Θέματα Αλγορίθμων

Αλγόριθμοι και Εφαρμογές στον Πραγματικό Κόσμο

Μεταπτυχιακό Μάθημα

4η Εβδομάδα: Βέλτιστες Διαδρομές & σε Χρονοεξαρτώμενα Δίκτυα

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Τμήμα Μηχανικών Η/Υ & Πληροφορικής

Πανεπιστήμιο Ιωαννίνων

Τετάρτη, 15-22 Μαρτίου 2017

Shortest Paths

... a fundamental problem in Computer Science

- **INPUT:**
  - Directed graph $G = (V, E)$.
  - Arc costs (distance, travel-time, fuel consumption, etc.): $\forall uv \in E,\ c[uv] \geq 0$.
  - Origin-destination pair: $(o, d) \in V \times V$.
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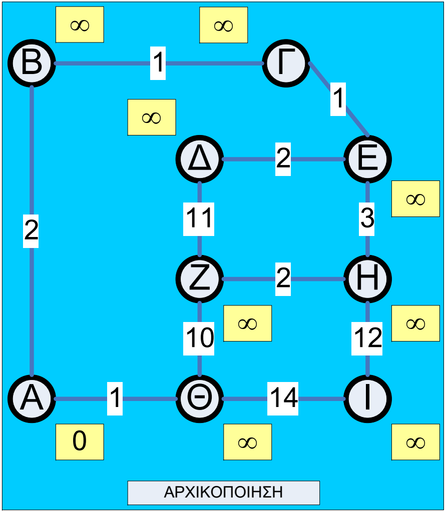
  - Arc costs usually represent *travel-times*.



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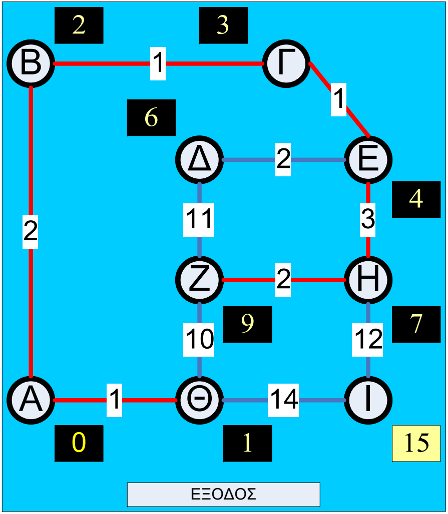

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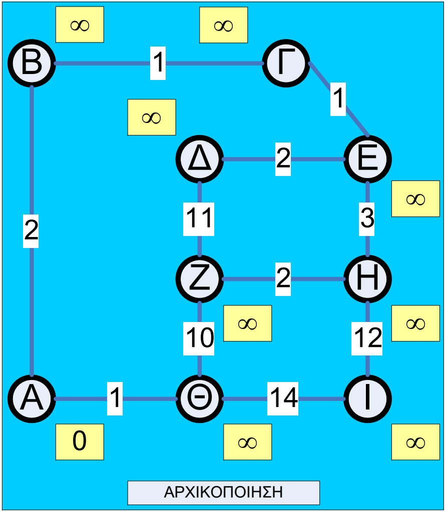
Dijkstra's Algorithm

Pseudocode

```
Dijkstra( G = (V,E), o ∈ V, d ∈ V, c : E → ℝ≥0 )
1.        for all v ∈ V do D[v] = ∞;
2.        D[o] = 0;
3.        Q.Insert(o, D[o]);                                   /* Q: priority queue */
4.        while !Q.IsEmpty() do
4.1.          v = Q.ExtractMin();                              /* v is the node with min tentative label */
4.2.          for all vw ∈ E(G) do                             /* scanning of node v */
4.2.1.            if D[w] > D[v] + c[vw]
4.2.2.            then                                         /* relaxation of arc vw */
4.2.2.1.              D[w] = D[v] + c[vw];
4.2.2.2.              if w ∈ Q then Q.DecreaseKey(w, D[w]);
4.2.2.3.              else Q.Insert(w, D[w]);
```

Dijkstra's Algorithm

Execution Example



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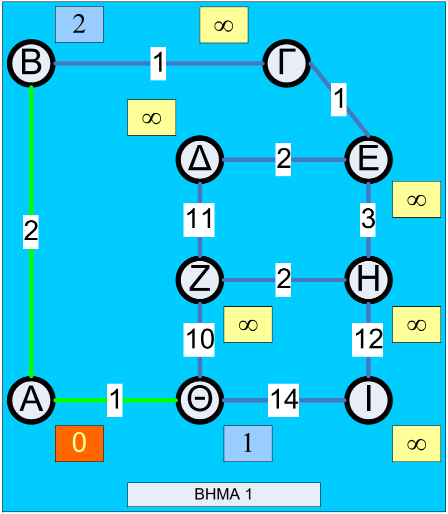

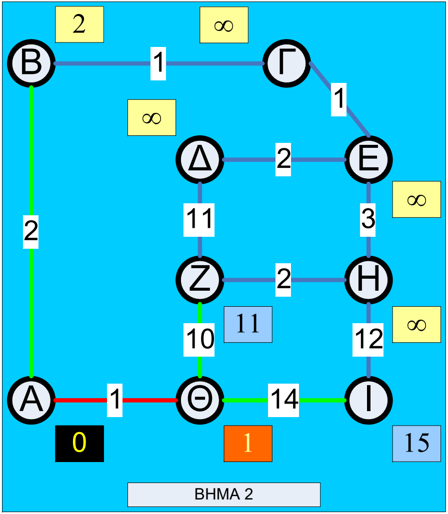

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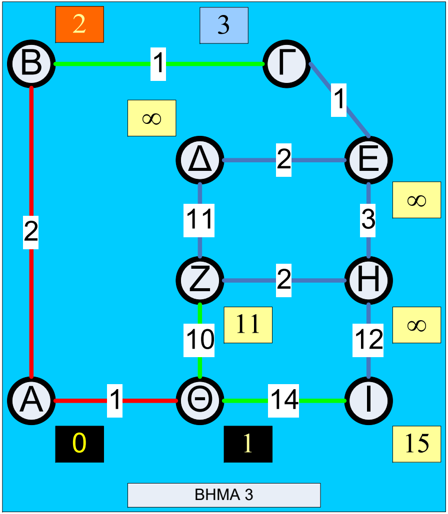
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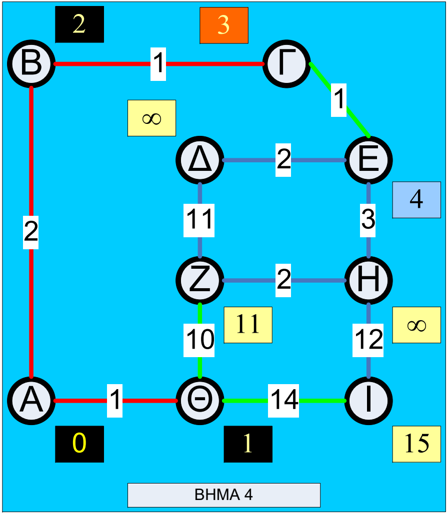
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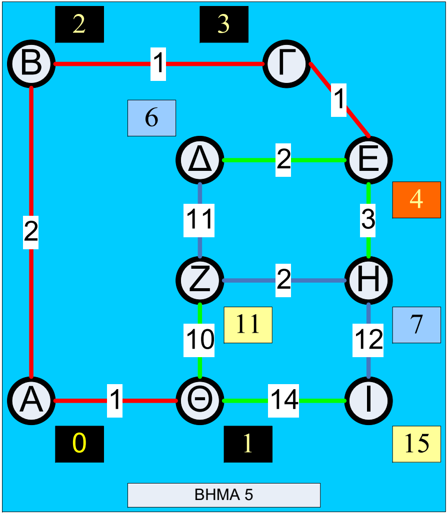

