$, and such that $\sum_{a_i \in A} s(a_i) = mb$, $1 \le i \le 3m$. We may suppose that, for each $a_i \in A$, $s(a_i) > m$ (if not, then we can multiply all $s(a_i)$ and b with $m + 1$).

We construct the following connected graph which is a permutation bipartite graph: Consider a set $M = \{m_1, m_2, \ldots, m_m\}$ of m vertices, a set $B = b_1, b_2, \ldots, b_b$ of b vertices, and add a vertex v that is connected to every vertex in the two sets. We add a set $M' = \{m'_1, m'_2, \ldots, m'_{m-1}\}$ of $m - 1$ vertices and a set $B' = \{b'_1, b'_2, \ldots, b'_{b-1}\}$ of $b - 1$ vertices we connect them to the vertices of M and B as follows: we connect each vertex m'_i, $1 \le i \le m - 1$, to the vertices $m_{i+1}, m_{i+2}, \ldots, m_m$, and each vertex b_i $1 \le i \le b - 1$, to the vertices $b'_{i+1}, b'_{i+2}, \ldots, b'_b$. Next we construct for every $a_i \in A$ a tree T_i of depth one with $s(a_i)$ leaves, namely $y^i_1, y^i_2, \ldots, y^i_{s(a_i)}$, and root x_i, that is, every leaf is adjacent to the root; note that there are $3m$ such trees $T_1, T_2, \ldots, T_{3m}$. Then we add a set $P = \{p_1, p_2, \ldots, p_{3m}\}$ of $3m$ vertices, and we connect each vertex p_i to the root x_i of the tree T_i, $1 \le i \le 3m$. We also connect p_i, $2 \le i \le 3m$, to the $s(a_{i-1})$ leaves of the tree T_{i-1}. The vertex p_1 is also connected to the vertices of M' and the vertex v. Additionally, for each vertex $p_i \in P$, $2 \le i \le 3m$, we add vertices v^i_j, $1 \le j \le m - 1 + b - s(a_{i-1}) + 1 + 3m - i$ and connect them to vertex p_i. We also add vertices v^1_j, $1 \le j \le b + 3m - 1$ and connect them to the vertex p_1; let G be the resulting graph. The graph G is a connected graph and it is illustrated in Fig. 2.

One can easily verify that the graph G is a bipartite graph; let X and Y be its two stable sets. It is easy to show that the graph $G = (X, Y; E)$ admits a strong ordering of its vertices, and, thus, it is a permutation bipartite graph. Let $\mathcal{X}$ and $\mathcal{Y}$ be the orderings of the vertices of X and Y, respectively. We define $\mathcal{X}$ and $\mathcal{Y}$ as follows:

$$\begin{aligned} \mathcal{X} &= \{b'_2, b'_3, \ldots, b'_b, v, m'_1, m'_2, \ldots, m'_{m-1}, v^1_1, v^1_2, \ldots, v^1_{b+3m-1}, \mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_5, \ldots, \mathcal{X}_{3m-2}, x_{3m}\} \\ \mathcal{Y} &= \{b_1, b_2, \ldots, b_b, m_1, m_2, \ldots, m_m, \mathcal{Y}_1, \mathcal{Y}_3, \mathcal{Y}_5, \ldots, \mathcal{Y}_{3m-2}, y^{3m}_1, y^{3m}_2, \ldots, y^{3m}_{s(a_{3m})}\} \end{aligned}$$

where $\mathcal{X}_i = \{x_i, p_{i+1}, y^{i+1}_1, y^{i+1}_2, \ldots, y^{i+1}_{s(a_{i+1})}, v^{i+2}_1, v^{i+2}_2, \ldots, v^{i+2}_{4m+b-s(a_{i+1})-i-2}\}$, $i = 1, 3, 5, \ldots, 3m - 2$, and $\mathcal{Y}_i = \{x_i, y^i_1, y^i_2, \ldots, y^i_{s(a_i)}, v^{i+1}_1, v^{i+1}_2, \ldots, v^{i+1}_{4m+b-s(a_i)-i-1}, x_{i+1}\}$, $i = 1, 3, 5, \ldots, 3m - 2$.

It is easy to see that the total number of edges in G is

$$\binom{m}{2} + \binom{b}{2} + m + b + 3m^2 + 3mb + 3m + mb + \sum_{i=1}^{3m-1} i = \binom{4m + b + 1}{2}$$

For every harmonious coloring of G and every pair of distinct colors i, j, $i \neq j$, there must be at most one edge with its endpoints colored with i and j. Thus, it follows that the harmonious chromatic number cannot be less than $4m + b + 1$, and if it is equal to $4m + b + 1$ then we have, for every pair of distinct colors i, j, $1 \le i, j \le 4m + b + 1$, a unique edge with its end-points colored with i and j.

