Recognizing HHDS-free Graphs

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Abstract: In this paper, we consider the recognition problem on a class of homogeneously orderable graphs, namely, the HHDS-free graphs. In particular, we prove properties and show that the recognition problem on this class of graphs has polynomial time complexity. We propose a simple $O(n^2m)$-time algorithm which determines whether a graph G on n vertices and m edges is HHDS-free. To the best of our knowledge, this is the first algorithm for recognizing this class of graphs.

Keywords: HHD-free graph, HHDS-free graph, sun, homogeneously orderable graph, perfectly orderable, recognition.

1 Introduction

In the late 1990s, Brandstädt, Dragan, and Nicolai [2] defined the class of graphs that admit a *homogeneous elimination order* and called them *homogeneously orderable* graphs (see also [3]). They show that the class of homogeneously orderable graphs contains the class of *homogeneous* graphs introduced by D'Atri, Moscarini, and Sassano [6]. The larger class of homogeneously orderable graphs seems to be more interesting for several reasons, among which algorithmic reasons–thus, e.g., the (cardinality) Steiner tree problem is solvable in polynomial time on homogeneously orderable graphs [6].

In this paper, we consider a class of homogeneously orderable graphs, namely, the HHDS-free graphs. A graph is *HHDS-free* if it contains no hole (i.e., a chordless cycle on ≥ 5 vertices), no house, no domino (D), and no sun S_k ($k \geq 3$) as induced subgraphs. In [2], Brandstädt, Dragan, and Nicolai proved that a graph G is HHDS-free if and only if every induced subgraph of G is homogeneously orderable. This result characterizes the hereditary homogeneously orderable graphs and shows that these graphs are the HHDS-free graphs; note that the class of homogeneously orderable graphs is not hereditary.

The definition of the homogeneously orderable graphs introduces this class of graphs as a common generalization of the classes of dually chordal and distance-hereditary graphs [2, 3]. Bandelt and Mulder [1] showed that a graph G is distance-hereditary if and only if it contains no house, no hole, no domino, and no gem as induced subgraphs, i.e., G is HHDG-free. Thus, distance-hereditary graphs are HHDS-free since every sun S_k ($k \geq 3$) contains a gem [2, 3]. It is important to note that the HHD-free graphs properly generalize the class of chordal (or triangulated) graphs [8]. In [11], Hoáng and Khouzam proved that the HHD-free graphs admit a *perfect order*, and thus are *perfectly orderable* [4, 13, 15]; the HHDS-free graphs are also perfectly orderable. A superclass of HHD-free graphs, which also properly generalizes the class of chordal graphs, is the class of HH-free graphs; a graph is HH-free if it contains no hole and no house as induced subgraphs. Chvátal conjectured and later Hayward [9] proved that the complement $\overline{G}$ of an HH-free graph G is also perfectly orderable.

In [2], the recognition complexity of HHDS-free graphs is posed as an open problem. Nevertheless, many recognition algorithms have been proposed for graph classes that are defined or characterized by forbidden induced holes, houses, or dominos (see [3, 8]). Indeed, Hoàng and Khouzam [11], while studying the class of brittle graphs (a well-known class of perfectly orderable graphs which contains the HHD-free graphs), showed that the HHD-free graphs can be recognized in $O(n^4)$ time, where n denotes the number of vertices of the input graph. An improved result was obtained by Hoàng and Sritharan [12] who presented an $O(n^3)$-time algorithm for recognizing HH-free graphs and showed that HHD-free graphs can be recognized in $O(n^3)$ time as well. One of the key ingredients in their algorithms is the reduction of a subproblem to the recognition of chordal graphs. Recently, Nikolopoulos and Palios [14] presented an $O(\min\{nm\alpha(n),\, nm+n^2\log n\})$-time and $O(n+m)$-space algorithm for recognizing HHD-free graphs.

The main result of this paper is that a graph G which is HHD-free is also HHDS-free if and only if there is no vertex v of G such that v belongs to a hole or is the top of a "building" in a graph which is a modification of G. This result enables us to describe an $O(n^2m)$-time algorithm for recognizing HHDS-free graphs, where n and m are the numbers of vertices and of edges of the input graph.

2 Theoretical Framework

We consider finite undirected graphs with no loops or multiple edges. Let G be such a graph; then, $V(G)$ and $E(G)$ denote the set of vertices and of edges of G respectively. The *neighborhood* $N(x)$ of a vertex $x \in V(G)$ is the set of all the vertices of G which are adjacent to x. The *closed neighborhood* of x is defined as $N[x] := N(x) \cup \{x\}$. We use $M(x)$ to denote the set $V(G) - N[x]$. The subgraph of a graph G induced by a subset S of G's vertices is denoted by $G[S]$. A subset $A \subseteq V(G)$ of vertices is a *clique*, if $G[A]$ is a complete subgraph of G. An *independent set* is a set of vertices no two of which are adjacent; it is also called a *stable set*.

A path $v_0v_1 \ldots v_k$ of a graph G is called *simple* if none of its vertices occurs more than once; it is called a *cycle* (*simple cycle*) if $v_0v_k \in E(G)$. A simple path (cycle) is *chordless* if $v_iv_j \notin E(G)$ for any two non-consecutive vertices v_i, v_j in the path (cycle). A chordless path (chordless cycle, respectively) on n vertices is commonly denoted by P_n (C_n, respectively). In particular, a chordless path on 4 vertices is denoted by P_4. If $abcd$ is a P_4 of a graph, then the vertices b and c are called *midpoints* and the vertices a and d *endpoints* of the P_4 $abcd$.

