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VARIATIONAL CALCULATIONS OF ASYMMETRIC NUCLEAR MATTER *

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Abstract: We report on variational calculations of the energy $E(\rho, \beta)$ of asymmetric nuclear matter having $\rho = \rho_n + \rho_p = 0.05$ to $0.35\ \mathrm{fm}^{-3}$, and $\beta = (\rho_n - \rho_p)/\rho = 0$ to 1. The nuclear hamiltonian used in this work consists of a realistic two-nucleon interaction, called v_{14}, that fits the available nucleon-nucleon scattering data up to 425 MeV, and a phenomenological three nucleon interaction adjusted to reproduce the empirical properties of symmetric nuclear matter. The variational many-body theory of symmetric nuclear matter is extended to treat matter with neutron excess. Numerical and analytic studies of the β-dependence of various contributions to the nuclear matter energy show that at $\rho < 0.35\ \mathrm{fm}^{-3}$ the β^4 terms are very small, and that the interaction energy $\mathrm{EI}(\rho, \beta)$ defined as $E(\rho, \beta) - T_F(\rho, \beta)$, where T_F is the Fermi-gas energy, is well approximated by $\mathrm{EI}_0(\rho) + \beta^2 \mathrm{EI}_2(\rho)$. The calculated symmetry energy at equilibrium density is 30 MeV and it increases from 15 to 38 MeV as ρ increases from 0.05 to $0.35\ \mathrm{fm}^{-3}$.

1. Introduction

Recently we [1]) [denoted by I henceforth] reported on variational calculations of symmetric nuclear matter with a hamiltonian consisting of a realistic two-nucleon interaction operator called v_{14}, and a phenomenological three-nucleon interaction (TNI). The v_{14} interaction operator is obtained by fitting the deuteron properties and the nucleon-nucleon scattering data in S, P, D and F waves up to 400 MeV [ref. [2])]. By itself the v_{14} interaction does not give satisfactory properties of nuclear matter. The TNI is obtained by requiring that the v_{14}+TNI model gives the correct energy, density and compressibility of nuclear matter. The contribution of TNI to the ground state energy of nuclear matter is small. The TNI is divided into two parts. One of them, called TNR, generates a density-dependent repulsive two-nucleon interaction that is added to the v_{14} interaction in nuclear matter calculations. The contribution of the other part is attractive, and it is represented by a function of density, $\mathrm{TNA}(\rho_n, \rho_p)$ as discussed in I. The v_{14}+TNI hamiltonian has also been used to study the equation of state of hot and cold nuclear and neutron matter [3]). In this paper we present our studies of the energy $E(\rho, \beta)$ of cold asymmetric nuclear matter having $(\rho_n - \rho_p)/\rho = \beta$. Symmetric-nuclear and neutron matter are the limiting cases having $\beta = 0$ and 1 respectively.

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The $E(\rho, \beta)$ is given by [1])

$$E(\rho,\beta) = T_F + E_{2B} + W_0(\mathrm{MB}) + W_s + W_c + W_F(\mathrm{MB}) + U + U_F + \mathrm{TNA}, \tag{1.1}$$

where T_F is the Fermi-gas kinetic energy, E_{2B} is the contribution of two-body clusters, and terms $W_0(\mathrm{MB})$ to U_F are contributions of many-body clusters. The terms E_{2B} to U_F are calculated with a density-dependent two-nucleon interaction that represents v_{14}+TNR. The $W_0(\mathrm{MB})$, W_s and W_c represent the bulk of many-body cluster contributions via Fermi-hypernetted-chain (FHNC), separable, and single-operator-chain (SOC) diagrams respectively; while $W_F(\mathrm{MB})$, U and U_F represent many-body cluster contributions to the kinetic energy. The values of these contributions in symmetric nuclear and pure neutron matter are given in table 1 at $\rho = 0.159$ fm^{-3} the assumed equilibrium density of nuclear matter.

TABLE 1

The breakdown of nuclear and neutron matter energy

$\rho(\mathrm{fm}^{-3})$	0.159	0.159
β	0	1.0
T_F	22.01	34.93
E_{2B}	−36.89	−19.48
$W_0(\mathrm{MB})$	−3.40	−2.05
W_c	2.94	1.99
W_s^*	5.07	1.53
$U+U_F+W_F(\mathrm{MB})$	0.38	0.046
TNA	−6.11	−2.04
E_{tot}	−16.00	14.94

The W_s^* is the sum of W_s, W_{cs}, $E_{LS}(\mathrm{MB})$ and $E_Q(\mathrm{MB})$ of I.

It is relatively simpler to calculate the contribution of many-particle isospin correlations in the symmetric nuclear matter. In this case only the so called C-part [4]) of the product of isospin operators is needed, and only closed rings of isospin operators have a non-zero C-part. Further the C-part of a product of $\tau_i \cdot \tau_j$ operators forming a single operator ring is independent of the order of the operators, and so it is simple to sum all single operator rings with chain equations [4]). All these simplifications are lost when we consider the $\beta \neq 0$ asymmetric matter. However calculating the energy of $\beta = 1$ neutron matter is simple. In this case $\tau_i \cdot \tau_j$ operators can be replaced by unity and the two nucleon correlation operator becomes a sum of central, spin-spin tensor and spin-orbit correlations.