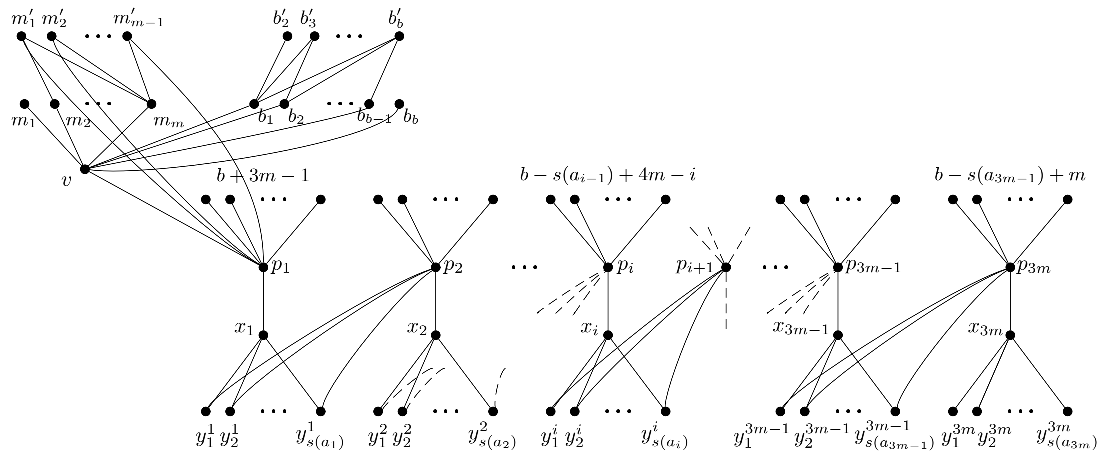

Figure 2: Illustrating the constructed connected permutation bipartite graph G.

Thus, we have an exact coloring of G; an *exact coloring* of G with k colors is a harmonious coloring of G with k colors in which, for each pair of colors i, j, there is exactly one edge ab such that a has color i and b has color j.

We now claim that the harmonious chromatic number of G is (less or equal to) $4m + b + 1$ if and only if A can be partitioned in m sets $A_1, \dots, A_m$ such that $\sum_{a \in A_j} s(a) = b$, for all j, $1 \le j \le m$.

($\Longleftarrow$) Suppose now a 3-partition of A in $A_1, \dots, A_m$ such that $\forall j : \sum_{a \in A_j} s(a) = b$ exists. We show how to find a harmonious coloring of G using $4m+b+1$ colors. We color the vertices of the sets M and M' with colors $1, 2, \dots, m$, the vertices of the sets B and B' with colors $m+1, m+2, \dots, m+b$, and vertex v with $m+b+1$. For convenience and ease of presentation, let $\mathcal{M}$ be the set containing colors $1, 2, \dots, m$, let $\mathcal{B}$ be the set containing colors $m+1, m+2, \dots, m+b$, and let $\mathcal{K}$ be the set containing colors $m+b+2, m+b+3, \dots, 4m+b+1$. If $a_i \in A_j$ then we color the vertex corresponding to a_i with color j. Each color $j \in \mathcal{M}$ is assigned to the three vertices x_i corresponding to three a_i that have together exactly b neighbors of degree 2. We assign to each one of these b neighbors a different color from $\mathcal{B}$, and next we assign to each vertex p_i of the set P a distinct color from $\mathcal{K}$. Recall that each vertex p_i, $1 \le i \le 3m$, is connected to $m + b + 1 + 3m - i$ vertices (see Fig. 1).

Next, we color the rest $m-1+b-s(a_{i-1})+1+3m-i$ neighbors of each p_i, $1 < i \le 3m$. We assign a distinct color from the set $\mathcal{M} \backslash c_i$ to $m-1$ neighbors of p_i, where c_i is the color previously assigned to the vertex x_i corresponding to a_i. We next assign a distinct color from the set $\mathcal{B} \backslash C_i$ to $b - s(a_{i-1})$ neighbors of p_i, where C_i is the set of the colors previously assigned to $s(a_{i-1})$ neighbors of the vertex x_{i-1} corresponding to a_{i-1}. Finally, we assign a different color to the rest $1 + 3m - i$ neighbors of p_i, using color $m + b + 1$ and the colors assigned to the vertices p_j, $i + 1 \le j \le 3m$. Note that, we have assigned a color to m neighbors of p_1, and, thus, in order to color the rest $b + 3m - 1$ neighbors of p_1, we use colors from $\mathcal{K}$ and $\mathcal{B}$. A harmonious coloring of G using $4m + b + 1$ colors results, and thus, the harmonious chromatic number of G is $4m + b + 1$.

($\Longrightarrow$) We next suppose that the harmonious chromatic number of G is (less or equal to) $4m+b+1$. Consider a harmonious coloring of G using $4m + b + 1$ colors. Without loss of generality we may suppose that the m vertices of the set M have distinct colors from $\mathcal{M}$, while the b vertices of the set B have distinct colors from $\mathcal{B}$. Also, without loss of generality, we color vertex v with color $m+B+1$, since v is adjacent to all the vertices of the two sets, and vertex p_1 with color $c_{p_1} = m + b + 2$. Note that p_1 is the vertex having the maximum degree, that is, $4m+B$, and, thus, color $m+b+2$ is adjacent to all colors, because we color all uncolored neighbors of p_1 with distinct colors from $\mathcal{M} \cup \mathcal{B} \cup \mathcal{K} \backslash c_{p_1}$. We claim that every vertex p_i, $1 < i \leq 3m$, takes a color from $\mathcal{K}$. Indeed, let $c_m \in \mathcal{M}$ be a color assigned to p_2. The degree of vertex p_2 is equal to $4m+b-1$. However, color c_m can be adjacent to $(m-1+b+3m+1)-(1+1) < 4m+b-1$ other colors, and, thus, we need one more color in order to color one more neighbor of p_2. Using similar arguments, we show that vertex p_2 cannot take a color from $\mathcal{B} \cup \{m+b+1, m+b+2\}$, and thus it takes a color from $\mathcal{K} \backslash c_{p_1}$. Recursively, as can easily be proved by induction on i, the same holds for all $p_i \in P$, $2 < i \leq 3m$, that is, p_i takes a color from $\mathcal{K} \backslash \mathcal{L}$, where $\mathcal{L}$ is the set containing colors $c_{p_1}, c_{p_2}, \ldots, c_{p_{i-1}}$, which are the colors already assigned to vertices p_j, $1 \leq j < i$. Note that, if $c_{\mathcal{K}}$ is a color from $\mathcal{K} \cup \{m+b+1\}$, then it cannot be assigned to any other vertex of G since any pair of colors $(c_{\mathcal{K}}, j)$, $1 \leq j \leq 4m+b+1$, already appears in the harmonious coloring. Recall that, for every pair of distinct colors i, j, $1 \leq i, j \leq 4m+b+1$, there is a unique edge with its end-points colored with i and j.

