The Harmonious Coloring Problem is NP-complete for Interval and Permutation Graphs

Katerina Asdre Kyriaki Ioannidou Stavros D. Nikolopoulos

Department of Computer Science, University of Ioannina
P.O.Box 1186, GR-45110 Ioannina, Greece
{katerina, kioannid, stavros}@cs.uoi.gr

Abstract: In this paper, we prove that the harmonious coloring problem is NP-complete for connected interval and permutation graphs. Given a simple graph G, a harmonious coloring of G is a proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number is the least integer k for which G admits a harmonious coloring with k colors. Extending previous work on the NP-completeness of the harmonious coloring problem when restricted to the class of disconnected graphs which are simultaneously cographs and interval graphs, we prove that the problem is also NP-complete for connected interval and permutation graphs.

Keywords: Harmonious coloring, harmonious chromatic number, achromatic number, interval graphs, permutation graphs, NP-completeness.

1 Introduction

Many NP-complete problems on arbitrary graphs admit polynomial time algorithms when restricted to the classes of interval graphs and cographs; NP-complete problems for these two classes of graphs that become solvable in polynomial time can be found in [1, 2, 5, 10, 13, 14]. However, the pair-complete coloring problem, which is NP-hard on arbitrary graphs [15], remains NP-complete when restricted to graphs that are simultaneously interval and cographs [3]. A *pair-complete coloring* of a simple graph G is a proper vertex coloring such that each pair of colors appears together on at least one edge, while the *achromatic number* $\psi(G)$ is the largest integer k for which G admits a pair-complete coloring with k colors. The achromatic number was introduced in [11, 12].

Bodlaender [3] provides a proof for the NP-completeness of the pair-complete coloring problem for disconnected cographs and interval graphs and extends his results for connected such graphs. His proof also establishes the NP-hardness of the harmonious coloring problem for disconnected interval graphs and cographs; a *harmonious coloring* of a simple graph G is a proper vertex coloring such that each pair of colors appears together on at most one edge, while the *harmonious chromatic number* $h(G)$ is the least integer k for which G admits a harmonious coloring with k colors [4]. Note that the problem of determining the harmonious chromatic number of connected cographs is trivial, since in such a graph each vertex must receive a distinct color as it is at distance at most 2 from all other vertices [4]. On the contrary, although the harmonious coloring problem is NP-complete for disconnected interval graphs, the complexity of the problem for connected interval graphs is not straightforward. Moreover, the NP-hardness of the pair-complete coloring problem for cographs also establishes the NP-hardness of the pair-complete coloring problem for the class of permutation graphs, and, also, the NP-hardness of the harmonious coloring problem when restricted to disconnected permutation graphs. However, the complexity of the harmonious coloring problem for connected permutation graphs has not been studied. Motivated by these issues we prove that the harmonious coloring problem is also NP-complete for connected interval and permutation graphs.

2 NP-completeness Results

The formulation of the harmonious coloring problem in [4] is equivalent to the following formulation.

Harmonious Coloring Problem
Instance: Graph $G = (V, E)$, positive integer $K \leq |V|$.
Question: Is there a positive integer $k \leq K$ and a proper coloring using k colors such that each pair of colors appears together on at most one edge?

We next prove our main result, that is, harmonious coloring is NP-complete for connected interval graphs; a graph G is an *interval graph* if its vertices can be put in one-to-one correspondence with a family of intervals on the real line such that two vertices are adjacent in G if and only if their corresponding intervals intersect.

Theorem 2.1. *Harmonious coloring is NP-complete when restricted to connected interval graphs.*

Proof. Harmonious coloring is obviously in NP. In order to prove NP-hardness, we use a transformation from 3-PARTITION.

Let a set $A = \{a_1, \ldots, a_{3m}\}$ of $3m$ elements, a positive integer B and let positive integer sizes $s(a_i)$ for each $a_i \in A$ be given, such that $\frac{1}{4}B < s(a_i) < \frac{1}{2}B$, and such that $\sum_{a_i \in A} s(a_i) = mB$, $1 \leq i \leq 3m$. We may suppose that, for each $a_i \in A$, $s(a_i) > m$ (if not, then we can multiply all $s(a_i)$ and B with $m + 1$).