With the hope of using the available $\beta = 0$ and 1 results in the calculation of $E(\rho, \beta)$ we studied the β-dependence of E_{2B}, $W_0(\mathrm{MB})$, W_c and W_s. That of E_{2B} and $W_0(\mathrm{MB})$ is studied numerically by calculating these at $\beta^2 = 0$ to 1 in steps of 0.1. It is difficult to sum W_c and W_s contributions by chain equations when $\beta \neq 0$ or 1. However three-body diagrams give the largest contribution to W_c and W_s. Their β-dependence is studied analytically and it seems that they can have significant β^0

and β^2 terms, but rather small $\beta^{n\geqq 4}$ terms. Even four-body chain diagrams do not give large β^4 contribution.

To a surprisingly high accuracy [0.3% (0.4%) at $\rho = 0.159$ (0.35)] the large $E_{2B}(\rho, \beta)$ can be reproduced by a sum of β^0 and β^2 terms. The calculated $W_0(\text{MB}, \rho, \beta)$ also has small $\beta^{n\geqq 4}$ dependence. The coefficient of the β^4 term of $W_0(\text{MB}, \rho, \beta)$ is practically zero at $\rho = 0.159$, and it is of the order of 1 MeV at $\rho = 0.35$ fm^{-3}. The TNA has by definition only β^0 and β^2 terms, and so, if we assume that the small U, U_F and $W_F(\text{MB})$ terms also have negligible $\beta^{n\geqq 4}$ dependence, it appears that the $E(\rho, \beta)$ may be well approximated by:

$$E(\rho, \beta) \simeq T_F(\rho, \beta) + \text{EI}_0(\rho) + \beta^2 \text{EI}_2(\rho). \tag{1.2}$$

This is the main result of this work.

The $\text{EI}_0(\rho)$ and $\text{EI}_2(\rho)$ obtained from the existing results [3]) for $E(\rho, \beta = 0, 1)$, and the symmetry energy $E_{\text{sym}}(\rho)$ are reported in sect. 2. Sect. 3 reports the two-body Euler-Lagrange equations for asymmetric matter, and the calculation of $E_{2B}(\rho, \beta)$. The generalization of FHNC equations to the case of asymmetric matter, and the calculation of $W_0(\text{MB}, \rho, \beta)$ is reported in sect. 4. The β-dependence of W_c and W_s is analysed in sect. 5. The main results are given in sect. 2; sects. 3–5 are rather technical, and assume familiarity with refs. [1,4]).

2. Results

The interaction energy of nuclear matter is defined as:

$$\text{EI}(\rho, \beta) = E(\rho, \beta) - T_F(\rho, \beta). \tag{2.1}$$

Assuming validity of the approximation (1.2) we have:

$$\text{EI}_0(\rho) = \text{EI}(\rho, 0), \tag{2.2}$$

$$\text{EI}_2(\rho) = \text{EI}(\rho, 1) - \text{EI}(\rho, 0). \tag{2.3}$$

The $\text{EI}_0(\rho)$ and $\text{EI}_2(\rho)$ are calculated using the $E(\rho, 0)$ and $E(\rho, 1)$ tabulated in ref. [3]). Five point interpolations were used to obtain $E(\rho, \beta = 0, 1)$ at the desired values of ρ. The $E_{\text{sym}}(\rho)$ is defined as:

$$E_{\text{sym}}(\rho) = \frac{1}{2} \left. \frac{\partial^2 E(\rho, \beta)}{\partial \beta^2} \right|_{\beta = 0} \tag{2.4}$$

and it is given by

$$E_{\text{sym}}(\rho) = \tfrac{5}{9} T_F(\rho, 0) + \text{EI}_2(\rho). \tag{2.5}$$

The $\text{EI}_0(\rho)$, $\text{EI}_2(\rho)$ and $E_{\text{sym}}(\rho)$ are tabulated in table 2.

TABLE 2

The coefficients $EI_0(\rho)$ and $EI_2(\rho)$ of the interaction energy of nuclear matter, and the symmetry energy of nuclear matter

ρ	EI_0	EI_2	E_{sym}
0.0492	−18.37	9.54	15.13
0.0676	−23.04	11.59	18.51
0.0975	−29.49	14.39	23.22
0.1257	−34.12	16.30	26.76
0.1589	−38.01	17.73	29.96
0.1975	−40.76	18.59	32.72
0.2234	−41.75	18.76	34.11
0.2515	−42.17	19.01	35.62
0.2767	−41.98	18.66	36.36
0.3034	−41.42	17.94	36.75
0.3497	−39.74	17.34	38.02

The calculated symmetry energy at the equilibrium density ($\rho = 0.16$ fm^{-3}) is 30 MeV. Empirically the symmetry energy is not very accurately determined. Its values range from 28–40 MeV [refs. [5,6]] in mass formulas. Our results for E_{sym} are in fair agreement with the results obtained by Fantoni and Rosati [7]) (31.1 MeV at $\rho = 0.17$ fm^{-3}) with the semi-realistic OMY potential, and by Seimens and Sjöberg [8]) (25–30 MeV at $k_\rho = 1.35$ fm^{-1}) with the Reid potential and lowest order Brueckner theory.