We now show that all the vertices of the set B' receive colors from $\mathcal{B}$. Since each vertex $u_i \in B'$, $2 \leq i \leq b$, is adjacent to at least one vertex in B, none of them can take color $m+b+1$. Let $u \in B'$ be one vertex taking a color from $\mathcal{M}$, and let d_u be its degree, while all the other vertices take colors from $\mathcal{B}$. The number of edges of G having one endpoint colored with a color from $\mathcal{M}$ that have not appeared yet is $mb - d_u$. Also, the number of edges of G having one endpoint colored with a color from $\mathcal{B}$ that have not appeared yet is mb. Thus, the number of pais that have not appeared yet in G, is $mb - d_u + mb - mb = mb - d_u$, while the number of uncolored edges is mb, that is, the edges of the form $x_i y_j^i$, $1 \leq i \leq 3m$, $1 \leq j \leq s(a_i)$. This implies that we need more colors, and consequently, all the vertices of the set B' receive colors from $\mathcal{B}$. Using similar arguments we can show that the vertices of the set M' receive colors from $\mathcal{M}$.

Note that pairs (μ, ν), $\mu \in \mathcal{M}$, $\nu \in \mathcal{B}$, have not appeared yet. Since every pair of colors must appear, we assign these pairs to the mB edges that have both endpoints uncolored. Note that these edges are the edges $x_i y_j^i$, $1 \leq i \leq 3m$, $1 \leq j \leq s(a_i)$, where x_i corresponds to a_i and y_j^i corresponds to the j-th neighbor of x_i having degree 2. The vertices x_i cannot take a color from $\mathcal{B}$, otherwise the $s(a_i) > m$ uncolored neighbors y_j^i cannot be colored with m colors from $\mathcal{M}$. Thus, vertices x_i are assigned a color from $\mathcal{M}$ and vertices y_j^i are assigned a color from $\mathcal{B}$ (recall that $\frac{b}{4} < s(a_i) < \frac{b}{2}$). Note that it is easy to assign a distinct color to the $4m+b-s(a_{i-1}-i$ neighbors of each p_i, $1 < i \leq 3m$ that have degree equal to one; recall that $m-1$ neighbors of p_1 belonging to the set M' are already assigned a color from $\mathcal{M}$. If c_{p_i} is the color of vertex p_i, we use distinct colors from $\mathcal{M} \cup \mathcal{B} \cup \mathcal{K} \backslash \{c_{x_i}, \mathcal{F}, \mathcal{L}, c_{p_i}\}$, where $\mathcal{F}$ is the set containing all colors already assigned to the $s(a_{i-1})+1$ neighbors of p_i and $c_{x_i} \in \mathcal{M}$ is the color already assigned to vertex x_i.

Finally, let $a_i \in A_j$ if and only if the vertex x_i (with neighbors y_j^i) is colored with color $j \in \mathcal{M}$. We claim that for all j, $\sum_{a \in A_j} s(a) = b$. Indeed, each color j must be adjacent to some colors from $\mathcal{B}$, and each color from $\mathcal{B}$ is assigned to exactly one vertex which is adjacent to all x_i colored with j. Hence, a correct 3-partition exists.

The theorem follows from the strong NP-completeness of 3-PARTITION, since the transformation can be done easily in polynomial time. ∎

We have shown that the connected permutation bipartite graph G presented in this paper, has $\binom{4m+b+1}{2}$ edges and $h(G) = 4m+b+1$. In [7] it was shown that if G is a graph with exactly $\binom{k}{2}$ edges, then a proper vertex coloring of G with k colors is pair-complete if and only if it is a harmonious coloring. Thus, if G is a graph with $\binom{k}{2}$ edges, then $\psi(G) = k$ if and only if $h(G) = k$ [4]. Consequently, for the graph G, which is a permutation bipartite graph, we have that $\psi(G) = 4m+b+1$

and, thus, our results also prove that the achromatic number is NP-complete for connected permutation bipartite graphs. Consequently, we can state the following theorem.