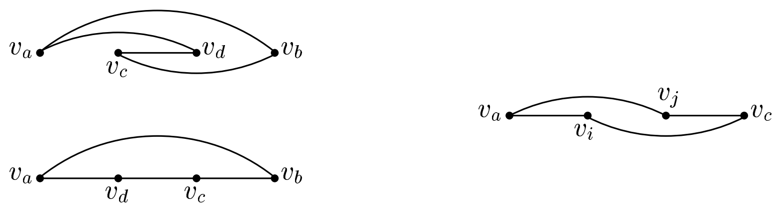
A graph G has a *perfect elimination ordering* if its vertices can be linearly ordered $(v_1, v_2, \ldots, v_n)$ so that each vertex v_i is simplicial in the graph $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$, for $1 \leq i \leq n$; a vertex of a graph is *simplicial* if its neighborhood induces a complete subgraph. It is well-known that a graph is *chordal* (or *triangulated*), if and only if it has a perfect elimination ordering; equivalently, a graph G is chordal if every cycle of length strictly greater than 3 possesses a chord, that is, an edge joining two nonconsecutive vertices of the cycle [3, 8, 16].

Definition 2.1 [5, 7]: A *sun* (or *trampoline*) is a chordal graph G on $2n$ vertices for some $n \geq 3$ whose vertex set can be partitioned into two sets, $U = \{u_0, u_1, \ldots, u_{n-1}\}$ and $W = \{w_0, w_1, \ldots, w_{n-1}\}$, such that W is independent and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j+1 \mod n$.

We prove the following two lemmas.

Lemma 2.1. *Let H be a graph whose vertices can be partitioned into two sets $U = \{u_0, u_1, \ldots, u_{k-1}\}$ and $W = \{w_0, w_1, \ldots, w_{k-1}\}$ of $k \geq 3$ vertices each, such that W is independent and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j+1 \mod k$. Then, H is a sun with partition sets U and W if and only if the subgraph $H[U]$ is chordal and the vertices $u_0, u_1, \ldots, u_{k-1}$ form a cycle $u_0u_1 \cdots u_{k-1}$.*

Proof: ($\Longrightarrow$) Since H is a sun, then H is chordal and thus the subgraph $H[U]$ is chordal as well. Moreover, for all $i = 0, 1, \ldots, k-1$, $u_iu_{i+1 \bmod k} \in E(H)$ since a chordless path shortcuting the



(a) (b)

Figure 1: Different cases for the C_4 $v_a v_b v_c v_d$.

path $u_{i+1 \bmod k} w_{i+1 \bmod k} \cdots u_{i-1} w_{i-1} u_i$ has to be of length 1; otherwise, the vertices of the path along with vertex w_i would induce a chordless cycle on 4 or more vertices, a contradiction to the chordality of H.

($\Longleftarrow$) Since $H[U]$ is chordal, the lemma follows easily from the fact that no w_i ($0 \leq i < k$) participates in a chordless cycle on 4 or more vertices since w_i's only neighbors, u_i and $u_{i+1 \bmod k}$, are adjacent in H. ∎

Lemma 2.2. *Let H be a graph that does not contain holes, $v_1, v_2, \ldots, v_k$ be vertices of H, and suppose that, for all $i = 1, 2, \ldots, k-1$, the adjacency of v_i to v_j, where $i < j \leq k$, implies the adjacency of v_i to all the vertices $v_{i+1}, v_{i+2}, \ldots, v_j$. Then, the subgraph of H induced by the vertices $v_1, v_2, \ldots, v_k$ is chordal.*

Proof: Since the graph H does not contain holes, we only need to show that the subgraph induced by the vertices $v_1, v_2, \ldots, v_k$ does not contain a C_4. Suppose for contradiction that it contained a C_4, say, $v_a v_b v_c v_d$, and suppose without loss of generality that $a = \min\{a, b, c, d\}$. Then, we distinguish the following cases:

(i) $b = \max\{a, b, c, d\}$: then, v_a is adjacent to v_b but is not adjacent to v_c and yet $c < b$ (see Figure 1(a)), a contradiction;

(ii) $c = \max\{a, b, c, d\}$: then, if $i = \min\{b, d\}$ and $j = \max\{b, d\}$, v_i is adjacent to v_c but is not adjacent to v_j and yet $i < j < c$ (see Figure 1(b)), a contradiction;

(iii) $d = \max\{a, b, c, d\}$: the case is similar to case (i) and leads to a contradiction.

In all cases, we reached a contradiction, which implies that the subgraph $H[\{v_1, v_2, \ldots, v_k\}]$ is chordal. ∎

Let G be a graph and let v be an arbitrary vertex of G. Let us define the following set of edges

$$E_v \;=\; \{\, xz \mid x, z \in M(v) \text{ and } \exists y \in M(v) \text{ such that } xyz \text{ is a } P_3 \text{ of } G \,\}$$

which we call *shortcuting* edges. Then, we construct the graph $\widehat{G}_v$ from G as follows:

- $V(\widehat{G}_v) \;=\; V(G)$
- $E(\widehat{G}_v) \;=\; E(G) \cup E_v$.

It is important to note that the definition of shortcuting edges implies that $E(G) \cap E_v = \emptyset$. If the graph G has n vertices and m edges, then the graph $\widehat{G}_v$ has n vertices and $O(n^2)$ edges.

Definition 2.2.