3. Calculation of $E_{2B}(\rho, \beta)$

The variational calculations use a variational wave function:

$$\Psi_V(\rho, \beta) = \{S \prod_{i<j} [\sum_{p=1,8} f^p(r_{ij}, d, d_t, \alpha) O^p_{ij}]\} \Phi(\rho, \beta). \tag{3.1}$$

The variational parameters d, d_t and α should be varied in principle to minimize the $E(\rho, \beta)$. However, the equilibrium values of d, d_t and α are not too different in nuclear and neutron matter. For example at $\rho = 0.159$ fm^{-3} the equilibrium values of d, d_t and α in nuclear and neutron matter are respectively 2.15, 3.44, 0.8 and 2.79, 3.44, 0.8, and the neutron matter energy obtained with these two sets of d, d_t and α is 14.9 and 14.6 MeV respectively. In the following sections we neglect the β-dependence of d, d_t and α and take their values from symmetric nuclear matter calculations. The results presented in sect. 2 tacitly assume that the small effect of the β-dependence of d, d_t and α on $E(\rho, \beta)$ is linear in β^2.

The $f^p(d, d_t, \alpha)$ are calculated from two-body Euler-Lagrange equations [eqs. (2.14)–(2.24) of I] which depend upon the Φ. Thus even for fixed values of d, d_t and α the f^p depend upon β and ρ. The β and ρ dependence of the f^p equations is contained in the functions $\phi^x_{T,S}(r,\rho,\beta)$ given by eqs. (2.8)–(2.11) of I. Here x = c, q or qq (for central, L^2 and L^4), and T, S are the pair isospin and spin. In asymmetric matter we have two Fermi momenta k_{Fn} and k_{Fp} for neutrons and protons, and it is convenient to define functions $\Psi^x_{\lambda\mu}(T,S,r)$:

$$(\Psi^x_{\lambda\mu}(T,S,r))^2 = \frac{1}{\Omega^2}\sum_{k_\lambda > k_{\mathrm{F}\lambda}}\sum_{k_\mu < k_{\mathrm{F}\mu}}[\psi^*(\boldsymbol{k}_\lambda\boldsymbol{k}_\mu)-(-1)^{T+S}\psi^*(\boldsymbol{k}_\mu\boldsymbol{k}_\lambda)]O^x_{12}\psi(\boldsymbol{k}_\lambda\boldsymbol{k}_\mu), \tag{3.2}$$

$$\psi(\boldsymbol{k}_\lambda\boldsymbol{k}_\mu) = \exp(i(\boldsymbol{k}_\lambda\cdot\boldsymbol{r}_1+\boldsymbol{k}_\mu\cdot\boldsymbol{r}_2)), \tag{3.3}$$

where λ and μ can be n or p for neutrons and protons. The $\phi^x_{T,S}(r,\rho,\beta)$ are given by

$$(\phi^x_{T,S}(r,\rho,\beta))^2 = (\Psi^x_{\mathrm{np}}(T,S,r))^2 + T\{(\Psi^x_{\mathrm{nn}}(T,S,r))^2+(\Psi^x_{\mathrm{pp}}(T,S,r))^2\}. \tag{3.4}$$

The explicit forms of $\Psi^x_{\lambda\mu}(T,S,r)$ are given below:

$$(\Psi^{\mathrm{c}}_{\lambda\mu}(T,S,r))^2 = \tfrac{1}{4}\rho_\lambda\rho_\mu\{1-(-1)^{T+S}l_\lambda l_\mu\}, \tag{3.5}$$

$$(\Psi^{\mathrm{q}}_{\lambda\mu}(T,S,r))^2 = \tfrac{1}{4}\rho_\lambda\rho_\mu\{\tfrac{1}{10}r^2(k^2_{\mathrm{F}\lambda}+k^2_{\mathrm{F}\mu})-(-1)^{T+S}\tfrac{1}{2}\boldsymbol{r}\cdot\nabla l_\lambda l_\mu\}, \tag{3.6}$$

$$(\Psi^{\mathrm{qq}}_{\lambda\mu}(T,S,r))^2 = \tfrac{1}{4}\rho_\lambda\rho_\mu\{r^4(\tfrac{1}{70}(k^4_{\mathrm{F}\lambda}+k^4_{\mathrm{F}\mu})+\tfrac{1}{25}k^2_{\mathrm{F}\lambda}k^2_{\mathrm{F}\mu}+\tfrac{1}{3}r^2(k^2_{\mathrm{F}\lambda}+k^2_{\mathrm{F}\mu}) \\ -(-1)^{T+S}\tfrac{1}{2}(\boldsymbol{r}\cdot\nabla)^2 l_\lambda l_\mu\}. \tag{3.7}$$

Here l_λ is the familiar Slater function $l(k_{\mathrm{F}\lambda}r)$.

The $f^p(\rho,\beta)$ are obtained by solving the eqs. (2.14)–(2.24) of I with the above $\Phi^x_{T,S}$. They do not exhibit significant β-dependence. The important correlations, such as f^{c} or f^{tr} change by $<2\%$ in going from $\beta = 0$ to 1. Nevertheless this β-dependence is taken into account in the following calculations.