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Dijkstra's Algorithm

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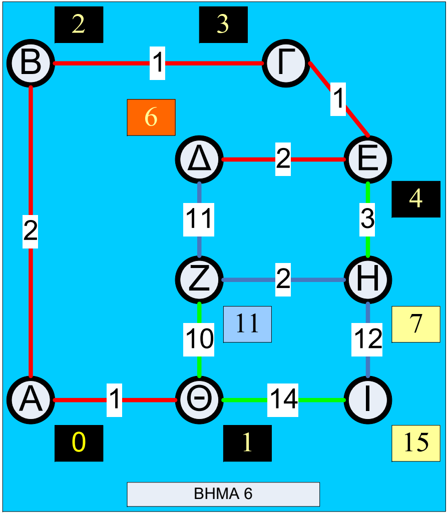

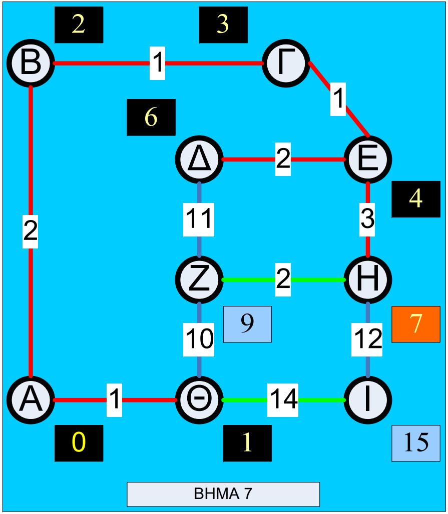
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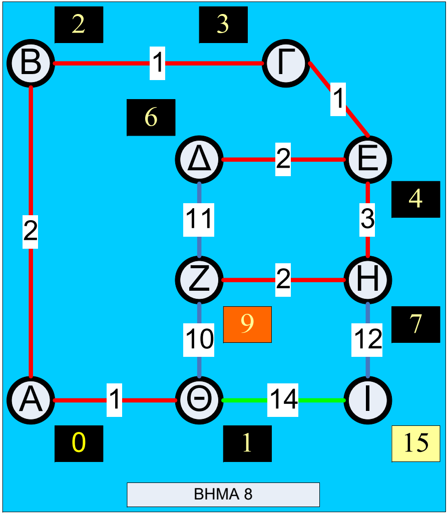
BHMA 7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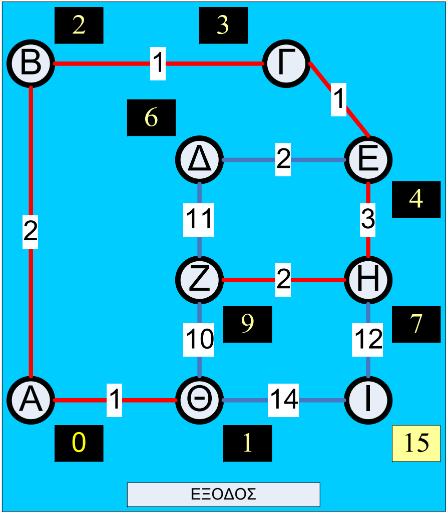
BHMA 8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- Correctness
  - **Labels:** Represent **upper bounds** on total costs (travel-times) from the origin towards each destination.
  - In each round the node v with *minimum tentative label* $D[v]$ is chosen for **finalization** of its label (not tentative anymore).
  - Non-negative arc-costs $\Rightarrow$ The label $D[v]$ to be finalized in each round is the exact min-cost from o to v (cannot be further improved).

Dijkstra's Algorithm

Analysis

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- Time Complexity /* depends on the choice of the priority queue */
  - $O(n)$ queue-insertion operations.
  - $O(n)$ queue-extract-minimum operations.
  - $O(m)$ queue-label correction operations (upon arc relaxations).

- Implementation of the priority queue with **Fibonacci Heaps**
  - $O(\log(n))$ *elementary operations* per extract-minimum operation.
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  ☺ Extremely simpler data structure than Fibonacci Heaps.

  ☺ Usually **faster in practice** (for large-scale, real-world instances).

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- Implementation of priority queue with **k-ary Heaps**
  - Each internal node has k children.
  - Fewer tree levels (than binary / fibonacci heaps), more nodes per level.

  ☺ Better exploitation of data locality.

  - Same time-complexity with Binary Heaps.

- Execution of Dijkstra for **Europe's road netwrok**, with respect to *arc-travel-times* metric:

Data Structure	Response to Queries (sec)
2-heap	12.38
4-heap	11.53
8-heap	11.52

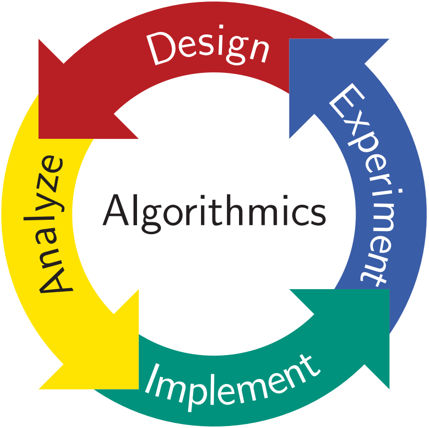
- Execution times on a *2.4GHz* AMD Opteron, with *16GB* RAM

(Microsoft Data Structures and Algorithms School (MIDAS), St. Petersburg (2010))

- Query times are for construction of a complete shortest-paths tree (SPT) from the origin towards **all reachable destinations**.

- Roughly half time for responding to **random** (o, d)-queries and *interrupting* Dijkstra upon scanning the destination vertex d.

Why Algorithm Engineering?



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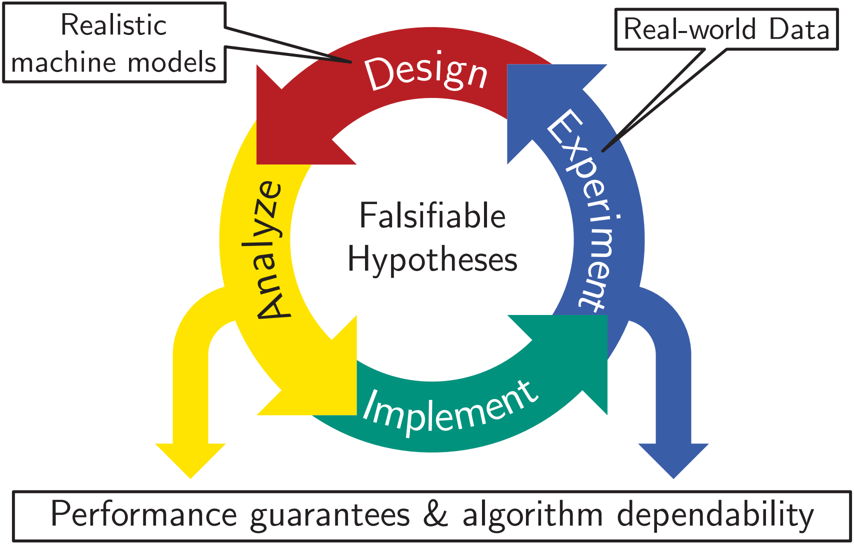

Challenge of Scale

... shortest paths in large-scale road networks

Algorithm Engineering

Shortest paths in road networks: A successful showcase (mostly) in static graphs...