Theorem 2.2. *The pair-complete coloring problem is NP-complete when restricted to connected permutation bipartite graphs.*

We have shown that harmonious coloring and pair-complete coloring are NP-complete problems for the class of permutation bipartite graphs. Consequently, the two problems are NP-complete for the class of biconvex bipartite graphs, which properly contains permutation bipartite graphs. A bipartite graph $G = (X, Y; E)$ is *convex* on the vertex set X if X can be ordered so that for each element y in the vertex set Y the elements of X connected to y form an interval of X; G is *biconvex* if it is convex on both X and Y. Consequently, we can state the following result.

Corollary 2.1. *The harmonious coloring problem and the pair-complete coloring problem are NP-complete for biconvex bipartite graphs.*

3 Convex Bipartite Graphs

We next prove that the k-path partition problem is NP-complete for *convex bipartite graphs*; recall that a bipartite graph $G = (X, Y; E)$ is convex on the vertex set X if X can be ordered so that for each element y in the vertex set Y the elements of X connected to y form an interval of X [17].

Theorem 3.1. *The k-path partition problem is NP-complete for convex bipartite graphs.*

Proof. The k-path partition problem is obviously in NP. In order to prove NP-hardness, we use a transformation from Bin-Packing.

Let a set $A = \{a_1, \ldots, a_n\}$ of n elements, a size $s(a_i) \in Z^+$ for each $a_i \in A$, a positive integer bin capacity B and a positive integer K.

We construct the following graph which is a convex bipartite graph: Consider an independent set $S^i = \{s^i_1, s^i_2, \ldots, s^i_{s(a_i)}\}$ of $s(a_i)$ vertices and an independent set $T^i = t^i_1, t^i_2, \ldots, t^i_{s(a_i)-1}$ of $s(a_i) - 1$ vertices for every $a_i \in A$, $1 \le i \le n$. We connect every $t^i_j \in T^i$ to vertices $s^i_j \in S^i$ and $s^i_{j+1} \in S^i$, $1 \le j \le s(a_i)$; let P_i, $1 \le i \le n$ be the resulting disconnected graphs, each containing $2s(a_i)-1$ vertices. Thus, we can associate each P_i with each $a_i \in A$. We add an independent set $C = \{c_1, c_2, \ldots, c_{n-K}\}$ of $n - K$ vertices and we connect each c_j, $1 \le j \le n - K$ to every vertex of all sets S^i, $1 \le i \le n$; let G be the resulting graph. The graph G is a connected graph and it is illustrated in Fig. 3.

One can easily verify that the graph G is a convex bipartite graph; we define the sets X and Y as follows:

$$\begin{aligned} X &= \{s^1_1, s^1_2, \ldots, s^1_{s(a_1)}, s^2_1, s^2_2, \ldots, s^2_{s(a_2)}, \ldots, s^n_1, s^n_2, \ldots, s^n_{s(a_n)}\} \\ Y &= \{t^1_1, t^1_2, \ldots, t^1_{s(a_1)-1}, c_1, c_2, \ldots c_{n-K}, t^2_1, t^2_2, \ldots, t^2_{s(a_2)-1}, \ldots, t^n_1, t^n_2, \ldots, t^n_{s(a_n)-1}\} \end{aligned}$$

Since X is ordered so that for each element y in the vertex set Y the elements of X connected to y form an interval of X, the constructed bipartite graph $G = (X, Y; E)$ of Fig. 3 is convex on the vertex set X.

We now claim that the graph G has a k-path partition into K paths of length at most $k = 2B - 2$ if and only if A can be partitioned into K disjoint sets $A_1, A_2, \ldots, A_K$ such that the sum of the sizes of the items in each A_i is B or less.

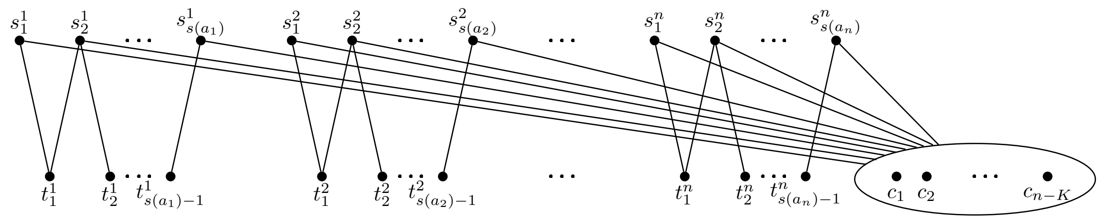

Figure 3: Illustrating the constructed convex bipartite graph G.

($\Longleftarrow$) Suppose now there exists a partition of A in $A_1, \ldots, A_K$ such that the sum of the sizes of the items in each A_i is B or less. We show how to find a k-path partition of G into K paths of length at most $k = 2B-2$. Let α_i be the number of items contained in each A_i, $1 \le i \le K$. We construct n paths of length $2s(a_j) - 2$, $1 \le j \le n$, that is, the paths $p_j = [s_1^j, t_1^j, s_2^j, t_2^j, s_3^j, \ldots, s_{s(a_j)-1}^j, t_{s(a_j)-1}^j, s_{s(a_j)}^j]$, $1 \le j \le n$. Note that each path p_j corresponds to each subgraph P_j of G. Then, we use $\alpha_i - 1$ vertices of the set C to connect the α_i paths corresponding to the elements of the set A_i into one path of length $\alpha_i - 2 - \alpha_i + 2\sum_{a \in A_i} s(a) \le 2B - 2$.