In symmetric nuclear matter only the C-parts of operator products contribute. In asymmetric matter terms linear in O^τ_{ij} also contribute. We define this term as the T_{ij} part of the ΠO^p,

$$\Pi O^p = C(\Pi O^p)+\sum_{i<j}T_{ij}(\Pi O^p)O^\tau_{ij}+\dots. \tag{3.8}$$

The T_{ij} parts, like the C-parts do not contain any σ or τ operators and

$$T_{ij}(\Pi O^p) = \tfrac{1}{3}C(O^\tau_{ij}\Pi O^p). \tag{3.9}$$

With these definitions $E_{2\mathrm{B}}(\rho,\beta)$ is given by:

$$E_{2\mathrm{B}} = \sum_{i,k=1,8}\sum_{j=1,14}\tfrac{1}{2}\rho\int \mathrm{d}^3r\langle C(f^i_{12}O^i_{12}H^j_{12}O^j_{12}f^k_{12}O^k_{12}) \\ +O^\tau_{12}T_{12}(f^i_{12}O^i_{12}H^j_{12}O^j_{12}f^k_{12}O^k_{12})\rangle_{\mathrm{dir}}$$

$$
\begin{aligned}
&- \sum_{n=1,4} \tfrac{1}{8}\rho \int \mathrm{d}^3 r \langle C(O^n_{12} f^i_{12} O^i_{12} H^j_{12} O^j_{12} f^k_{12} O^k_{12}) \\
&\qquad\qquad + O^{\tau}_{12} T_{12} (O^n_{12} f^i_{12} O^i_{12} H^j_{12} O^j_{12} f^k_{12} O^k_{12}) \rangle_{\mathrm{ex}} \\
&+ \sum_{n=1,4} \tfrac{1}{4}\rho \frac{\hbar^2}{m} \int \mathrm{d}^3 r \langle C(O^n_{12} f^i_{12} O^i_{12} \nabla f^k_{12} O^k_{12}) \cdot \nabla \\
&\qquad\qquad + O^{\tau}_{12} T_{12} (O^n_{12} f^i_{12} O^i_{12} \nabla f^k_{12} O^k_{12}) \cdot \nabla \rangle_{\mathrm{ex}} . \qquad (3.10)
\end{aligned}
$$

The three integrals above correspond to the contributions of W_0 diagrams 1.1 and 1.2 and W_F diagram 1.3 of fig. 1 respectively. The C and T_{12} parts can be expressed

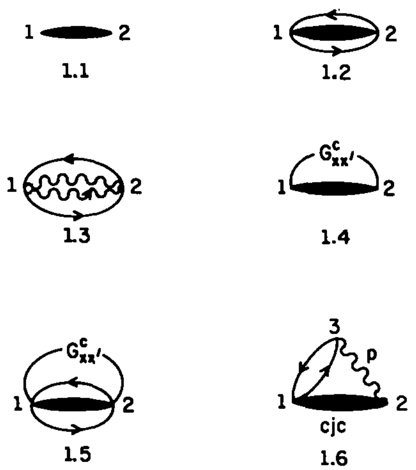

Fig. 1. Diagrams 1.1–3 give the two-body energy, while 1.4 and 1.5 illustrate the W_0(MB) contribution. The three-body diagram 1.6 forms part of the W_c contribution.

TABLE 3

The expectation values of L^n and τL^n operators

$y_n = \rho_n/\rho$	$y_p = \rho_p/\rho$
$l_c = y_n l_n + y_p l_p$	$l_\tau = y_n l_n - y_p l_p$
$\langle 1 \rangle_{\mathrm{dir}} = 1$	$\langle \tau \rangle_{\mathrm{dir}} = \beta^2$
$\langle L^2 \rangle_{\mathrm{dir}} = \frac{1}{5} r^2 (y_n k^2_{\mathrm{Fn}} + y_p k^2_{\mathrm{Fp}})$	$\langle L^2 \tau \rangle_{\mathrm{dir}} = \frac{1}{5} r^2 \beta (y_n k^2_{\mathrm{Fn}} - y_p k^2_{\mathrm{Fp}})$
$\langle L^4 \rangle_{\mathrm{dir}} = 2\langle L^2 \rangle_{\mathrm{dir}} + A + B$	$\langle L^4 \tau \rangle_{\mathrm{dir}} = 2\langle L^2 \tau \rangle_{\mathrm{dir}} + A - B$
$A = \frac{12}{175} r^4 \{ y_n^2 k^4_{\mathrm{Fn}} + y_p^2 k^4_{\mathrm{Fp}} \}$	
$B = \frac{1}{175} r^4 y_n y_p \{ 5(k^4_{\mathrm{Fn}} + k^4_{\mathrm{Fp}}) + 14 k^2_{\mathrm{Fn}} k^2_{\mathrm{Fp}} \}$	
$\langle 1 \rangle_{\mathrm{ex}} = l_c^2$	$\langle \tau \rangle_{\mathrm{ex}} = l_\tau^2$
$\langle L^2 \rangle_{\mathrm{ex}} = \langle \boldsymbol{r} \cdot \nabla \rangle_{\mathrm{ex}} = \frac{1}{2} \boldsymbol{r} \cdot \nabla l_c^2$	$\langle L^2 \tau \rangle_{\mathrm{ex}} = \langle \boldsymbol{r} \cdot \nabla \tau \rangle_{\mathrm{ex}} = \frac{1}{2} \boldsymbol{r} \cdot \nabla l_\tau^2$
$\langle L^4 \rangle_{\mathrm{ex}} = 2\langle L^2 \boldsymbol{r} \cdot \nabla \rangle_{\mathrm{ex}} = \frac{1}{2} (\boldsymbol{r} \cdot \nabla)^2 l_c^2$	$\langle L^4 \tau \rangle_{\mathrm{ex}} = 2\langle \tau L^2 \boldsymbol{r} \cdot \nabla \rangle_{\mathrm{ex}} = \frac{1}{2} (\boldsymbol{r} \cdot \nabla)^2 l_\tau^2$