Continent-sized road networks: Millions of intersections

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Most Popular Speedup Techniques

- Arc Flags (Lauther (2004), Köhler et al. (2006), Bauer & Delling (2008))
- A^* with Landmarks (Goldberg & Harrelson (2005))
- Reach (Gutman (2004), Goldberg et al. (2006))
- Highway Hierarchies (Sanders & Schultes (2005))
- Contraction Hierarchies (Geisberger et al. (2008))
- Transit Node Routing (Bast et al. (2006))

Distance Oracles

Another success story in static graphs...

Distance Oracles: Create (**offline**) data structures that require *reasonable space* requirements and allow answering in **real-time** to arbitrary queries *efficiently*, with provable approximation guarantees (stretch).

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  - $O(n+m)$ size.
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Provide smooth tradeoffs among space / query time / stretch!!!

Distance Oracles

Theoretical bounds for static graphs...

Reference	Setting	Stretch	Query	Space
(TZ05)	weighted graph	$2k-1, k \geq 2$	$O(k)$	$O\left(kn^{1+1/k}\right)$
(WN13)	weighted graph	$2k-1, k \geq 2$	$O(\log(k))$	$O\left(kn^{1+1/k}\right)$
(Che13)	weighted graph	$2k-1, k \geq 2$	$O(1)$	$O\left(kn^{1+1/k}\right)$
(AG13)	sparse weighted graph	$1+\epsilon$	$o(n)$	$o(n^2)$
(Kle02) (Tho04)	planar weighted digraph	$1+\epsilon$	$O(\epsilon^{-1})$	$O\left(\frac{n\log(n)}{\epsilon}\right)$
(MN06)	metric	$O(k)$	$O(1)$	$O\left(kn^{1+1/k}\right)$
(BGKRL11)	Doubling metric, dynamic	$1+\epsilon$	$O(1)$	$\epsilon^{-O(ddim)}n + 2^{O(ddim\log(ddim))}n$

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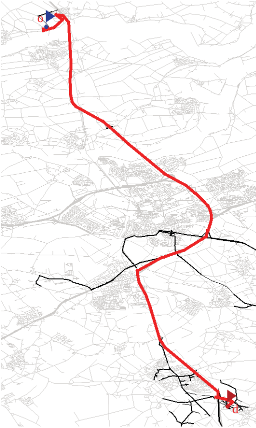

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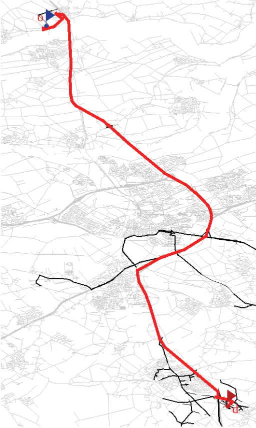

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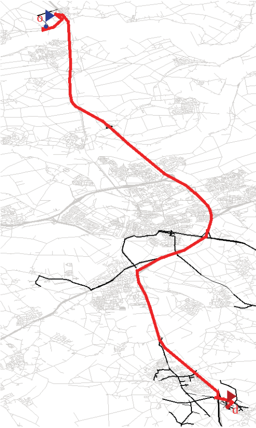

Speedup Techniques / Distance Oracles

Goal...

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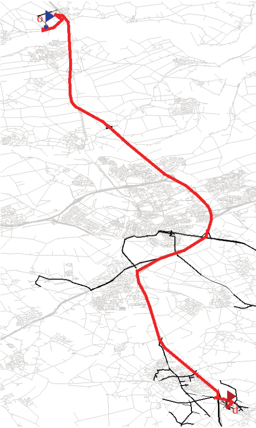

- *Assessment Criteria* of Speedup Techniques / Distance Oracles:
  - Preprocessing time / space.
  - Query (response) time to arbitrary requests.
  - Stretch (approximation guarantee).

Speedup Techniques / Distance Oracles

Generic idea...

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   - Consider (small) sets of *boundary vertices* in a partition of the graph into roughly equal-sized cells.
   - Randomly select *landmark* vertices.
   - Consider nearest *access points* (hubs) per vertex.
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2. **Metric-dependent preprocessing:** Equip the network with selective **distance summaries**, e.g., boundary-to-boundary, hub-to-cell, landmark-to-all distances, etc.
3. **Query Algorithm:** Respond fast to queries, based on the (possibly metric-independent) preprocessing and/or the precomputed metric-dependent distance summaries.

- Extremely successful theme in static graphs.
  - In theory (oracles):
    - PRE-Space: Subquadratic (sometimes quasi-linear).
    - QUE-Time: Constant / sublinear in graph size.
    - Stretch: Small (sometimes PTAS).
  - In practice (speedups):
    - PRE-Space: A few GBs (sometimes less than 1 GB).
    - QUE-Time: Milliseconds (sometimes microseconds).
    - Stretch: Exact distances (in most cases).

Time Dependent

Shortest Path

... a more realistic but also more involved problem

Why Time-Dependent Shortest Paths?

Real-life networks: Elements demostrate temporal behavior.

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Real-life networks: Elements demostrate temporal behavior.

- Graph elements **added/removed** in real-time. /* Dynamic Shortest Path */
- Metric demonstrates **stochastic behavior**. /* Sthochastic Shortest Path */
- Graph is **fixed**, metric **changes with the value of a parameter** $\gamma \in [0, 1]$ in a predetermined fashion. /* Parametric Shortest Path */
- Graph is **fixed**, metric **changes over time** in a predetermined fashion. /* Time-Dependent Shortest Path */

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  Time-Dependent Shortest Path */

  - Arcs are allowed to become **occasionally unavailable** (e.g., due to periodic maintenance, saving consumption of resources, etc), for predetermined unavailability time-intervals (discrete domain).
  - Arc lengths (e.g., traversal-time / consumption) **change with departure-time from tail** which is treated as a real-valued variable (functions with continuous domain, but not necessarily continuous range).

Why Time-Dependent Shortest Paths?

Real-life networks: Elements demostrate temporal behavior.

- Graph is fixed, metric changes with the value of a parameter $\gamma \in [0, 1]$ in a predetermined fashion. /* Parametric Shortest Path */
- Graph is fixed, metric changes over time in a predetermined fashion. /* Time-Dependent Shortest Path */
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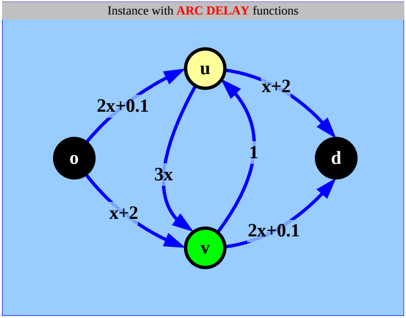