($\Longrightarrow$) We next suppose that G has a $(2B-2)$-path partition into K paths. Since the set X contains $\sum_{i=1}^{n} s(a_i)$ vertices and the set Y contains $\sum_{i=1}^{n} s(a_i) - K$ vertices, then a minimum path partition cannot contain less than K paths. Moreover, since each vertex $t_j^i \in T$ $(1 \le i \le n,\ 1 \le j \le s(a_i) - 1)$ sees only the vertices s_j^i and s_{j+1}^i of X, a path containing vertices of the subgraph P_i can be connected to a path containing vertices of the subgraph $P_{i'}$ only through a vertex of the set C, which contains $n - K$ vertices. We claim that, in order to obtain a path partition of no more than K paths, we first have to construct n paths $p_i = [s_1^i, t_1^i, s_2^i, t_2^i, s_3^i, \ldots, s_{s(a_i)-1}^i, t_{s(a_i)-1}^i, s_{s(a_i)}^i]$, $1 \le i \le n$, and then we have to connect them using vertices of C in such a way that no path contains more than $2B - 1$ vertices; note that both endpoints of each path p_i are in X and each p_i corresponds to a subgraph P_i. Indeed, let q_i be a subpath of p_i and let p_j be the $n-1$ paths corresponding to the $n-1$ subgraphs P_j, where $p_j = [s_1^j, t_1^j, s_2^j, t_2^j, s_3^j, \ldots, s_{s(a_j)-1}^j, t_{s(a_j)-1}^j, s_{s(a_j)}^j]$, $1 \le j \le n$ and $i \ne j$. Then, there exist vertices of the subgraph P_i that are not included in the path q_i, which form a path q_i'. Thus, we have to connect $n+1$ paths using $n-K$ vertices of the set C, which results to $K+1$ paths, a contradiction. Consequently, in order to obtain a path partition of no more than K paths, we first have to construct n paths p_i, $1 \le i \le n$, corresponding to the subgraphs P_i, and then we have to connect them using vertices of C in such a way that no path contains more than $2B-1$ vertices. Let $P' = \{p_1', p_2', \ldots, p_K'\}$ be the set of the paths of the $(2B-2)$-path partition of G. Each one of these K paths contains at most B vertices of X and if a vertex s_ℓ^i, $\ell \in [1, s(a_i)]$ belongs to a certain path then all vertices s_j^i, $1 \le j \le s(a_i)$, belong to the same path. Consequently, the set A can be partitioned into K disjoint sets $A_1, A_2, \ldots, A_K$ such that the sum of the sizes of the items in each A_i is B or less.

The theorem follows from the strong NP-completeness of Bin-Packing, since the transformation can be done easily in polynomial time. ∎

4 Quasi-Threshold Graphs

A graph G is called *quasi-threshold*, or QT-graph for short, if G contains no induced subgraph isomorphic to P_4 or C_4 (cordless path or cycle on 4 vertices); for definition and optimization problems on this class see [12, 16, 18, 20, 21]. The class of quasi-threshold graphs is a subclass of the class of cographs and contains the class of threshold graphs [6, 12]; see Fig. 1.

4.1 Structural properties

Let G be a QT-graph with vertex set $V(G)$ and edge set $E(G)$. The neighborhood $N(x)$ of a vertex $x \in V(G)$ is the set of all the vertices of G which are adjacent to x. The closed neighborhood of x is defined as $N[x] := \{x\} \cup N(x)$. The subgraph of a graph G induced by a subset S of the vertex set $V(G)$ is denoted by $G[S]$. For a vertex subset S of G, we define $G - S := G[V(G) - S]$.

The following lemma follows immediately from the fact that for every subset $S \subset V(G)$ and for a vertex $x \in S$, we have $N_{G[S]}[x] = N[x] \cap S$ and that $G - S$ is an induced subgraph.

Lemma 4.1. [16, 21]: *If G is a QT-graph, then for every subset $S \subset V(G)$, both $G[S]$ and $G[V(G)-S]$ are also QT-graphs.*

The following theorem provides important properties for the class of QT-graphs. For convenience, we define

$$cent(G) = \{x \in V(G) \mid N[x] = V(G)\}.$$

Theorem 4.1. [16, 21]: *The following three statements hold.*

(i) A graph G is a QT-graph if and only if every connected induced subgraph $G[S], S \subseteq V(G)$, satisfies $cent(G[S]) \neq \emptyset$.

(ii) A graph G is a QT-graph if and only if $G[V(G) - cent(G)]$ is a QT-graph.

(iii) Let G be a connected QT-graph. If $V(G) - cent(G[S]) \neq \emptyset$, then $G[V(G) - cent(G)]$ contains at least two connected components.

Let G be a connected QT-graph. Then $V_1 := cent(G)$ is not an empty set by Theorem 4.1. Put $G_1 := G$, and $G[V(G) - V_1] = G_2 \cup G_3 \cup \cdots \cup G_r$, where each G_i is a connected component of $G[V(G) - V_1]$ and $r \geq 3$. Then since each G_i is an induced subgraph of G, G_i is also a QT-graph, and so let $V_i := cent(G_i) \neq \emptyset$ for $2 \leq i \leq r$. Since each connected component of $G_i[V(G_i) - cent(G_i)]$ is also a QT-graph, we can continue this procedure until we get an empty graph. Then we finally obtain the following partition of $V(G)$:

$$V(G) = V_1 + V_2 + \cdots + V_k, \quad \text{where } V_i = cent(G_i).$$

Moreover we can define a partial order $\preceq$ on the set $\{V_1, V_2, \ldots, V_k\}$ as follows:

$$V_i \preceq V_j \ \text{ if } \ V_i = cent(G_i) \ \text{ and } \ V_j \subseteq V(G_i).$$

It is easy to see that the above partition of the vertex set $V(G)$ of the QT-graph G possesses the following properties.