▷ We collectively call a house or a building a *generalized building* or *g-building* for short.

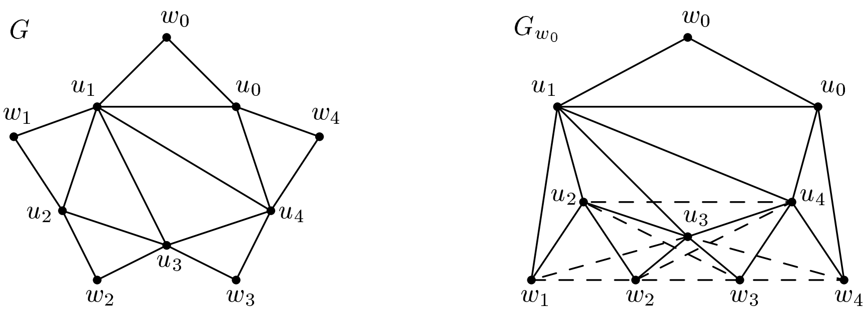


Figure 2

- ▷ If v is the top of the house or the building, then v is the *top* of the g-building. If the vertex v at the top is adjacent to vertices u, w in the g-building, we say that the *roof* of the g-building is $(v; u, w)$. The vertices of the g-building that do not belong to its roof form a chordless path which we call the g-building's *base*.
- ▷ A g-building is *shorter* than an other g-building if it involves fewer vertices.

Our HHDS-free graph recognition algorithm relies on the following theorem.

Theorem 2.1. *Let G be an HHD-free graph. The graph G contains a sun if and only if there exists a vertex v such that the graph $\widehat{G}_v$ defined above with respect to v contains a house or a building with v at its top.*

Proof: ($\Longrightarrow$) Suppose that the graph G contains a sun induced by the sets of vertices $U = \{u_0, u_1, \ldots, u_{k-1}\}$ and $W = \{w_0, w_1, \ldots, w_{k-1}\}$, where $k \geq 3$ (see Definition 2.1). Then, in the graph $\widehat{G}_{w_0}$, the vertices $w_0, u_0, u_1, w_1, w_2, \ldots, w_{k-1}$ induce a house or a building with vertex w_0 at its top (see Figure 2 for an example where $k = 5$; dashed edges indicate shortcuting edges); note that $u_0u_1 \in E(G)$ (see Lemma 2.1), that the vertices u_0 and u_1 are not adjacent to any of the vertices $w_1, w_2, \ldots, w_{k-2}$, and $w_2, w_3, \ldots, w_{k-1}$ respectively, and that, for all $i = 1, 2, \ldots, k-2$, the vertices w_i and w_{i+1} induce a shortcuting edge.

($\Longleftarrow$) Suppose that there exists a vertex v which is the top of a house or a building in $\widehat{G}_v$, i.e., v is the top of a g-building. Then, the following holds:

> **Fact 1.** If the vertex v is the top of a g-building in the graph $\widehat{G}_v$, with roof $(v; u, w)$, then every edge in the base of the *shortest* g-building with roof $(v; u, w)$ is a shortcuting edge.

Fact 1 is established in Lemma 2.3. Thus, if the shortest g-building with roof $(v; u, w)$ has base $p_1p_2 \cdots p_k$, then each p_ip_{i+1} $(1 \leq i \leq k-1)$ is a shortcuting edge; let us replace each such edge with the corresponding P_3 $p_iq_ip_{i+1}$ in G. Then, as in the proof of Lemma 2.3, we can show that, for $i = 1, 2, \ldots, k-1$, the vertex q_i is not adjacent to any of the vertices in $\{p_1, p_2, \ldots, p_{i-1}, p_{i+2}, \ldots, p_k\}$, which implies that the q_is are all distinct (note that the q_is may be arbitrarily adjacent to one other); the situation is depicted in Figure 3 where dashed lines indicate potential edges.

Additionally, vertex u is adjacent to at least one of the vertices $q_1, q_2, \ldots, q_{k-1}$. If u were not adjacent to any of them, then if x is the leftmost neighbor of w among $q_1, q_2, \ldots, q_{k-1}, p_k$ and if ρ is a chordless path shortcuting the path $p_1q_1p_2q_2 \cdots x$, the vertices v, u, w, and the vertices of the path ρ induce a house or a building in G (with v at its top), which contradicts the fact that the graph G is HHD-free. Thus, u is adjacent to at least one q_i. In fact, we can show the following:

> **Fact 2.** There exists an integer r, where $1 \leq r \leq k-1$, such that the vertex u is adjacent to precisely $q_1, q_2, \ldots, q_r$ among the q_is, otherwise the graph G contains a sun.

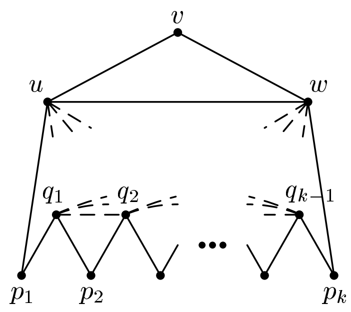


Figure 3

Fact 2 is established in Lemma 2.6 (case (b)) with the aid of Lemma 2.5: since u is adjacent to both p_1 and a vertex q_i, then Lemma 2.5 implies that it is also adjacent to q_1; then, Lemma 2.6 (case (b)) implies that if $r = \max\{\, j \mid uq_j \in E(G)\,\}$ and if there exists a vertex q_i $(2 \le i \le r-1)$ which is not adjacent to u, then the graph G contains a sun, as desired.