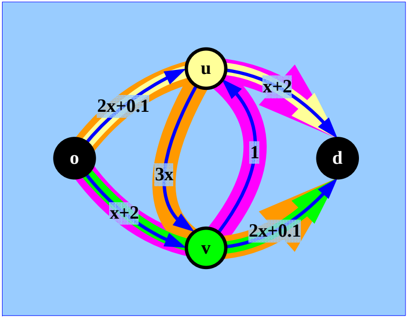

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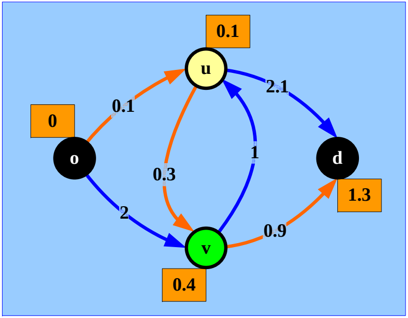

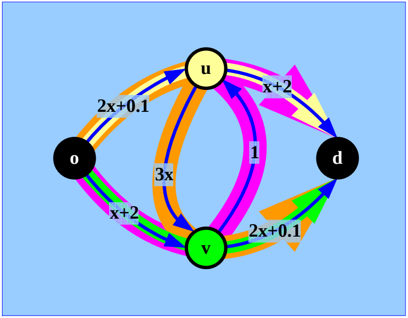

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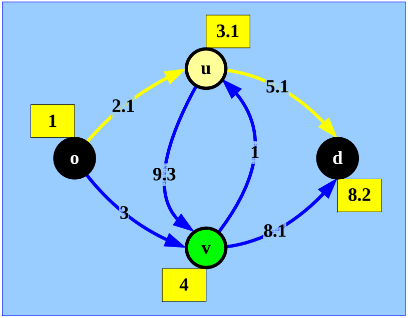

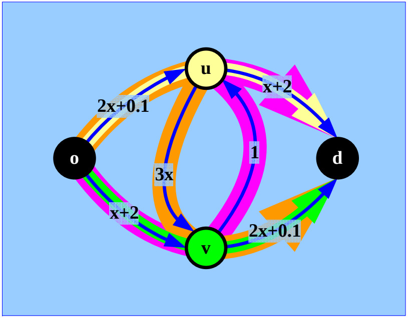

Q1 How would you commute **as fast as possible** from *o* to *d*, for a given departure time (from *o*)? Eg: $\boxed{t_o = 1}$

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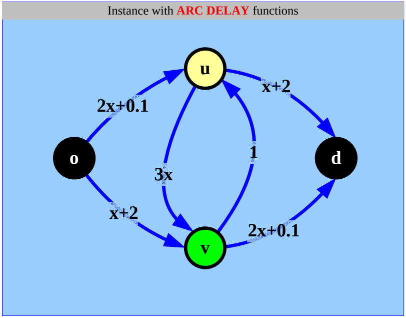

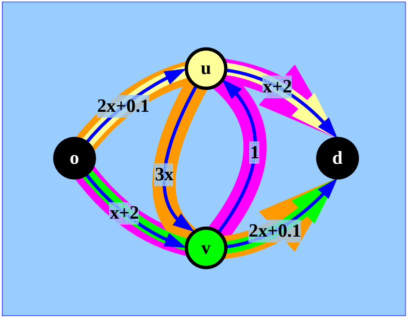

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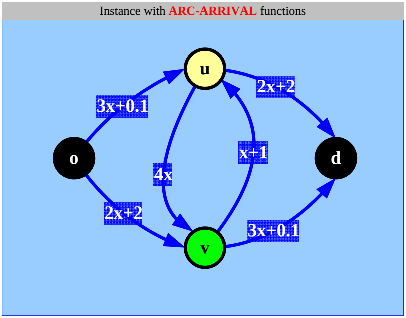
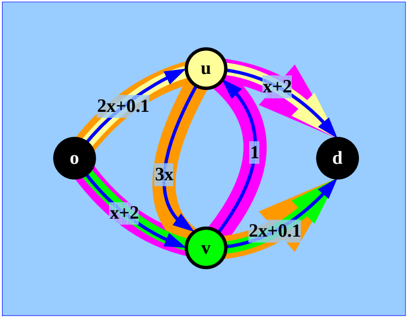
Q2 What if you are not sure about the departure time?





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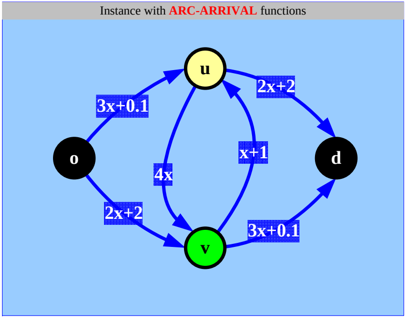

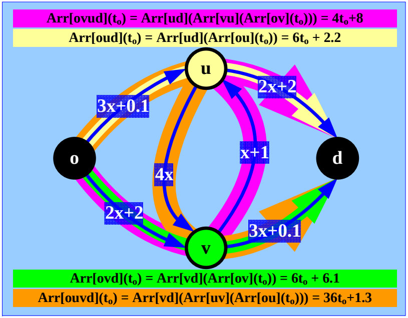

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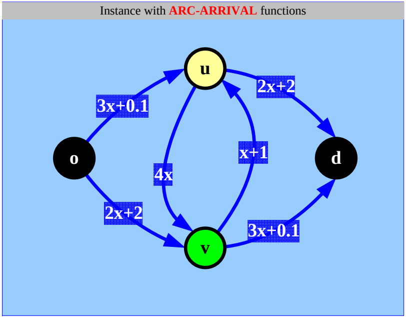

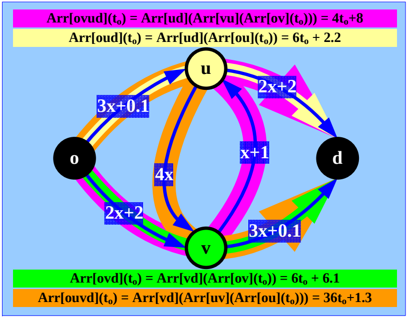

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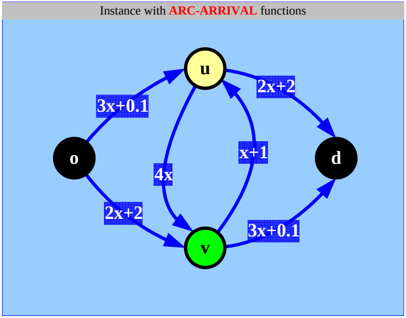
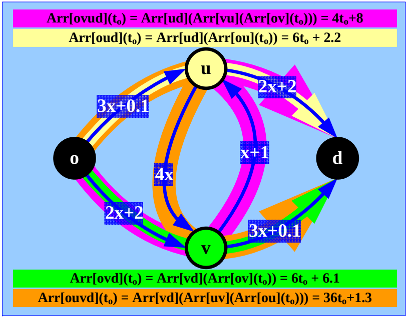
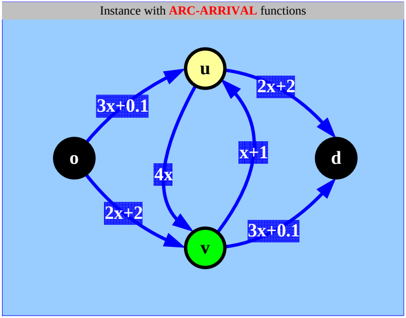
A2

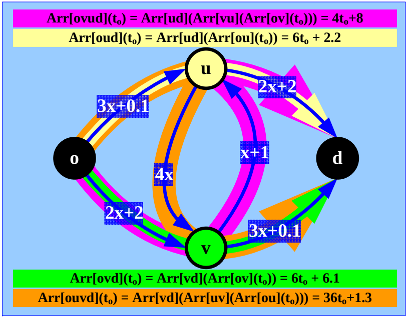
$$\text{shortest } od\text{–path} = \begin{cases} \textbf{orange path}, \text{ if} & t_o \in [0, 0.03] \\ \textbf{yellow path}, \text{ if} & t_o \in [0.03, 2.9] \\ \textbf{purple path}, \text{ if} & t_o \in [2.9, +\infty) \end{cases}$$





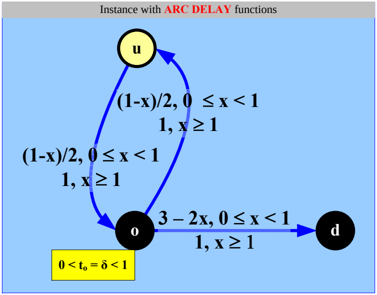
Q3 Would **waiting-at-nodes** be worth it?