Extending the result of Bodlaender [3], we construct the following connected graph which is an interval and a permutation graph: Consider a clique with m vertices, a clique with B vertices, and add a vertex v that is connected to every vertex in the two cliques; let G_1 be the resulting graph. Next we construct for every $a_i \in A$ a tree T_i of depth one with $s(a_i)$ leaves and root x_i, that is, every leaf is adjacent to the root; note that there are $3m$ such trees $T_1, T_2, \ldots, T_{3m}$. Then we construct a path $P = [v_1, v_2, \ldots, v_{3m}]$ of $3m$ vertices, and we connect each vertex v_i of the path P to all the vertices of the tree T_i, $1 \leq i \leq 3m$. Additionally, for each vertex $v_i \in P$, we add $m - 1 + B - s(a_i) + i - 1$ vertices and connect them to vertex v_i; let G_2 be the resulting graph. Note that the graph $G_1 \cup G_2$ is disconnected. Finally, we add an edge to the graph $G_1 \cup G_2$ connecting vertices v_1 and v and let G be the resulting graph. The graph G is a connected graph and it is illustrated in Fig. 1.

One can easily verify that G is an interval graph. A clique can be represented as a number of intervals that share at least one point in common. Two cliques sharing a vertex u can be represented as a number of intervals such that one of them, which corresponds to u, shares at least one point with the intervals corresponding to the vertices of each clique. Thus, the vertices of G can be put in one-to-one correspondence with a family of intervals on the real line such that two vertices are adjacent in G if and only if their corresponding intervals intersect.

It is easy to see that the total number of edges in G is

$$\binom{m}{2} + \binom{B}{2} + m + B + 3m + mB + 3m + mB + 3m(m-1) + 2mB + \sum_{i=1}^{3m} i = \binom{4m + B + 1}{2}$$

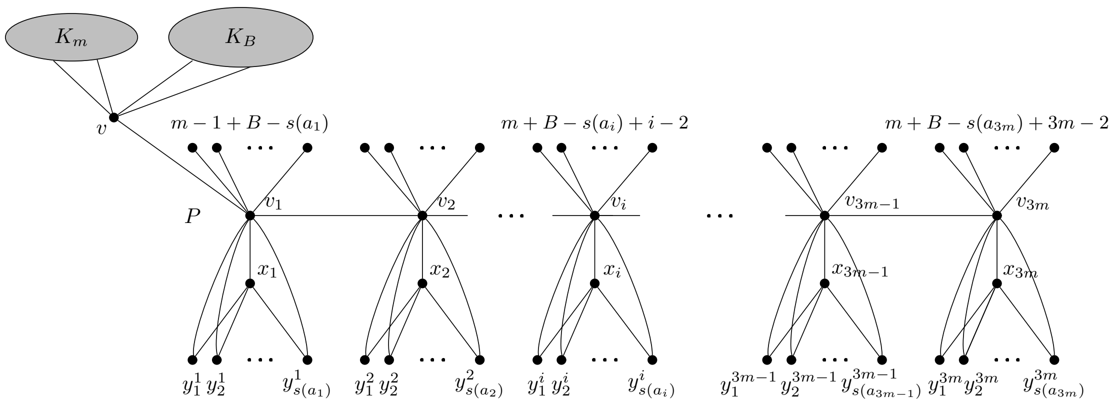

Figure 1: Illustrating the constructed connected interval and permutation graph G.

For every harmonious coloring of G and every pair of distinct colors i, j, $i \neq j$, there must be at most one edge with its endpoints colored with i and j. Thus, it follows that the harmonious chromatic number cannot be less than $4m + B + 1$, and if it is equal to $4m + B + 1$ then we have, for every pair of distinct colors i, j, $1 \leq i, j \leq 4m + B + 1$, a unique edge with its end-points colored with i and j. Thus, we have an exact coloring of G; an *exact coloring* of G with k colors is a harmonious coloring of G with k colors in which, for each pair of colors i, j, there is exactly one edge (a, b) such that a has color i and b has color j.

We now claim that the harmonious chromatic number of G is (less or equal to) $4m + B + 1$ if and only if A can be partitioned in m sets $A_1, \ldots, A_m$ such that $\sum_{a \in A_j} s(a) = B$, for all j, $1 \leq j \leq m$.