So, let us consider the case where the vertex u is adjacent to all the vertices $q_1, q_2, \ldots, q_r$, where $1 \le r \le k-1$. Similarly, we assume that there exists an integer ℓ, where $1 \le \ell \le k-1$, such that the vertex w is adjacent to all the vertices $q_\ell, q_{\ell+1}, \ldots, q_{k-1}$. Then, it has to be that $r \ge \ell$; if $r < \ell$, then the vertices v, u, w, and the vertices of a chordless path shortcuting the path $q_r p_{r+1} q_{r+1} \cdots p_\ell q_\ell$ induce a house or a building in G, a contradiction. In fact, $r = k-1$ and $\ell = 1$, i.e., the vertices u, w are adjacent to all the vertices $q_1, q_2, \ldots, q_{k-1}$. Suppose for contradiction that $r \le k-2$; then, because $r \ge \ell$, the vertex w is adjacent to both q_{k-2} and q_{k-1}. If $k = 2$, then the vertices u, p_1, q_1, p_2, w induce a house in G (with vertex p_2 at its top), a contradiction. If $k \ge 3$, then $q_{k-2}q_{k-1} \notin E(G)$; otherwise, the vertices $p_{k-2}, q_{k-2}, q_{k-1}$ would induce a P_3 in G whose vertices are non-neighbors of v, that is, $p_{k-2}q_{k-1}$ would be a shortcuting edge in $\widehat{G}_v$, which would imply that the vertices $v, u, p_1, p_2, \ldots, p_{k-2}, q_{k-1}, w$ would induce a g-building in $\widehat{G}_v$ with roof (v, u, w), a contradiction to the minimality of the g-building induced by $v, u, p_1, p_2, \ldots, p_k, w$. But then, the vertices $w, q_{k-2}, p_{k-1}, q_{k-1}, p_k$ induce a house in G (with vertex p_k at its top), a contradiction. Therefore, the assumption that $r \le k-2$ led us to a contradiction no matter whether $k = 2$ or $k \ge 3$. Hence, $r = k-1$, i.e., vertex u is adjacent to all the vertices $q_1, q_2, \ldots, q_{k-1}$; similarly, vertex w is adjacent to all these vertices as well.

If there exists a vertex q_i which is adjacent to a vertex q_j but is not adjacent to a vertex $q_{j'}$, where $i < j' < j \le k-1$, then clearly $k \ge 4$ and Lemma 2.6 along with Lemma 2.5 imply that the graph G contains a sun: since q_i is adjacent to both p_{i+1} and q_j, then Lemma 2.5 implies that it is also adjacent to q_{i+1} (note that the graph G is HHD-free and contains the path $p_{i+1}, q_{i+1}, p_{i+2}, q_{i+2}, \ldots, p_j, q_j$, with chords only between q_is, and the vertex q_i is not adjacent to any of $p_{i+2}, p_{i+3}, \ldots, p_j$); then, Lemma 2.6 (case (b)) implies that if there exists a vertex $q_{j'}$ $(i+1 \le j' \le \hat{\jmath}-1)$ which is not adjacent to q_i, where $\hat{\jmath} = \max\{\, t \mid q_i q_t \in E(G)\,\}$, then the graph G contains a sun.

Now, if for all $i = 1, 2, \ldots, k-2$, the adjacency of q_i to a vertex q_j, where $i < j \le k-1$, implies the adjacency of q_i to each of $q_{i+1}, q_{i+2}, \ldots, q_j$, then Lemma 2.2 implies that the subgraph of G induced by the vertices $w, u, q_1, q_2, \ldots, q_{k-1}$ is chordal; recall that $uw \in E(G)$ and both u and w are adjacent to all the vertices $q_1, q_2, \ldots, q_{k-1}$. Additionally, we take advantage of the fact that u is adjacent to all the vertices $q_1, q_2, \ldots, q_{k-1}$ in order to show by induction on i that $q_i q_{i+1} \in E(G)$ for all $i = 1, 2, \ldots, k-2$. For the basis step, we observe that if $q_1 q_2 \notin E(G)$ then the vertices u, p_1, q_1, p_2, q_2 induce a house in G (with vertex p_1 at its top), a contradiction. For the inductive step, we assume that $q_{j-1}q_j \in E(G)$ where $j \ge 2$, and suppose for contradiction that $q_j q_{j+1} \notin E(G)$; if $q_{j-1}q_{j+1} \notin E(G)$, then the vertices $u, q_{j-1}, q_j, p_{j+1}, q_{j+1}$ induce a house in G with vertex q_{j-1} at its top (Figure 4(a)), which leads to a contradiction, whereas if $q_{j-1}q_{j+1} \in E(G)$, then the vertices $q_{j-1}, p_j, q_j, p_{j+1}, q_{j+1}$ induce a house in G with vertex p_j at its top (Figure 4(b)), a contradiction again. Therefore, $q_j q_{j+1} \in E(G)$, and from the induction, $q_i q_{i+1} \in E(G)$ for all $i = 1, 2, \ldots, k-2$.