as sum over terms containing 1, L^2, L^4, $r \cdot \nabla$ and $L^2 r \cdot \nabla$ operators. The required expectation values $\langle\ \rangle_{\mathrm{dir}}$ and $\langle\ \rangle_{\mathrm{ex}}$ of these operators in asymmetric matter are given in table 3, where O_{12}^{τ} is abbreviated by τ.

The calculated $E_{2\mathrm{B}}(\rho, \beta)$ at $\rho = 0.159\ \mathrm{fm}^{-3}$ is given in table 4. It is almost exactly linear in β^2. The absence of significant β^4 terms in $E_{2\mathrm{B}}$ can be understood as follows. The f^{p}'s have little β-dependence, so the β-dependence of $E_{2\mathrm{B}}$ must come from the expectation values in table 3. $\langle 1 \rangle_{\mathrm{dir}}$ and $\langle \tau \rangle_{\mathrm{dir}}$ do not have β^4 terms, while the β^4 terms in $\langle L^2 \rangle_{\mathrm{dir}}$ and $\langle L^2 \tau \rangle_{\mathrm{dir}}$ are very small as can be seen by expanding these in powers of β:

$$\langle L^2 \rangle_{\mathrm{dir}} \propto 1 + \tfrac{5}{9}\beta^2 + \tfrac{5}{9}\tfrac{1}{27}\beta^4 + \ldots, \tag{3.11}$$

$$\langle L^2 \tau \rangle_{\mathrm{dir}} \propto \beta^2 - \tfrac{1}{27}\beta^4 + \ldots. \tag{3.12}$$

The contribution of $\langle L^4 \rangle$ terms is very small and it does not give significant β^4 terms. So it is understandable that the direct part of $E_{2\mathrm{B}}$ has no significant $\beta^{n \geq 4}$ dependence.

The exchange part of $E_{2\mathrm{B}}$ involves l_{c} and l_{τ} functions which may be expanded in powers of β as follows. Let k_{F} denote the Fermi momentum of nuclear matter at density ρ, and $x = k_{\mathrm{F}} r$. We get

$$l_{\mathrm{c}}(r) = \sum_{n=0,\infty} \frac{1}{(2n)!} \frac{1}{3^{2n}} \beta^{2n} x^{6n-3} \left(\frac{1}{x^2} \frac{\mathrm{d}}{\mathrm{d}x} \right)^{2n} (x^3 l(x)), \tag{3.13}$$

$$l_{\tau}(r) = \sum_{n=0,\infty} \frac{1}{(2n+1)!} \frac{1}{3^{2n+1}} \beta^{2n+1} x^{6n} \left(\frac{1}{x^2} \frac{\mathrm{d}}{\mathrm{d}x} \right)^{2n+1} (x^3 l(x)). \tag{3.14}$$

The above can be rewritten as

$$l_{\mathrm{c}}(r) = l_0(x) + l_2(x)\beta^2 + l_4(x)\beta^4 + \ldots, \tag{3.15}$$

$$l_{\tau}(r) = \beta l_1(x) + \beta^3 l_3(x) + \ldots, \tag{3.16}$$

where $l_i(x)$ are functions independent of β. Since the l_{c} and l_{τ} are multiplied by short ranged functions with a typical range of ~ 1.5 fm their main contribution comes from small distances. However $l_{i \geq 2}(x)$ are quite small at small r as can be seen from fig. 2, and hence the exchange part of $E_{2\mathrm{B}}$ is also quite linear in β^2.

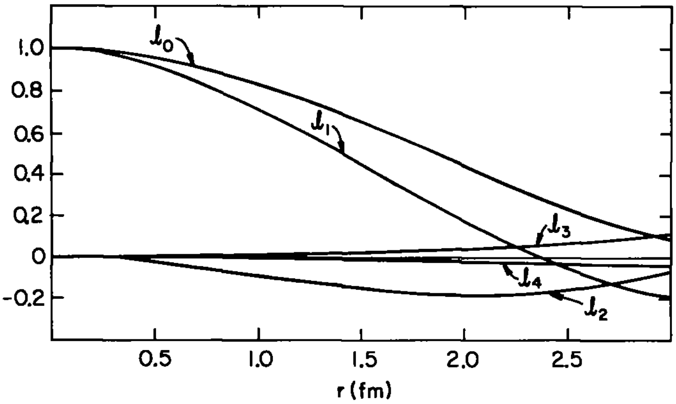

Fig. 2. The functions l_{1-4} at $k_F = 1.33\ \mathrm{fm}^{-1}$.