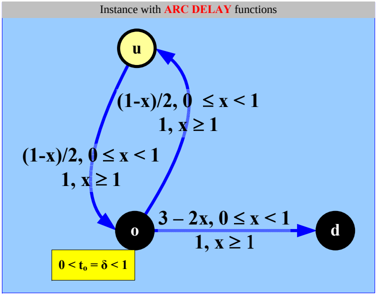
Q3 Would **waiting-at-nodes** be worth it?

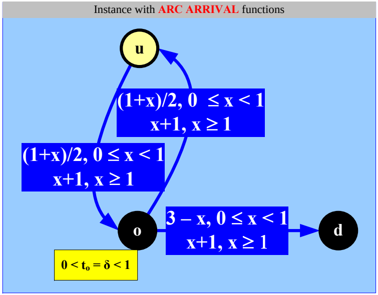
A3 NO, since arrival-time functions are *non-decreasing* functions of departure-time from origin.



Q4 Would **waiting-at-nodes** be worth it in this case?

...waiting at nodes...





Q4 Would **waiting-at-nodes** be worth it in this case?

A4 YES, because arrival-time function is *decreasing* in x: Wait until time 1 and then traverse od, if already present at o at time $t_o < 1$. Otherwise, traverse od immediately.

Waiting Policies

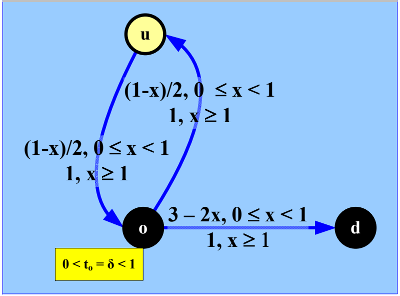
Unrestricted Waiting (UW) Unlimited waiting is allowed at every node along an *od*-path.

Origin Waiting (OW) Unlimited waiting is only allowed at the origin node of each *od*-path.

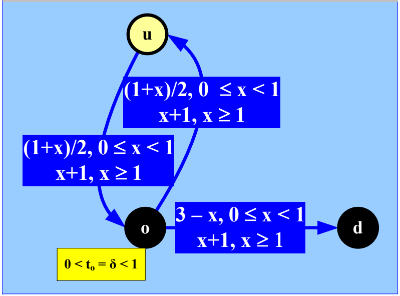
Forbidden Waiting (FW) No waiting is allowed at any node of each *od*-path.

> Depending on the *waiting policy*, the scheduler has to decide not only for an optimal connecting path (that assures the earliest arrival at the destination), but also for the appropriate optimal waiting times at the nodes along this path.

Instance with **ARC DELAY** functions

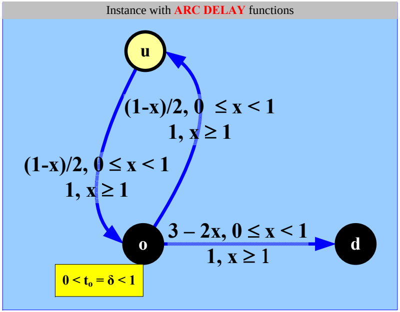


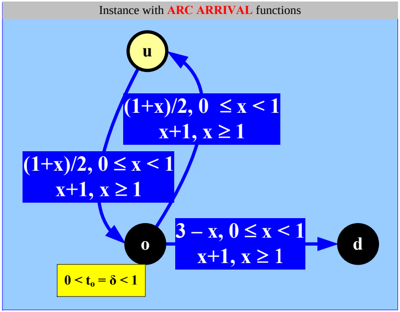
Instance with **ARC ARRIVAL** functions



Q5 What if **waiting-at-nodes** is **forbidden**?

....forbidden waiting times....

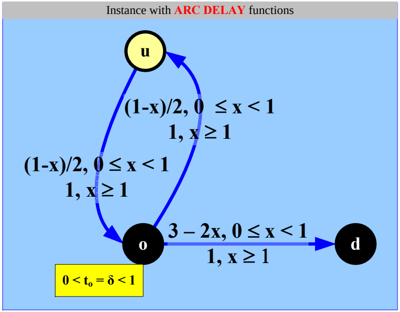


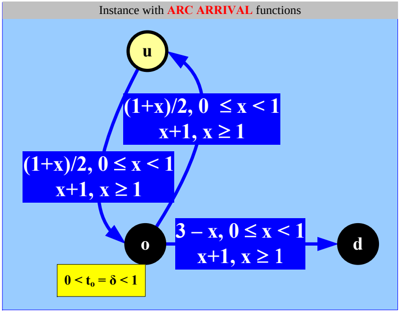


Q5 What if waiting-at-nodes is forbidden?

A5 An infinite, non-simple TD shortest *od*-path with finite delay.

o					d
δ					$3-\delta$





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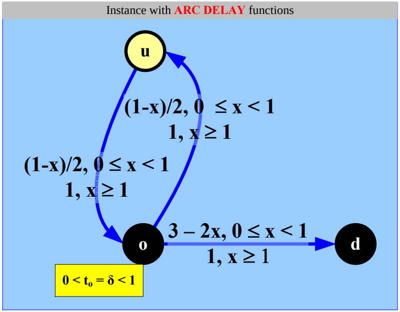
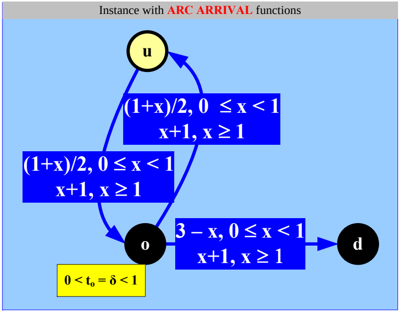
o	u	o			d
δ	$\frac{1+\delta}{2}$	$\frac{3+\delta}{4}$			$3 - \frac{3+\delta}{4} > 2$

TDSP :: EXAMPLE 3

....forbidden waiting times....