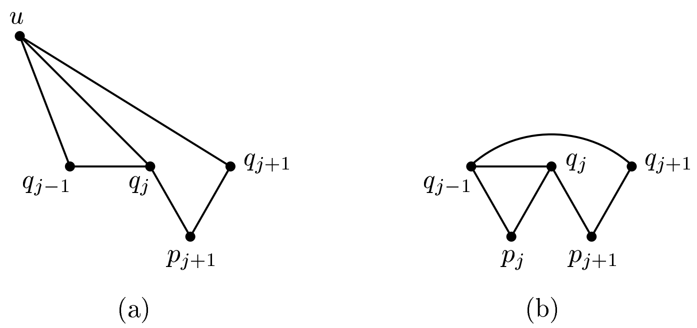


Figure 4

This result, the chordality of the subgraph $G[\{w, u, q_1, q_2, \ldots, q_{k-1}\}]$, the fact that $uw \in E(G)$, $uq_1 \in E(G)$, and $wq_{k-1} \in E(G)$, and Lemma 2.1 imply that the subgraph of G induced by the vertices $v, u, p_1, q_1, p_2, q_2, \ldots, p_{k-1}, q_{k-1}, p_k, w$ is a sun with partition sets $U = \{u, q_1, q_2, \ldots, q_{k-1}, w\}$ and $W = \{v, p_1, p_2, \ldots, p_k\}$. ∎

Lemma 2.3. *Let G be an HHD-free graph, v a vertex of G, and $\widehat{G}_v$ be the auxiliary graph defined above with respect to v. If the vertex v is the top of a g-building in the graph $\widehat{G}_v$ and if u and w are the neighbors of v in the g-building, then every edge in the base of the shortest g-building with roof $(v; u, w)$ is a shortcuting edge.*

Proof: Let the shortest g-building with roof $(v; u, w)$ have base $p_1 p_2 \cdots p_k$, where $k \geq 2$ (Figure 5(a)). Since G does not contain a house or a hole, the path $p_1 \cdots p_k$ contains shortcuting edges; let us replace each shortcuting edge $p_i p_{i+1}$ $(1 \leq i < k)$ by the corresponding P_3 $p_i q_i p_{i+1}$ of G. Then, each such vertex q_i is not adjacent to any vertex in $\{p_1, \ldots, p_{i-1}, p_{i+2}, \ldots, p_k\}$: if q_i were adjacent to p_j, for some $j \in \{1, 2, \ldots, i-1\}$ then the vertices p_j, q_i, p_{i+1} would induce a P_3 in G, and thus $p_j p_{i+1}$ would be a shortcuting edge, which would imply that the vertices $v, u, p_1, \ldots, p_j, p_{i+1}, \ldots, p_k, w$ would induce a g-building with roof $(v; u, w)$ in $\widehat{G}_v$, in contradiction to the minimality of the g-building induced by $v, u, p_1, p_2, \ldots, p_k, w$; a similar argument leads to a contradiction if we assume that q_i were adjacent to p_j, for some $j \in \{i+2, i+3, \ldots, k\}$. The fact that q_i is not adjacent to any vertex in $\{p_1, \ldots, p_{i-1}, p_{i+2}, \ldots, p_k\}$ also implies that the vertices q_i are all different.

We will show next that every edge $p_i p_{i+1}$ is a shortcuting edge. Suppose for contradiction that $p_i p_{i+1}$ is not a shortcuting edge; hence, it is an edge of G instead. Consider a chordless path ρ in G shortcuting the path $p_1, \ldots, p_i$ (possibly containing q_js) and a chordless path ρ' shortcuting the path $p_{i+1}, \ldots, p_k$ (again possibly containing q_js). We show that the concatenation of the path ρ, the edge $p_i p_{i+1}$, and the path ρ' forms a chordless path in G. If there were a chord, this would have been an edge $q_\ell q_r$, where $\ell < i$ and $r \geq i+1$. Let us consider the edge $q_\ell q_r$ that minimizes the

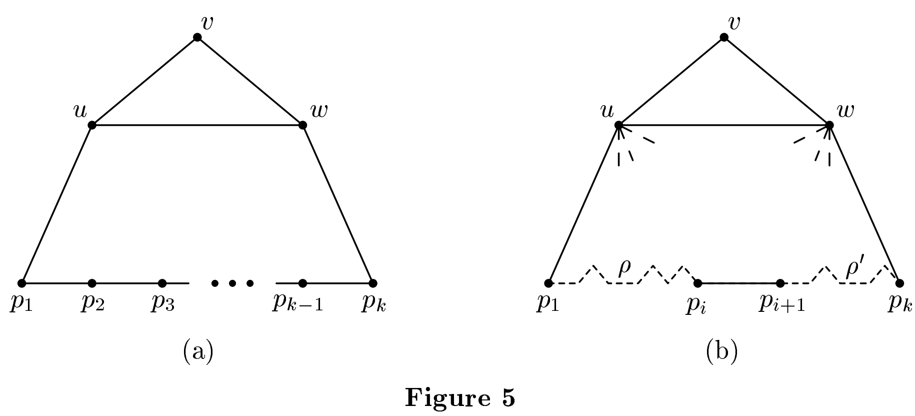


Figure 5

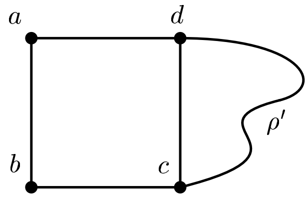


Figure 6: The C_4 *abcd* and the path ρ'.