($\Longleftarrow$) Suppose now a 3-partition of A in $A_1, \ldots, A_m$ such that $\forall j : \sum_{a \in A_j} s(a) = B$ exists. We show how to find a harmonious coloring of G using $4m + B + 1$ colors. We color the vertices of the first clique with colors $1, 2, \ldots, m$, the vertices of the second clique with $m+1, m+2, \ldots, m+B$, and vertex v with $m + B + 1$. For convenience and ease of presentation, let $\mathcal{M}$ be the set containing colors $1, 2, \ldots, m$, let $\mathcal{B}$ be the set containing colors $m+1, m+2, \ldots, m+B$, and let $\mathcal{K}$ be the set containing colors $m+B+2, m+B+3, \ldots, 4m+B+1$. If $a_i \in A_j$ then we color the vertex corresponding to a_i with color j. Each color $j \in \mathcal{M}$ is assigned to the three vertices corresponding to three a_i that have together exactly B neighbors of degree 2. We assign to each one of these B neighbors a different color from $\mathcal{B}$, and next we assign to each vertex v_i of the path P a distinct color from $\mathcal{K}$. Recall that each vertex v_i, $1 < i < 3m$, is connected to two other vertices of P, i.e., v_{i-1} and v_{i+1}, and $m + B + i - 1$ more vertices, vertex v_1 is connected to v_2, v and $m + B$ other vertices, while vertex v_{3m} is connected to v_{3m-1} and $m + B + 3m - 1$ more vertices (see Fig. 1).

Next, we color the rest $m - 1 + B - s(a_i) + i - 1$ neighbors of each v_i. We assign a distinct color from the set $\mathcal{M} \backslash c_i$ to $m - 1$ neighbors of v_i, where c_i is the color previously assigned to the vertex corresponding to a_i. We next assign a distinct color from the set $\mathcal{B} \backslash C_i$ to $B - s(a_i)$ neighbors of v_i, where C_i is the set of the colors previously assigned to $s(a_i)$ neighbors of the vertex corresponding to a_i. Finally, we assign a different color to the rest $i - 1$ neighbors of v_i, $3 \leq i \leq 3m$, using color $m + b + 1$ and the colors assigned to the vertices v_j, $1 \leq j \leq i - 2$. Note that, in order to color the $m + B - s(a_2)$ neighbors of v_2, we only need to use color $m + B + 1$ and colors from $\mathcal{M}$ and $\mathcal{B}$, while for the $m - 1 + B - s(a_1)$ neighbors of v_1 we only use colors from $\mathcal{M}$ and $\mathcal{B}$. A harmonious coloring of G using $4m + B + 1$ colors results, and thus, the harmonious chromatic number of G is $4m + B + 1$.

($\Longrightarrow$) We next suppose that the harmonious chromatic number of G is (less or equal to) $4m+B+1$. Consider a harmonious coloring of G using $4m+B+1$ colors. Without loss of generality we may suppose that the m vertices of the first clique have distinct colors from $\mathcal{M}$, while the B vertices of the second clique have distinct colors from $\mathcal{B}$. Also, without loss of generality, we color vertex v with color $m+B+1$ since v is adjacent to all the vertices of the two cliques. Since v_{3m} is the vertex having the maximum degree, that is, $4m+B$, it has to take a color from $\mathcal{K}$. Indeed, if it takes a color from $\mathcal{M}$, then none of its neighbors can take a color from $\mathcal{M}$ and we cannot color $4m+B$ vertices using only $4m+B+1-m$ colors. Using similar arguments, we cannot color vertex v_{3m} using a color from $\mathcal{B}$ or the color $m+B+1$. Thus, without loss of generality, we assign to v_{3m} the color $4m+B+1$. We color all its neighbors with distinct colors from $\mathcal{M}\cup\mathcal{B}\cup\{m+B+1\}\cup\mathcal{K}\backslash\{4m+B+1\}$. Note that, vertex v_{3m-1} takes a color from $\mathcal{K}\backslash\{4m+B+1\}$; let $4m+B$ be this color. Indeed, using similar arguments, it cannot take a color from $\mathcal{M}\cup\mathcal{B}\cup\{m+B+1\}\cup\{4m+B+1\}$. Note that, color $4m+B+1$ cannot be assigned to any other vertex of G since any pair of colors $(4m+B+1, j)$, $1 \leq j \leq 4m+B$, already appears in the harmonious coloring. Recall that, for every pair of distinct colors i, j, $1 \leq i, j \leq 4m+B+1$, there is a unique edge with its end-points colored with i and j. Recursively, as can easily be proved by induction on i, the same holds for all $v_i \in P$, $1 \leq i \leq 3m-2$, that is, v_i takes a color from $\mathcal{K}\backslash\mathcal{L}$, where $\mathcal{L}$ is the set containing colors $m+B+1+i+1, m+B+1+i+2, \ldots, 4m+B+1$, which are the colors already assigned to vertices v_j, $i < j \leq 3m$.