Theorem 4.2. [16, 21]: *Let G be a connected QT-graph, and let $V(G) = V_1 + V_2 + \cdots + V_k$ be the partition defined above; in particular, $V_1 := cent(G)$. Then this partition and the partially ordered set $(\{V_i\}, \preceq)$ have the following properties:*

(P1) If $V_i \preceq V_j$, then every vertex of V_i and every vertex of V_j are joined by an edge of G.

(P2) For every V_j, $cent(G[\{\bigcup V_i \mid V_i \preceq V_j\}]) = V_j$.

(P3) For every two V_s and V_t such that $V_s \preceq V_t, G[\{\bigcup V_i \mid V_s \preceq V_i \preceq V_t\}]$ is a complete graph. Moreover, for every maximal element V_t of $(\{V_i\}, \preceq)$, $G[\{\bigcup V_i \mid V_1 \preceq V_i \preceq V_t\}]$ is a maximal complete subgraph of G.

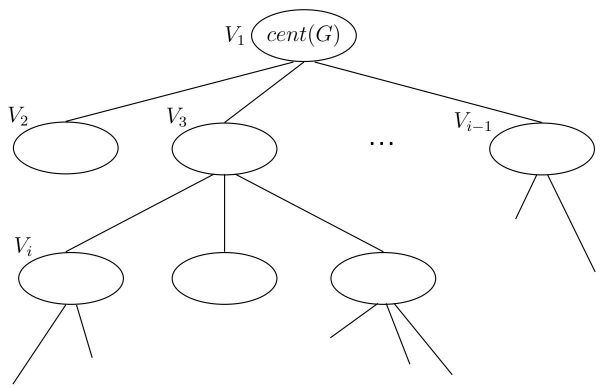

Figure 4: The typical structure of the cent-tree $T_c(G)$ of a QT-graph.

The results of Theorem 4.2 provide structural properties for the class of QT-graphs. We shall refer to the structure that meets the properties of Theorem 4.2 as *cent-tree* of the graph G and denote it by $T_c(G)$. The cent-tree $T_c(G)$ (see Fig. 4) of a QT-graph is a rooted tree; it has nodes $V_1, V_2, \ldots, V_k$, root $V_1 := cent(G)$, and every node V_i is either a leaf or has at least two children. Moreover, $V_s \preceq V_t$ if and only if V_s is an ancestor of V_t in $T_c(G)$. Thus, we can state the following result.

Corollary 4.1. *A graph G is a QT-graph if and only if G has a cent-tree $T_c(G)$.*

Observation 4.1. Let G be a QT-graph and let $V = V_1 + V_2 + \cdots + V_k$ be the above partition of $V(G)$; $V_1 := cent(G)$. Let $S = \{v_s, v_{s+1}, \ldots, v_t, \ldots, v_q\}$ be a stable set such that $v_t \in V_t$ and V_t is a maximal element of $(V_i, \preceq)$ or, equivalently, V_t is a leaf node of $T_c(G)$, $s \leq t \leq q$. It is easy to see that S has the maximum cardinality $\alpha(G)$ among all the stable sets of G. On the other hand, the sets $\{\bigcup V_i \mid V_1 \preceq V_i \preceq V_t\}$, for every maximal element V_t of $(V_i, \preceq)$, provide a clique cover of size $\kappa(G)$ which is the smallest possible clique cover of G; that is $\alpha(G) = \kappa(G)$. Based on the Theorem 4.2 or, equivalently, on the properties of the cent-tree of G, it is easy to show that the clique number $\omega(G)$ equals the chromatic number $\chi(G)$ of the graph G; that is, $\chi(G) = \omega(G)$. □

4.2 NP-completeness results

In order to prove the NP-completeness of the pair-complete coloring problem for cographs and interval graphs, Bodlaender [3] constructs an instance of a disconnected graph which is simultaneously a cograph and an interval graph and modifies it in order to obtain a connected instance of a graph which remains a cograph and an interval graph. One can easily verify that the constructed graphs are also quasi-threshold graphs. Thus, his proof also establishes the NP-hardness of the pair-complete coloring problem for the class of quasi-threshold graphs, as well as the NP-hardness of the harmonious coloring problem for disconnected quasi-threshold graphs. Consequently, we state the following result.

Corollary 4.2. *The pair-complete coloring problem is NP-complete for quasi-threshold graphs; the harmonious coloring problem is NP-complete for disconnected quasi-threshold graphs.*

We next prove that the k-path partition problem is NP-complete for quasi-threshold graphs.

Theorem 4.3. *The k-path partition problem is NP-complete for quasi-threshold graphs.*

Proof. The k-path partition problem is obviously in NP. In order to prove NP-hardness, we use a transformation from 3-PARTITION.