difference $r-\ell$; then, the vertices of the path ρ from q_ℓ to p_i, and the vertices of the path ρ' from p_{i+1} to q_r induce a cycle in G. In fact, they induce a chordless cycle due to the minimality of $q_\ell q_r$ and the chordlessness of ρ and ρ', and since p_i sees none of the vertices of ρ' except for p_{i+1} and p_{i+1} sees none of the vertices of ρ except for p_i. Additionally, because G contains no hole, it must be the case that $\ell = i-1$ and $r = i+1$, i.e., the vertices $q_\ell, p_i, p_{i+1}, q_r$ form a C_4. Then, since the vertices q_ℓ, q_r, p_{r+1} induce a P_3 in G and thus the edge $q_\ell p_{r+1}$ is a shortcuting edge in $\widehat{G}_v$, if neither u nor w see q_ℓ then the vertices $v, u, p_1, p_2, \ldots, p_\ell, q_\ell, p_{r+1}, p_{r+2}, \ldots, p_k, w$ would form a g-building in $\widehat{G}_v$ with roof $(v; u, w)$ which is shorter than the g-building induced by $v, u, p_1, \ldots, p_k, w$, in contradiction to the minimality of the latter g-building. Hence, at least one of u, w sees q_{ell}, and similarly at least one of u, w sees q_r. On the other hand, neither u nor w see both q_ℓ and q_r, since G does not contain a house. Therefore, either u sees q_ℓ and w sees q_r or u sees q_r and w sees q_ℓ; in either case, the vertices v, u, q_ℓ, q_r, w induce a house (recall that $uw \in E(G)$); a contradiction. Thus, no chord exists, and the concatenation of the path ρ, the edge $p_i p_{i+1}$, and the path ρ' forms a chordless path π in G (Figure 5(b)).

The vertex u is not adjacent to any vertex in the path ρ'. If it were, let t' be the leftmost such vertex; clearly, $t' \neq p_{i+1}$. Moreover, let t be the rightmost vertex of ρ which is adjacent to u; t is well defined since $up_1 \in E(G)$ and $t \neq p_i$. But then, the vertex u and the vertices in the part of the path π from t to t' induce a hole in G, which leads to a contradiction; thus, u is not adjacent to any vertex in ρ'. Similarly, w is not adjacent to any vertex in ρ. But then G contains a hole: it is induced by the vertices u, w, and the vertices of the path π from the rightmost neighbor of u in ρ (which is to the left of p_i) to the leftmost neighbor of w in ρ' (which is to the right of p_{i+1}). This however contradicts the fact that G is HHD-free, and therefore we conclude that the path $p_1 p_2 \cdots p_k$ of the shortest g-building consists entirely of shortcuting edges. ∎

Lemma 2.4. *Let G be a graph which contains a C_4 abcd and a path ρ from c to d (different from the path cd) whose vertices other than its endpoints c and d are adjacent neither to a nor to b. Then, the graph G contains a hole, a house, or a domino.*

Proof: Since we are dealing with simple graphs, the length of a chordless path ρ' shortcuting the path ρ, where we ignore the chord cd, would be at least equal to 2. Then, if the length of ρ' is 2 or 3, the vertices of ρ' along with the vertices a and b induce a house or a domino in G respectively, whereas if the length of ρ' is greater than or equal to 4, then the vertices of ρ' induce a hole in G because of the edge cd (see Figure 6). ∎

Lemma 2.5. *Let G be an HHD-free graph which contains a path $p_s, q_s, p_{s+1}, q_{s+1}, \ldots, p_t, q_t$, where $t \geq s+1$, with chords only between q_is, and let x be a vertex of G which is adjacent to p_s and is not adjacent to any of $p_{s+1}, p_{s+2}, \ldots, p_t$. If the vertex x is adjacent to q_t, then it is also adjacent to q_s.*

Proof: Suppose for contradiction that $xq_s \notin E(G)$. Let $q_{t'}$ be the leftmost among the vertices $q_{s+1}, q_{s+2}, \ldots, q_t$ which is adjacent to x; the vertex $q_{t'}$ is well defined since x is adjacent to q_t. Then, $q_s q_{t'} \in E(G)$, otherwise the length of a chordless path in G shortcuting the path $q_s p_{s+1} q_{s+1} \cdots p_t q_t$ would be of length at least 2 and the vertices of the path along with x and p_s would induce a hole in G, a contradiction. But then, the vertices $x, p_s, q_s, q_{t'}$ induce a C_4 in G and G contains the path $q_s p_{s+1} q_{s+1} \cdots p_{t'} q_{t'}$ whose vertices other than its endpoints are adjacent neither to x nor to

p_s. Thus, Lemma 2.4 applies implying that the graph G contains a hole, a house, or a domino; this however leads to a contradiction since G is HHD-free. Therefore, the vertex x is adjacent to q_s. ∎

Lemma 2.6. *Let G be an HHD-free graph which contains a path $q_s, p_{s+1}, q_{s+1}, \ldots, p_t, q_t$, where $t \geq s+2$, with chords only between q_is, and let x be a vertex of G which is adjacent to q_s and q_t, and is not adjacent to any of $p_{s+1}, p_{s+2}, \ldots, p_t$.*

(a) Suppose that the vertex x is not adjacent to the vertices $q_{s+1}, q_{s+2}, \ldots, q_{t-1}$, and that for $i = s, s+1, \ldots, t-1$, if the vertex q_i is adjacent to q_j (where $i < j \leq t$) then it is adjacent to all the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$. Then, the vertices $x, q_s, p_{s+1}, q_{s+1}, \ldots, p_t, q_t$ induce a sun in G.

(b) If there exists a vertex q_i ($s+1 \leq i \leq t-1$) which is not adjacent to x, then the graph G contains a sun.