Note that pairs (μ, ν), $\mu \in \mathcal{M}$, $\nu \in \mathcal{B}$, have not appeared yet. Since every pair of colors must appear, we assign these pairs to the mB edges that have both endpoints uncolored. Note that these edges are the edges (x_i, y_j^i), $1 \leq i \leq 3m$, $1 \leq j \leq s(a_i)$, where x_i corresponds to a_i and y_j^i corresponds to the j-th neighbor of x_i having degree 2. The vertices x_i cannot take a color from $\mathcal{B}$, otherwise its $s(a_i) > m$ uncolored neighbors y_j^i cannot be colored with m colors from $\mathcal{M}$. Thus, vertices x_i are assigned a color from $\mathcal{M}$ and vertices y_j^i are assigned a color from $\mathcal{B}$ (recall that $\frac{B}{4} < s(a_i) < \frac{B}{2}$). Note that the only uncolored vertices are $m-1+B-s(a_i)+i-1$ neighbors of each v_i, $1 \leq i \leq 3m$. In order to color $m-1+B-s(a_i)$ of the uncolored neighbors of v_i, we use distinct colors from $(\mathcal{M}\cup\mathcal{B})\backslash\mathcal{F}$, where $\mathcal{F}$ is the set containing all colors already assigned to the $s(a_i)+1$ neighbors of v_i. In order to color the last $i-1$ uncolored neighbors of v_i, $i > 1$, we can only use colors from $\mathcal{K}\backslash\mathcal{L}\backslash\{m+B+1+i, m+B+i\}$ because the only unused pairs are $(m+B+1+i, j)$, where $m+B+1 \leq j \leq m+B+1+i-2$.

Finally, let $a_i \in A_j$ if and only if the vertex x_i (with neighbors y_j^i) is colored with color $j \in \mathcal{M}$. We claim that for all j, $\sum_{a \in A_j} s(a) = B$. Indeed, each color j must be adjacent to some colors from $\mathcal{B}$, and each color from $\mathcal{B}$ is assigned to exactly one vertex which is adjacent to all x_i colored with j. Hence, a correct 3-partition exists.

The theorem follows from the strong NP-completeness of 3-PARTITION, since the transformation can be done easily in polynomial time. ∎

We can easily show that the interval graph G illustrated in Fig. 1 is also a permutation graph. The graph G is an interval graph if and only if it is a chordal graph and the graph $\overline{G}$ is a comparability graph [9]. Moreover, one can easily verify that G admits an acyclic transitive orientation and, thus, it is a comparability graph. Since G and $\overline{G}$ are comparability graphs, it follows that G is a permutation graph [9]. Consequently, we can state the following theorem.

Theorem 2.2. *Harmonious coloring is NP-complete when restricted to connected permutation graphs.*

3 Concluding Remarks

We have shown that the connected interval graph G presented in this paper, which is also a permutation graph, has $\binom{4m+B+1}{2}$ edges and $h(G) = 4m+B+1$. In [6] it was shown that if G is a graph with exactly $\binom{k}{2}$ edges, then a proper vertex coloring of G with k colors is pair-complete if and only if it is a harmonious coloring. Thus, if G is a graph with $\binom{k}{2}$ edges, then $\psi(G) = k$ if and only if $h(G) = k$ [4]. Consequently, for the graph G, which is simultaneously an interval and a permutation graph, we have that $\psi(G) = 4m + B + 1$ and, thus, our results could be also used to prove that the achromatic number is NP-complete for connected interval and permutation graphs.

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