4. Calculation of central correlation chains and W_0(MB)

The central correlation chains are treated in the FHNC approximation. In asymmetric matter we must keep track of neutron and proton exchange loops separately. This is done by classifying the chains as: G_{dd}, G^{n}_{de}, G^{p}_{de}, G^{nn}_{ee}, G^{np}_{ee}, G^{pp}_{ee}, G^{n}_{cc} and G^{p}_{cc}. The subscripts, d for direct, e for closed exchange loop and c for incomplete chain of exchanges, specify the exchange patterns at the ends of the chain. The superscripts, n for neutron and p for proton, specify the type of exchange loops at the ends. The G^{n}_{de} has a neutron exchange loop at one end, G^{np}_{ee} has a neutron loop at one end and a proton loop at the other, and G^{p}_{cc} has incomplete proton exchange chain etc.

The derivation of the equations for the G's is quite straightforward, and hence we merely give the results. In the following equation λ, μ can be n or p.

We define the generalized Slater functions:

$$L_\mu = -l_\mu + 2G^{\mu}_{cc} \tag{4.1}$$

and partial distribution functions:

$$g_{dd} = (f^c)^2 \exp(G_{dd}), \tag{4.2}$$

$$g^{\mu}_{de} = g_{dd} G^{\mu}_{de}, \tag{4.3}$$

$$g^{\mu\lambda}_{ee} = g_{dd}(G^{\mu}_{de}G^{\lambda}_{de} + G^{\mu\lambda}_{ee} - \tfrac{1}{2}L^2_\mu \delta_{\lambda\mu}), \tag{4.4}$$

$$g^{\mu}_{cc} = \tfrac{1}{2} g_{dd} L_\mu \tag{4.5}$$

and the link functions:

$$X_{dd} = g_{dd} - G_{dd} - 1, \tag{4.6}$$

$$X^{\mu}_{\rm de} = g^{\mu}_{\rm de} - G^{\mu}_{\rm de}, \tag{4.7}$$

$$X^{\mu}_{\rm cc} = \tfrac{1}{2}(g_{\rm dd} - 1)L_{\mu}, \tag{4.8}$$

$$X^{\lambda\mu}_{\rm ee} = g^{\lambda\mu}_{\rm ee} - G^{\lambda\mu}_{\rm ee}. \tag{4.9}$$

The integral operators that join links at neutron or proton vertices are given by:

$$\Theta_{\mu}(X(r_{ik}); Y(r_{kj})) = \rho_{\mu} \int {\rm d}^3 r_k X(r_{ik}) Y(r_{kj}). \tag{4.10}$$

The chain equations are obtained as

$$G_{\rm dd} = \sum_{\nu = \rm n,p} [\Theta_{\nu}(X_{\rm dd} + X^{\nu}_{\rm de}; g_{\rm dd} - 1) + \Theta_{\nu}(X_{\rm dd}; g^{\nu}_{\rm de})], \tag{4.11}$$

$$G^{\mu}_{\rm de} = \sum_{\nu = \rm n,p} [\Theta_{\nu}(X_{\rm dd} + X^{\nu}_{\rm de}; g^{\mu}_{\rm de}) + \Theta_{\nu}(X_{\rm dd}; g^{\nu\mu}_{\rm ee})], \tag{4.12}$$

$$G^{\lambda\mu}_{\rm ee} = \sum_{\nu = \rm n,p} [\Theta_{\nu}(X^{\lambda}_{\rm ed} + X^{\lambda\nu}_{\rm ee}; g^{\mu}_{\rm de}) + \Theta_{\nu}(X^{\lambda}_{\rm ed}; g^{\nu\mu}_{\rm ee})], \tag{4.13}$$

$$G^{\mu}_{\rm cc} = \Theta_{\mu}(X^{\mu}_{\rm cc}; g^{\mu}_{\rm cc}). \tag{4.14}$$

These chain functions are used to calculate the FHNC contribution to $W_0(\rm MB)$ represented by diagrams of type 1.4, 1.5 of fig. 1. The main contribution of these diagrams is thought to come from terms having i, j and $k \leqq 6$. The terms having i, j or $k > 6$ give relatively smaller contribution to $W_{\rm 2B}$, and we neglect their contribution to $W_0(\rm MB)$ in the present work.

With the $l_{\rm c}$, l_{τ}, $y_{\rm p}$ and $y_{\rm n}$ of table 3 we define:

$$G_{\rm c,de} = y_{\rm n} G^{\rm n}_{\rm de} + y_{\rm p} G^{\rm p}_{\rm de}, \tag{4.15a}$$

$$G_{\tau,\rm de} = y_{\rm n} G^{\rm n}_{\rm de} - y_{\rm p} G^{\rm p}_{\rm de}, \tag{4.15b}$$

$$G_{\rm c,ee} = y^2_{\rm n} G^{\rm nn}_{\rm ee} + y^2_{\rm p} G^{\rm pp}_{\rm ee} + 2y_{\rm p} y_{\rm n} G^{\rm np}_{\rm ee}, \tag{4.16}$$