Proof: (a) First, the set $\{q_s, q_{s+1}, \ldots, q_t\}$ contains at least 3 vertices. Next, due to the property of the q_is, Lemma 2.2 implies that the subgraph of G induced by the vertices $q_s, q_{s+1}, \ldots, q_t$ is chordal. In light of Lemma 2.1 and of the fact that the vertex x is adjacent to q_s and q_t only, and each vertex p_i ($s+1 \leq i \leq t$) is adjacent to q_{i-1} and q_i only, we need only prove that the vertices $q_s, q_{s+1}, \ldots, q_t$ induce a cycle $q_s q_{s+1} \cdots q_t$ in G.

We begin by showing that the vertex q_s is adjacent to at least one vertex in $\{q_s, q_{s+1}, \ldots, q_t\}$; if it were not, then the vertices x, q_s, p_{s+1}, and the vertices of a chordless path shortcuting the path $q_{s+1}, p_{s+2}, q_{s+2}, \ldots, p_t, q_t$ would induce a hole in G, a contradiction. If q_ℓ is that vertex, i.e., $q_s q_\ell \in E(G)$, then $q_s q_t \in E(G)$: this is trivially true if $q_\ell = q_t$; if $q_\ell \neq q_t$, then because the graph G contains the path $x, q_t, p_t, q_{t-1}, \ldots, p_{\ell+1}, q_\ell$, where $\ell \leq t-1$, with chords only between q_is, and the vertex q_s is adjacent to x and q_ℓ but is not adjacent to any of $p_t, p_{t-1}, \ldots, p_{\ell+1}$, Lemma 2.5 applies implying that q_s is adjacent to q_t in G. From this fact and from the property of the vertices q_i ($s \leq i < t$) that the adjacency of q_i to a q_j, where $i < j \leq t$, implies the adjacency of q_i to all the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$, we conclude that q_s is adjacent to all the vertices $q_{s+1}, q_{s+2}, \ldots, q_t$; this in turn enables us to additionally show (by induction on i) that $q_i q_{i+1} \in E(G)$ for all $i = s+1, s+2, \ldots, t-1$. For the basis step, we note that if $q_{s+1} q_{s+2} \notin E(G)$, then the vertices $q_s, p_{s+1}, q_{s+1}, p_{s+2}, q_{s+2}$ induce a house in G with vertex p_{s+1} at its top, a contradiction. For the inductive step, assume that $q_{j-1} q_j \in E(G)$ where $j \geq s+1$. We show that $q_j q_{j+1} \in E(G)$; if not, then the vertices $q_s, q_{j-1}, q_j, p_{j+1}, q_{j+1}$ induce a house in G with vertex q_{j-1} at its top, a contradiction. Our inductive proof is complete implying that $q_i q_{i+1} \in E(G)$ for all $i = s+1, s+2, \ldots, t-1$; then, because $q_s q_{s+1} \in E(G)$ and $q_s q_t \in E(G)$, we have that the vertices $q_s, q_{s+1}, \ldots, q_t$ indeed induce a cycle $q_s q_{s+1} \cdots q_t$ in G.

(b) Since the vertex x is adjacent to q_s and q_t, and is not adjacent to a vertex in $\{q_{s+1}, q_{s+1}, \ldots, q_{t-1}\}$, we can find vertices q_ℓ, q_r, where $s \leq \ell < r \leq t$, such that x is adjacent to q_ℓ and q_r but is not adjacent to any of $q_{\ell+1}, q_{\ell+2}, \ldots, q_{r-1}$. Then, if for each vertex q_i ($\ell \leq i \leq r-1$), the adjacency of q_i to a vertex q_j, where $i < j \leq r$, implies the adjacency of q_i to all the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$, Lemma 2.6 (case (a)) applies implying that the vertices $x, q_\ell, p_{\ell+1}, q_{\ell+1}, \ldots, p_r, q_r$ induce a sun in G. Suppose now that there exists a vertex q_i ($\ell \leq i \leq r-1$) which is adjacent to a vertex q_j and is not adjacent to a vertex $q_{j'}$, where $i < j' < j \leq r$. Let us collect all such vertices in a (non-empty) set S. Then, for each vertex q_i in S, we can find indices ℓ_i and r_i where $i < \ell_i < r_i \leq r$, such that q_i is adjacent to q_{ℓ_i} and q_{r_i} but is not adjacent to any of the vertices $q_{\ell_i+1}, q_{\ell_i+2}, \ldots, q_{r_i-1}$, and the difference $r_i - \ell_i$ is minimized. Let $q_{\hat{\imath}}$ be a vertex in S such that $r_{\hat{\imath}} - \ell_{\hat{\imath}} = \min_{q_i \in S}\{r_i - \ell_i\}$; the minimality of $q_{\hat{\imath}}$ implies that for $i = \ell_{\hat{\imath}}, \ell_{\hat{\imath}}+1, \ldots, r_{\hat{\imath}}-1$, if the vertex q_i is adjacent to q_j (where $i < j \leq r_{\hat{\imath}}$) then it is adjacent to all the vertices $q_{i+1}, q_{i+2}, \ldots, q_j$. This, the fact that the graph G contains the path $q_{\ell_{\hat{\imath}}}, p_{\ell_{\hat{\imath}}+1}, q_{\ell_{\hat{\imath}}+1}, \ldots, p_{r_{\hat{\imath}}}, q_{r_{\hat{\imath}}}$, where $r_{\hat{\imath}} \geq \ell_{\hat{\imath}}+2$, with chords only between q_is, and the fact that vertex $q_{\hat{\imath}}$ is adjacent to $q_{\ell_{\hat{\imath}}}$ and $q_{r_{\hat{\imath}}}$ but is not adjacent to any of $q_{\ell_{\hat{\imath}}+1}, q_{\ell_{\hat{\imath}}+2}, \ldots, q_{r_{\hat{\imath}}-1}$ imply that Lemma 2.6 (case (a)) applies and therefore the vertices $q_{\hat{\imath}}, q_{\ell_{\hat{\imath}}}, p_{\ell_{\hat{\imath}}+1}, q_{\ell_{\hat{\imath}}+1}, \ldots, p_{r_{\hat{\imath}}}, q_{r_{\hat{\imath}}}$ induce a sun in G. ∎