Let a set $A = \{a_1, \ldots, a_{3m}\}$ of $3m$ elements, a positive integer B and let positive integer sizes $s(a_i)$ for each $a_i \in A$ be given, such that $\frac{1}{4}B < s(a_i) < \frac{1}{2}B$, and such that $\sum_{a_i \in A} s(a_i) = mB$, $1 \leq i \leq 3m$. We may suppose that, for each $a_i \in A$, $s(a_i) > m$ (if not, then we can multiply all $s(a_i)$ and b with $m + 1$).

We construct the following graph which is a quasi-threshold graph: Consider a graph $G(V \cup C, E)$ having a clique $K_{a_i}(V_{a_i}, E_{a_i})$ on $s(a_i)$ vertices for each $a_i \in A$ such that $V_{a_i} \cap V_{a_j} = \emptyset$, $i \neq j$, and $V = \bigcup_{a_i \in A} V_{a_i}$. There are no edges in G between vertices in different cliques. In addition, G has $2m$ "connector" vertices $C = \{v_1, v_2, \ldots, v_{2m}\}$ which form a clique in G. Every $v_i \in C$ is connected to every $u \in V$. It is clear that G is a quasi-threshold graph.

We now claim that A has a 3-PARTITION, that is, A can be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{a \in A_i} s(a) = B$ for $1 \leq i \leq m$, if and only if G has a partition into m paths of length $k = B + 2$. Notice that the constraints on the item sizes ensure that each S_i must have exactly three elements from A.

($\Longrightarrow$) If A has a 3-PARTITION $A_i = \{x_i, y_i, z_i\}$, $1 \leq i \leq m$, then we can use the two elements $v_{2i-1}, v_{2i} \in C$ to connect the corresponding subgraphs K_{x_i}, K_{y_i} and K_{z_i} into a path $V_{x_i}, v_{2i-1}, V_{y_i}, v_{2i}$, V_{z_i} of length $B + 2$.

($\Longleftarrow$) We next suppose that G has a $(B+2)$-path partition into m paths, $P_1, P_2, \ldots, P_m$. Since G has $m(B + 2)$ vertices, each P_i must contain exactly $B + 2$ vertices. Because of the size constraints, each P_i must contain at least two connector vertices from C.

We claim that, in order to obtain a path partition of no more than m paths, we first have to construct $3m$ paths $p_1, p_2, \ldots, p_{3m}$ corresponding to the $3m$ cliques, and then we have to connect them using vertices of C in such a way that each path P_i, $1 \leq i \leq m$, contains exactly $B + 2$ vertices. Indeed, let q_k be a subpath of path p_k corresponding to clique K_{a_k} and let p_j be the $3m - 1$ paths corresponding to the rest $3m - 1$ cliques. Then, there exist vertices of clique K_{a_k} that are not included in the path q_k, which form a path q'_k. Thus, we have to connect $3m + 1$ paths using $2m$ vertices of the set C, which results to $m + 1$ paths, a contradiction. Consequently, in order to obtain a path partition of m paths, we first have to construct $3m$ paths p_i, $1 \leq i \leq 3m$, corresponding to the cliques K_{a_i}, and then we have to connect them using vertices of C in such a way that each path contains exactly $B + 2$ vertices.

Since we have $3m$ paths, corresponding to $3m$ cliques, and $2m$ connectors, each P_i must contain exactly two connector vertices. We claim that none of the paths P_i contains an edge between two vertices of clique C. Indeed, let P_k be a path containing an edge from clique C, that is, it contains two vertices of C. Since $s(a_i) < \frac{B}{2}$, $1 \leq i \leq 3m$, if P_k contains paths from two cliques, then its length is less than $B + 2$. Thus, at least one more connector vertex from C is needed in order to connect at least one more path p_j to the path P_k. Consequently, we have a path, that is, P_k, using at least three connector vertices of C, a contradiction. Therefore, none of the paths P_i contains an edge between two vertices of clique C.

Since each P_i must contain exactly two connector vertices, no path P_i can have vertices from more than three cliques K_{a_i}. Since the length of each P_i is $B + 2$, each P_i must cover the vertices of exactly three cliques K_{a_i} and the sizes of the corresponding three elements of A must add up to B. Consequently, the set A can be partitioned into m disjoint sets $A_1, A_2, \ldots, A_m$ such that the sum of the sizes of the items in each A_i is equal to B.

The theorem follows from the strong NP-completeness of 3-PARTITION, since the transformation can be done easily in polynomial time. ∎

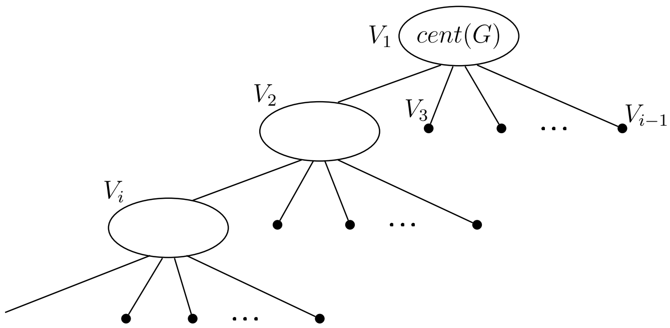

Figure 5: The typical structure of the cent-tree $T_c(G)$ of a threshold graph.

Since the class of quasi-threshold graphs is a subclass of interval graphs, which is a subclass of chordal graphs, the proof of the NP-hardness of the k-path partition problem for quasi-threshold graphs also establishes the NP-hardness of this problem for the class of interval and chordal graphs. Thus, we can state the following result.