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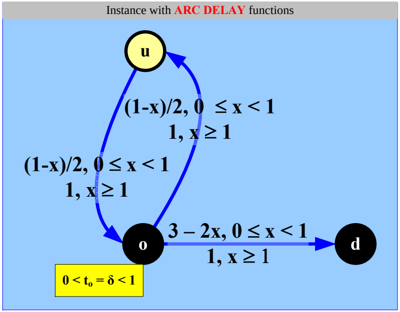
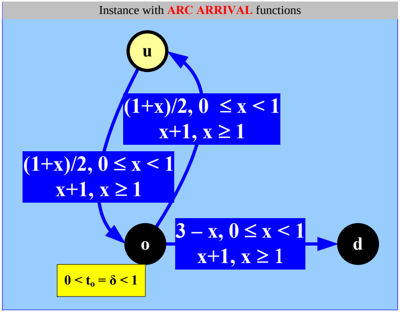
o	u	o	u	o	d
δ	$\frac{1+\delta}{2}$	$\frac{3+\delta}{4}$	$\frac{7+\delta}{8}$	$\frac{15+\delta}{16}$	$3 - \frac{15+\delta}{16} > 2$





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o	u	o	presence at o after $k \uparrow \infty$ visits of u	d
δ	$\frac{1+\delta}{2}$	$\frac{3+\delta}{4}$	$\lim_{k\uparrow\infty} \frac{2^{2k}-1+\delta}{2^{2k}} = 1$	$t_d \downarrow 2$

Subpath optimality and *shortest path simplicity* are **not guaranteed** for TDSP, if waiting-at-nodes is forbidden.

Inexistence of Optimal Waiting Times

Q Do optimal waiting times at nodes always exist?

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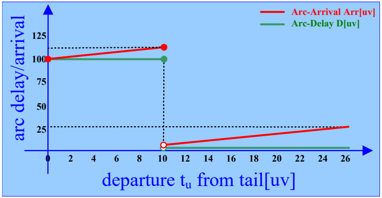
$$D[uv](t_u) = \begin{cases} 100, & t_u \leq 10, \\ 1, & t_u > 10 \end{cases}$$

$$Arr[uv](t_u) = \begin{cases} t_u + 100, & t_u \leq 10, \\ t_u + 1, & t_u > 10 \end{cases}$$



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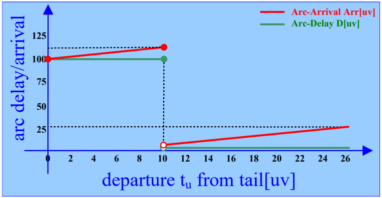

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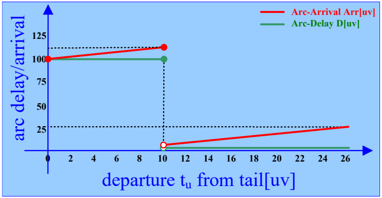

- **Reason:** *Pathological discontinuity* of the delay / arrival-time function.
- **Solution:** Optimal waiting times always exist for continuous functions, and for (possibly discontinuous) pwl functions for which

  if $\lim_{t \downarrow t_u} D[uv](t) < \lim_{t \uparrow t_u} D[uv](t)$
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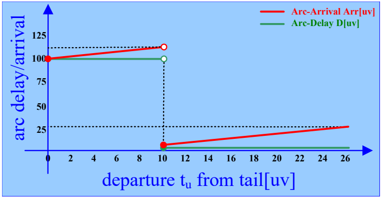

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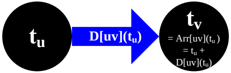
From now on we assume that *optimal waiting times* at nodes always exist and are polynomial-time computable.

FIFO vs non-FIFO Arc Delays

- **(Strict) FIFO Arc-Delays:** The slopes of all the *arc-delay* functions are at least equal to (greater than) -1.

Equivalently: *Arc-arrival* functions are **non-decreasing** (aka **no-overtaking** property).



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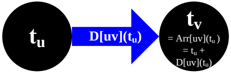

- **Non-FIFO Arc-Delays:** Possibly preferrable to wait for some period at the tail of an arc, before trespassing it. E.g.:
  - Wait for the next *(faster) IC train*, than use the (immediately available) *(slower) local train*.

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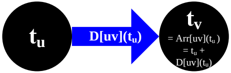

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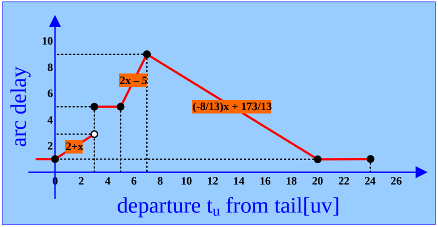

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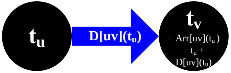

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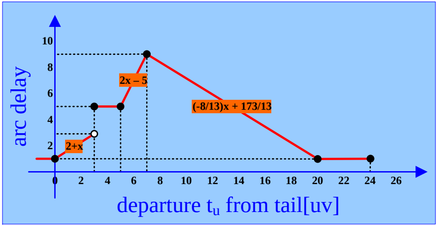

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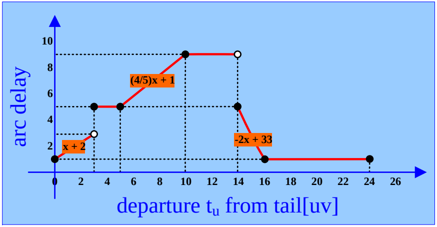
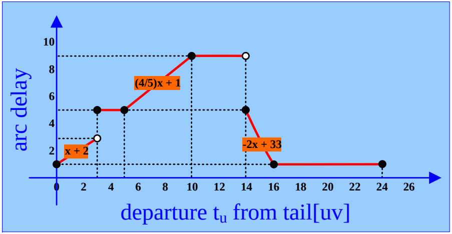
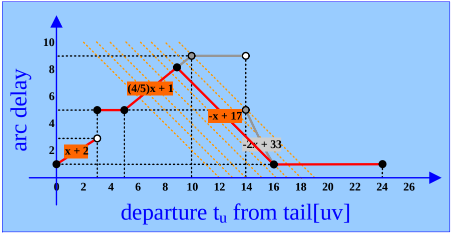


Non-FIFO arc delay example

Non-FIFO+UW Arc ⇔ FIFO Arc

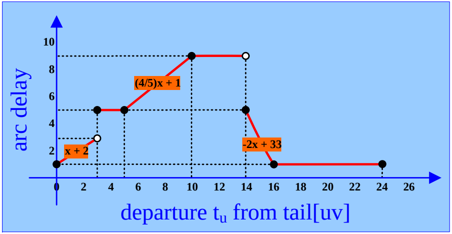


Non-FIFO+UW arc delay function



Equivalent FIFO (+FW) arc delay function

Non-FIFO+UW Arc ⇔ FIFO Arc



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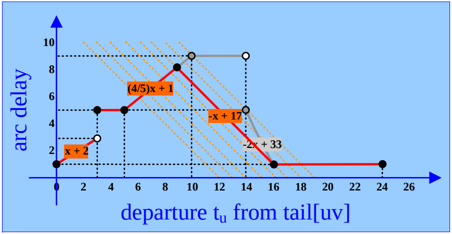

Equivalent FIFO (+FW) arc delay function

- A ``scan'' of the line with slope -1 *from right to left* suffices.