3 The Algorithm

The recognition algorithm takes advantage of Theorem 2.1. We start by checking whether the input graph G is HHD-free. If it is not, then clearly G is not HHDS-free. Otherwise, for each vertex v of G, we construct the auxiliary graph $\widehat{G}_v$ and check whether v is the top of a house or a building in $\widehat{G}_v$; if this is so for any vertex v, then G is not HHDS-free. We note that in order to check whether v is the top of a house or a building in $\widehat{G}_v$, we use the Algorithm Not-in-HHB [14] which for a graph H and a vertex x returns true if and only if the vertex x belongs to a hole or is the top of a house or a building in H; Lemma 3.1 proves that v does not belong to a hole in $\widehat{G}_v$ if G is HHD-free.

Lemma 3.1. *Let G be an HHD-free graph, v a vertex of G, and $\widehat{G}_v$ be the auxiliary graph defined in Section 2 with respect to v. Then, the vertex v does not belong to a hole in the graph $\widehat{G}_v$.*

Proof: Suppose that v belongs to a hole $vup_1 \cdots p_k w$ in $\widehat{G}_v$, where $k \geq 2$. As the graph G does not contain holes, the path $p_1 p_2 \cdots p_k$ definitely contains shortcuting edges. If we replace each of these shortcuting edges by the corresponding P_3 in G, we obtain a path in G from p_1 to p_k; let $a_1 a_2 \cdots a_t$ be a chordless such path, where $a_1 = p_1$ and $a_t = p_k$. Let a_s be the leftmost vertex in the path which is adjacent to w in G; the vertex a_s is well defined since w is adjacent to a_t, and $s \geq 2$ since w is not adjacent to a_1. Then, u must be adjacent to a_s; if not, then if a_r is the rightmost vertex in $a_1 a_2 \cdots a_{s-1}$ which is adjacent to u, then the vertices $v, u, a_r, a_{r+1}, \ldots, a_s, w$ induce a hole in G, a contradiction. Moreover, $s \geq 3$ and u cannot be adjacent to a_{s-1}, otherwise the vertices v, u, a_{s-1}, a_s, w would induce a house in G (with vertex a_{s-1} at its top), a contradiction. Then, if u is adjacent to a_{s-2}, the graph G contains a domino (induced by the vertices $v, u, a_{s-2}, a_{s-1}, a_s, w$), otherwise it contains a hole since u is adjacent to a_1; in all cases, we get a contradiction, which implies that v cannot belong to a hole in $\widehat{G}_v$. ∎

In detail, the recognition algorithm works as follows:

Algorithm Rec-HHDS-free

Input: an undirected graph G.

Output: "true," if G is an HHDS-free graph; otherwise, "false."

1. **if** G is not HHD-free
   then return "false;"

2. **for** each vertex v of G **do**
   - 2.1 construct the auxiliary graph $\widehat{G}_v$;
   - 2.2 **if** v is the top of a house or a building in $\widehat{G}_v$
     then return "false;" {*G contains a sun*}

3. **return** "true."

The correctness of the algorithm follows from Theorem 2.1.

Time and Space Complexity. Let n and m be the number of vertices and edges of the input graph G. Step 1 can be executed in $O(\min\{nm\alpha(n),\, nm + n^2 \log n\})$ time and $O(n+m)$ space [14]. In Step 2, the construction of the auxiliary graph $\widehat{G}_v$ can be done in $O(nm)$ time and requires $O(n^2)$ space. Then, checking whether vertex v is the top of a house or a building is done by means of the Algorithm Not-in-HHB [14], which for a graph on N vertices and M edges takes $O(N + \min\{M\alpha(N),\, M + N \log N\})$ time and $O(N+M)$ space; since $\widehat{G}_v$ has n vertices and $O(n^2)$ edges, Substep 2.2 takes $O(n^2)$ time and space. Thus, the entire execution of Step 2 for all the vertices of G takes $O(n^2 m)$ time and $O(n^2)$ space. Step 3 takes constant time and space.

Therefore, we obtain the following theorem.

Theorem 3.1. *Let G be an undirected graph on n vertices and m edges. Then, it can be determined whether G is an HHDS-free graph in $O(n^2 m)$ time and $O(n^2)$ space.*

4 Concluding Remarks

We have presented a recognition algorithm for the class of HHDS-free graphs running in $O(n^2 m)$ time with $O(n^2)$ space. To the best of our knowledge, it is the first algorithm for recognizing the class of HHDS-free graphs. The proposed recognition algorithm can be augmented to provide a certificate (an induced house, hole, domino, or sun) whenever it decides that the input graph is not HHDS-free; the proof of Theorem 2.1 is constructive and helps find a sun in linear additional time and space whenever a vertex v is found to be the top of a house or a building in the graph $\widehat{G}_v$.

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