Corollary 4.3. *The k-path partition problem is NP-complete for interval and chordal graphs.*

5 Threshold Graphs

In this section we study the pair-complete coloring problem on threshold graphs and describe a linear-time algorithm based on structural properties of the class of threshold graphs.

The concept of threshold graph was introduced by Chvátal and Hammer in 1977 [5]. A graph G is a *threshold graph* [5, 6, 12] if and only if G does not contain $2K_2$, P_4 or C_4 as induced subgraphs. There exists an alternative equivalent definition [19]: A graph is threshold if there exists a partition of $V(G)$ into disjoints sets K, I and an ordering $\{u_l, u_2, \ldots, u_n\}$ of the nodes in I such that K induces a clique in G, I is a stable set of vertices and $N_G(u_1) \subseteq N_G(u_2) \subseteq \cdots \subseteq N_G(u_n)$. A partition of $V(G)$ satisfying the above definition will be called a (K, I) *partition* of G.

5.1 A tree structure

The class of threshold graphs is a subclass of quasi-threshold graphs; see Fig. 1. Consequently, for a threshold graph G there is a tree structure which meets the properties of G, that is, the cent-tree $T_c(G)$ which is similar to the cent-tree of a QT-graph; see Fig. 4. Since a threshold graph G does not contain an induced subgraph isomorphic to $2K_2$, each non-leaf vertex V_i has $k_i \geq 2$ children, where at most one of them is a non-leaf child while the rest $k_i - 1$ children are leaves containing only one vertex; see Fig. 5. Note that the cent-tree $T_c(G)$ of a threshold graph G represents a (K, I) *partition* of G; equivalently, given a (K, I) *partition* of G, we can construct the cent-tree $T_c(G)$.

5.2 Pair-complete coloring problem: a polynomial solution

The pair-complete coloring problem on a threshold graph G can be solved in linear time using its cent-tree $T_c(G)$; see Fig. 5. The vertices V_i of the leftmost path of the tree form a clique and thus each vertex $v_i \in V(G)$ belonging to this path must receive a distinct color. If n' is the number of the vertices of G that belong to the leftmost path of $T_c(G)$, then we claim that the vertices of G take colors from the set $C = \{1, 2, \dots, n'\}$ and the achromatic number $\psi(G)$ is $\psi(G) = n'$. Indeed, let $C' \subset C$ be the set of the colors assigned to the leftmost leaf of $T_c(G)$ and let $c'_i \in C'$. If we assign a new color, say, $n' + 1$, to an uncolored vertex of $T_c(G)$ then the pair $(n' + 1, c'_i)$ cannot appear, which is a contradiction. Consequently, we use the set C to assign colors to the uncolored leaves of $T_c(G)$ in such a way that no vertex $v_i \in V(G)$ takes a color already assigned to an ancestor that belongs to the leftmost path.

Note that, if n' is the number of the vertices of G that belong to the leftmost path of $T_c(G)$, then n' equals the clique number $\omega(G)$, and, thus, $\psi(G) = \omega(G)$. Furthermore, based on the properties of the cent-tree $T_c(G)$, it is easy to show that the clique number equals the chromatic number $\chi(G)$ of the graph G; that is, $\chi(G) = \omega(G)$. Thus, we propose the following linear-time algorithm which holds for connected and disconnected threshold graphs:

Algorithm Pair_Complete_Coloring

Input: a threshold graph G;

Output: a pair-complete coloring of G having $\psi(G) = \omega(G)$;

1. Construct the cent-tree $T_c(G)$ of G;
2. Color the vertices of the leftmost path (clique) of $T_c(G)$ with distinct colors from the set $C = \{1, 2, \dots, \psi(G)\}$.
3. Color each leaf vertex of $T_c(G)$ using a color already assigned to the sibling vertex that belongs to the leftmost path of $T_c(G)$ and contains a clique.
4. If there are any isolated vertices, color them using a color from the set C.

It is worth noting that a disconnected threshold graph includes only one connected component having more than one vertex; each one of the rest of the connected components consists of only one vertex; otherwise there would exist a subgraph isomorphic to $2K_2$. Consequently, we can color the isolated vertices using one color we have already used. Thus, the fourth step of the algorithm is performed when the graph is disconnected. In conclusion, we state the following theorem:

Theorem 5.1. *Let G be a threshold graph. The pair-complete coloring problem is solved in linear time on G and the achromatic number is $\psi(G) = \omega(G)$.*

6 Concluding Remarks

We have studied the complexity of the harmonious coloring problem and the pair-complete coloring problem on subclasses of bipartite graphs. Specifically, we have proved that both problems are NP-complete for the class of connected permutation bipartite graphs and, thus, they are NP-complete for the class of biconvex bipartite graphs. Apart from the NP-completeness results, we have proposed a linear-time algorithm for the pair-complete coloring problem on a subclass of chordal graphs namely threshold graphs.

We have also studied the complexity of the k-path partition problem and proved that it is NP-complete for the class of convex bipartite graphs. Given that this problem is polynomially solvable for permutation bipartite graphs, we have sharpened the demarcation line between polynomially solvable and NP-hard cases of the k-path partition problem. The status of the problem remains open for the class of biconvex bipartite graphs; this class properly contains permutation bipartite graphs and is a proper subclass of convex bipartite graphs.

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