$$G_{\tau,\rm ee} = y^2_{\rm n} G^{\rm nn}_{\rm ee} + y^2_{\rm p} G^{\rm pp}_{\rm ee} - 2y_{\rm p} y_{\rm n} G^{\rm np}_{\rm ee}, \tag{4.17}$$

$$L_{\rm c} = y_{\rm n} L_{\rm n} + y_{\rm p} L_{\rm p}, \tag{4.18}$$

$$L_{\tau} = y_{\rm n} L_{\rm n} - y_{\rm p} L_{\rm p}, \tag{4.19}$$

$$h^{\rm c} = \exp(G^{\rm c}_{\rm dd}). \tag{4.20}$$

The contribution of W_0(MB) diagrams 1.4 and 1.5 of fig. 1 is given by

$$W_0(\mathrm{MB})_{1.4} = \tfrac{1}{2}\rho \sum_{i,j,k=1,6} \int \mathrm{d}^3r\{C(f^iO^iH^jO^jf^kO^k)$$
$$\times\{h^c[1+2G_{c,\mathrm{de}}+G_{c,\mathrm{ee}}+(G_{c,\mathrm{de}})^2]-1\}$$
$$+T(f^iO^iH^jO^jf^kO^k)[h^c[\beta^2+2G_{\tau,\mathrm{de}}+G_{\tau,\mathrm{ee}}+(G_{\tau,\mathrm{dc}})^2]-\beta^2]\}, \quad (4.21)$$

$$W_0(\mathrm{MB})_{1.5} = -\tfrac{1}{8}\rho \sum_{i,j,k=1,8}\ \sum_{n=1,4} \int \mathrm{d}^3r\{C(O^nf^iO^iH^jO^jf^kO^k)(h^cL_c^2-l_c^2)$$
$$+T(O^nf^iO^iH^jO^jf^kO^k)(h^cL_\tau^2-l_\tau^2)\}. \quad (4.22)$$

The calculated W_0(MB) at $\rho = 0.159\ \mathrm{fm}^{-3}$ is given in table 4, it is quite linear in β^2.

TABLE 4

The calculated E_{2B} and W_0(MB) at $\rho = 0.159\ \mathrm{fm}^{-3}$ at various values of β^2

β^2	E_{2B}	W_0(MB)	$E_{2B}+W_0$(MB)	$A+B\beta^2$
0	−36.88	−3.08	−39.97	*
0.1	−35.14	−3.02	−38.17	−38.13
0.2	−33.40	−2.95	−36.36	−36.29
0.3	−31.65	−2.88	−34.54	−34.45
0.4	−29.90	−2.81	−32.71	−32.61
0.5	−28.13	−2.74	−30.88	−30.77
0.6	−26.36	−2.67	−29.03	−28.93
0.7	−24.57	−2.60	−27.18	−27.09
0.8	−22.77	−2.53	−25.31	−25.25
0.9	−20.96	−2.46	−23.41	*
0.99	−19.29	−2.39	−21.68	−21.76

The coefficients A and B of the last column are determined from $E_{2B}+W_0$(MB) at $\beta^2 = 0$ and 0.9.

Note that the W_0(MB) in table 1 differs from that in table 4 at $\beta = 0$ and 1. In the calculation of nuclear and neutron matter [3]) we include (i) some of the terms having one or more of i, j and $k = 7, 8$ and (ii) single operator rings in the links of FHNC [4]). Both these small effects are neglected here.

5. β-dependence of W_c and W_s

The three-body diagrams give the largest contribution to W_c and W_s, and hence we first discuss their β-dependence. Any product Π of any number of O_{12}, O_{23} and O_{31} operators can be reduced by repeated use of the Pauli identity to the form:

$$\Pi = C(\Pi)+\sum_{i<j\leq 3} T_{ij}(\Pi)\tau_i\tau_j+B\tau_1\cdot\tau_2 x\tau_3+\text{terms having } \sigma_{i=1,3} \text{ operators.} \quad (5.1)$$

In direct diagrams the $\langle \Pi \rangle$ is simply

$$\langle \Pi \rangle = \langle C(\Pi) \rangle + \beta^2 \sum_{i<j\leqq 3} \langle T_{ij}(\Pi) \rangle ; \tag{5.2}$$

the $\tau_1 \cdot \tau_2 \times \tau_3$ gives zero contribution. Hence, to the extent the β-dependence of the f^p can be neglected, the contributions of all three-body direct W_c and W_s diagrams have only β^0 and β^2 terms.

Exchange three-body diagrams involve the functions l_c and l_τ, and in principle their contribution can have $\beta^{n\geqq 4}$ terms. However they appear to be small. For example the contribution of the important W_c exchange diagram 1.6 of fig. 1, is given by ($j, p = \sigma$ or t)

$$-\tfrac{1}{8}\rho^2 A^j \int f_{12}^{c^2} v_{12}^{j\tau} F_{23}^{p\tau} \xi_{13\,2}^{\sigma p j} (3l_c^2 - l_\tau^2)_{13} \mathrm{d}^3 r_{12} \mathrm{d}^3 r_{13}. \tag{5.3}$$

To extract the β-dependence we expand $3l_c^2 - l_\tau^2$ in powers of β^2:

$$3l_c^2 - l_\tau^2 = 3l_0^2 + (6l_0 l_2 - l_1^2)\beta^2 + (3l_2^2 - 2l_1 l_3 + 6l_0 l_4)\beta^4. \tag{5.4}$$

The function multiplying β^4 is much smaller than that multiplying β^2 in the above expansion, and so we may expect these contributions to have only β^0 and β^2 terms. At $k_F = 1.33\ \mathrm{fm}^{-1}$ and $r < 2$ fm the coefficient of β^4 in (5.4) is $<5\,\%$ of that of the β^2 term.