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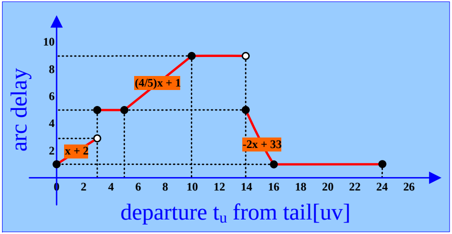

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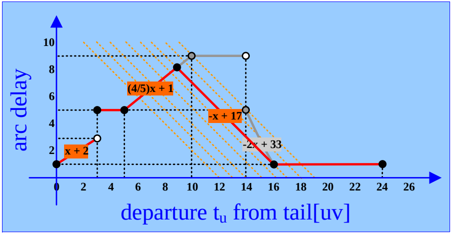

Equivalent FIFO (+FW) arc delay function

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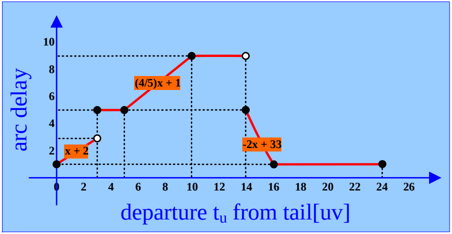
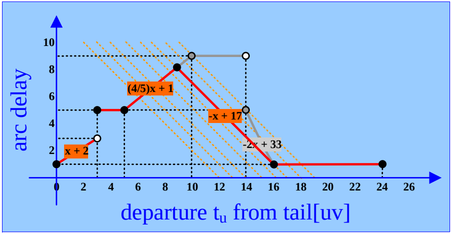
  - **Shortcircuit** pieces of the arc-delay function lying above the line of slope -1.



Non-FIFO+UW arc delay function



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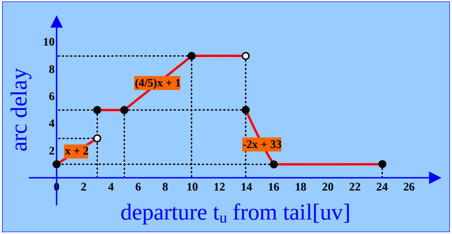

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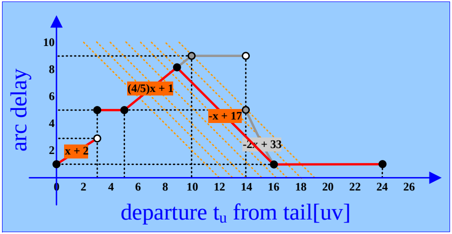

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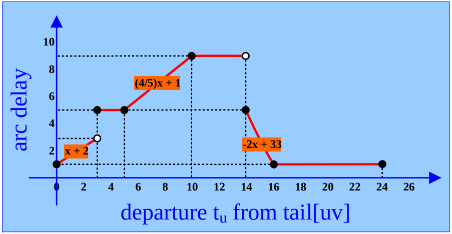

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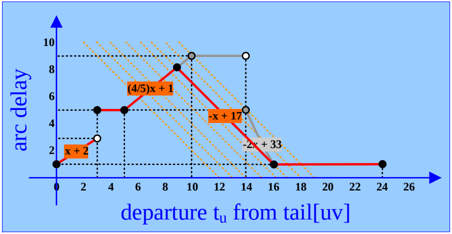

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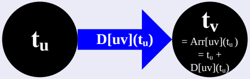
- Interested in programming the transformation? Let me know!

DEFINITION: Time-Dependent Shortest Paths

INPUT:

- *Directed* graph $G = (V, A)$ with **succinctly represented arc-travel-time** functions $(D[a])_{a\in A}$. $(Arr[a] = ID + D[a])_{a\in A}$.



Variants of Time-Dependent Shortest Path

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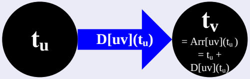

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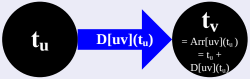

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GOAL1: For departure-time t_o from o, determine $t_d = Arr[o,d](t_o)$.

GOAL2: Provide a **succinct representation** of $Arr[o,d]$ (or $D[o,d]$).

Why Care for Both Goals?

1. Not always sure when to depart (still think about it)! Possessing the entire *distance function* $D[o,d]$ allows for easy answers (e.g., via look-ups) in several queries for varying departure times, or even finding the *minimum travel / ealriest-arrival time* within a window of possible departure times.

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    - ☹ Complexity is **polynomial** in the number of ``elementary'' *functional operations*. i.e., (EVAL, LINEAR COMBINATION, MIN, COMPOSITION)
    - ☹ Not so ``elementary'' operations after all (see next slides)!!!

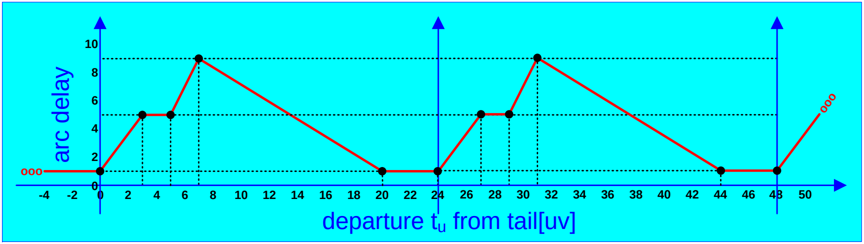
Algorithms for TDSP

... in FIFO, continuous, pwl instances

Input/Output Data

PWL Arc Delays

Forward Description (as function of departure times from origin)



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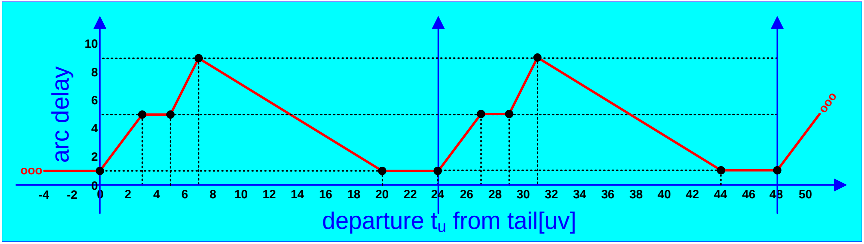
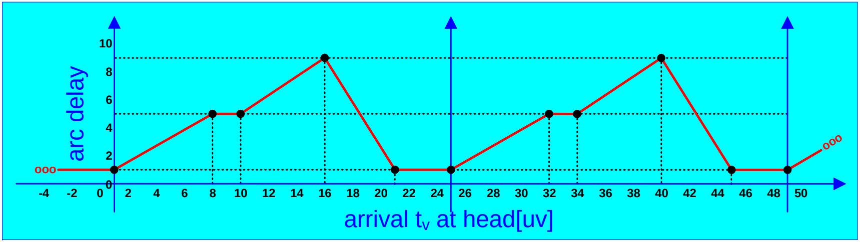
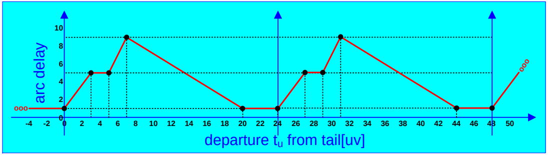


Reverse Description (as function of arrival times at destination)



How to Store/Access PWL Arc Delays



- Exploit *periodicity* and *piecewise-linearity*:

$$\forall t_u \in \mathbb{R},\ \overrightarrow{D}[uv](t_u) \;=\; \begin{cases} \frac{4}{3}t_u + 1, & 0 \le t_u \textbf{mod}\ T \le 3 \\ 5, & 3 \le t_u \textbf{mod}\ T \le 5 \\ 2t_u - 5, & 5 \le t_u \textbf{mod}\ T \le 7 \\ -\frac{8}{13}t_u + \frac{173}{13}, & 7 \le t_u \textbf{mod}\ T \le 20 \\ 1, & 20 \le t_u \textbf{mod}\ T \le 24 \end{cases}$$

- Representation: Array of **(slope-constant-dep.time UB) triples** equipped with advanced (binary/predecessor) *search capabilities*.