It can be shown that in three-body W_c or W_s diagrams with exchanges, the β^4 terms can come only through the $l_{i\geqq 2}$ functions. The contribution of such diagrams has a product of operators Π that can be reduced to the form 5.1. It is convenient to include in this product the spin exchange operators $\frac{1}{2}(1+\sigma_i \cdot \sigma_j)$, but treat the isospin exchange explicitly. One then has to consider the eight different possibilities in which the particles 1, 2 and 3 in the right-hand Ψ are nnn, nnp, npn, pnn, ppp, ppn, pnp and npp. The contribution for any given possibility can be written as an integral containing the β-dependence via the y_n, y_p, l_n and l_p. The sum of the contributions of all the eight possibilities can be expressed, as is done to obtain eq. (5.4), with an integral containing l_c, l_τ and β. To the extent all but l_0 and l_1 can be neglected the contribution has only β^0 and β^2 terms.

Four-body W_c and W_s diagrams can give β^4 terms. However, their contribution is small (~ 1 MeV) and so their β^4 terms should also be small. The β-dependence of four-body spin-isospin and tensor-isospin direct single-operator-chain diagrams is easy to calculate with the following method. The β-dependence comes from the product Π of the four $\tau_i \cdot \tau_j$ operators in the chain. The contribution depends upon the order of the four $\tau_i \cdot \tau_j$ operators in Π, and the total contribution is the sum of their contributions in all possible orders. Let us consider the simple order:

$$\tau_1 \cdot \tau_2 \tau_2 \cdot \tau_3 \tau_3 \cdot \tau_4 \tau_4 \cdot \tau_1 = \tau_{1i}\tau_{1l}\tau_{2i}\tau_{2j}\tau_{3j}\tau_{3k}\tau_{4k}\tau_{4l}. \tag{5.5}$$

In asymmetric matter the $\langle \tau_i \tau_j \rangle$ is given by the matrix Q_{ij}

$$\langle \tau_i \tau_j \rangle = Q_{ij} = \begin{bmatrix} 1 & i\beta & 0 \\ -i\beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.6}$$

$$\langle \tau_j \tau_i \rangle = Q^*_{ij}, \tag{5.7}$$

and the contribution of term (5.5) is given by $\mathrm{Tr}\,(Q^* QQQ)$.

Both Q and Q^* can be diagonalized simultaneously to the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\beta & 0 \\ 0 & 0 & 1+\beta \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\beta & 0 \\ 0 & 0 & 1-\beta \end{bmatrix}$$

and so the Tr of $(n-s)$ Q-matrices and s Q^*-matrices is given by:

$$\mathrm{Tr}\,(Q^{n-s}(Q^*)^s) = 1+(1+\beta)^s(1-\beta)^{n-s}+(1-\beta)^s(1+\beta)^{n-s}. \tag{5.8}$$

The expectation value of a product of n $\tau_a \cdot \tau_b$ operators forming an SOR in any order has the form (5.8) with a value of s, in range 1 to $n-1$ determined by the order. Thus the expectation value of the symmetrized product is given by:

$$\langle S(\tau_1 \cdot \tau_2 \tau_2 \cdot \tau_3 \ldots \tau_n \cdot \tau_1)\rangle = \sum_{s=1,n-1} P^n_s(1+(1+\beta)^s(1-\beta)^{n-s}+(1-\beta)^s(1+\beta)^{n-s}), \tag{5.9}$$

where P^n_s gives the probability of all orders whose expectation value contains s Q^*-matrices. The P^n_s can be obtained from the recursion relation:

$$P^n_s = \frac{1}{n-1}[sP^{n-1}_s + (n-s)P^{n-1}_{s-1}], \tag{5.10}$$

$$P^2_0 = P^2_2 = 0, \qquad P^2_1 = 1. \tag{5.11}$$

Eq. (5.9) gives the β-dependence of four-body spin-isospin and tensor-isospin direct single operator rings as

$$\langle S(\tau_1 \cdot \tau_2 \tau_2 \cdot \tau_3 \tau_3 \cdot \tau_4 \tau_4 \cdot \tau_1\rangle = 3-\tfrac{8}{3}\beta^2+\tfrac{2}{3}\beta^4. \tag{5.12}$$

The coefficient of β^4 is non-zero, but it is reasonably small. Hence we may expect that the order of magnitude of the β^4 term is probably smaller than the contribution of four-body terms (which is ~1 MeV at $k_F = 1.33\ \mathrm{fm}^{-1}$). Earlier calculations [7,8]) also find that the β^4 term in nuclear matter energy at $k_F = 1.33\ \mathrm{fm}^{-1}$ is less than 1 MeV.

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