$\left(\frac{4}{3}, 1, 3\right)$	$(0, 5, 5)$	$(2, -5, 7)$	$\left(-\frac{8}{13}, \frac{173}{13}, 20\right)$	$(0, 1, 24)$

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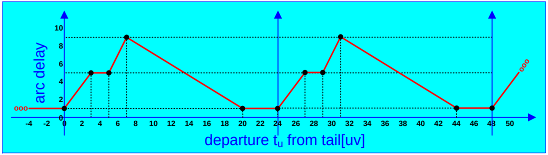

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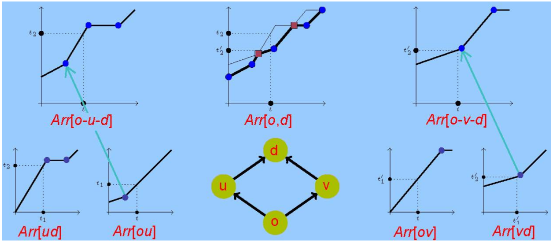
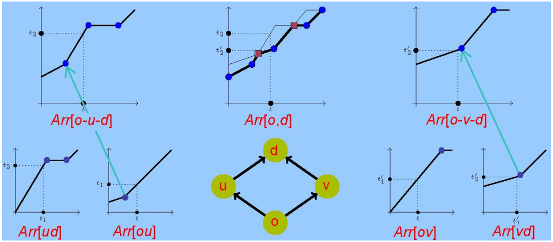
- Representation: Array of **(dep.time-delay) pairs** equipped with advanced (binary/predecessor) *search capabilities*.

$(0,1)$	$(3,5)$	$(5,5)$	$(7,9)$	$(20,1)$



- **Primitive Breakpoint (PB):** Departure-time b'_e from $head[e]$ at which $D[e]$ changes slope (assume $K \in O(m)$ PBs in total).



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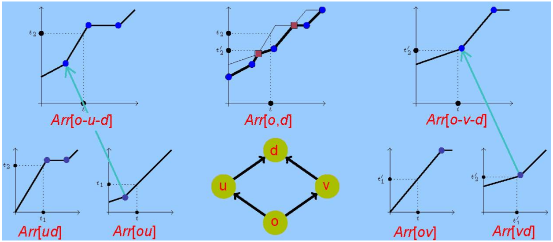
Piecewise Linearity of Path / Earliest Arrivals



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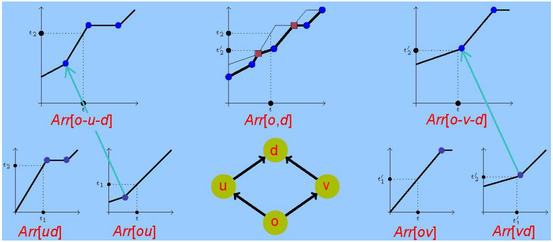

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- **Minimization Breakpoint (MB):** Departure-time b_v from origin o s.t. $Arr[o, v]$ **changes slope** due to application of MIN.
- Periodicity of arc-delays implies periodicity of earliest-arrival function $Arr[o, d]$.

Known Issues wrt Representations

- **Same representation** both for arc-arrival (or delay) functions and earliest-arrival (or shortest-travel-time) functions.
  - Convenient for handling artificial arcs (representing shortest-travel-time functions) in *overlay abstractions* of the road network.

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- We need only $O\left(\frac{1}{\varepsilon} \cdot \log\left(\frac{D_{\max}[o,d]}{D_{\min}[o,d]}\right)\right)$ breakpoints for a $(1+\varepsilon)$ **upper approximation** $\overline{D}[o,d]$ of $D[o,d]$, for the case of continuous, piecewise-linear arc-delays.

Complexity of TDSP

Lower Bound: $|BP(Arr_{\mathrm{pwl}}[o,d])| = n^{\Omega(\log n)}$ (I)

A Useful Observation (L2.1-2.2 in FHS11)

For any pair of **monotone, pwl** functions f and g, both their composition $f \circ g$ and their minimum $\min\{f, g\}$ are also **monotone, pwl** functions.

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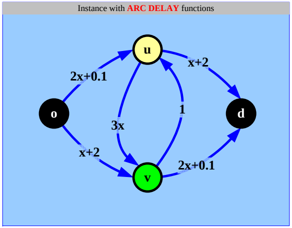
Parametric Shortest Path (PSP): A Similar (but different) Problem

- INPUT: $G = (V, A)$, $o, d \in V$. A *linear length function* $\ell[a](\gamma) = \lambda[a] \cdot \gamma + \mu[a]$ per edge $a \in A$ (negative lengths are **allowed**).
- DEFINITIONS:
  - **Path-length:** $\forall p \in G, L[p](\gamma) = \sum_{a \in p} \ell[a](\gamma)$.
  - **Min-length:** $\forall o, d \in V, L[o,d](\gamma) = \min_{p \in P_{o,d}}\{L[p](\gamma)\}$.
- GOAL1: Compute $L[o,d]$ for a **given value** of γ.
- GOAL2: Succinctly represent $L[o,d]$ for **all (real) values** of γ.

TDSP vs PSP?



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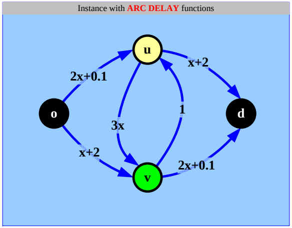
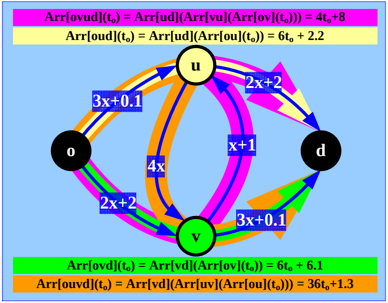
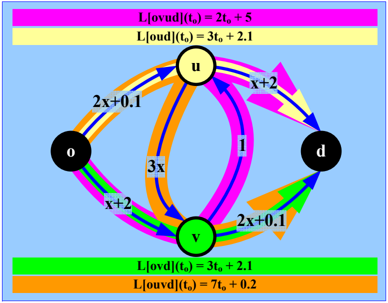




TDSP: Arc-arrival composition along paths



PSP: Arc-length addition along paths

Lower Bound: $|BP(Arr_{\mathrm{pwl}}[o,d])| = n^{\Omega(\log n)}$ (II)

Known Fact (Carstensen (1984), Mulmuley-Shah (2000))

There exists (linear) PSP-instance with $n^{\Omega(\log n)}$ BPs in $L[o,d]$.

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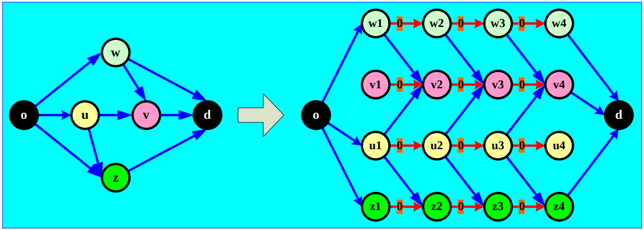
Main Steps for TDSP Lower Bound:

1. Assure *non-negativity of lengths* in the PSP instance, in the **departure-time interval of interest**.
2. Scale properly the PSP instance.
3. Consider the corresponding TDSP instance, with parameter γ handled as departure time from the origin o.
4. Prove that $L[o,d]$ (for PSP instance) and $D[o,d]$ (for TDSP instance) have (almost) the **same number** of BPs.

1. Construct a **layered-graph**, in a **path-length-preserving** manner:



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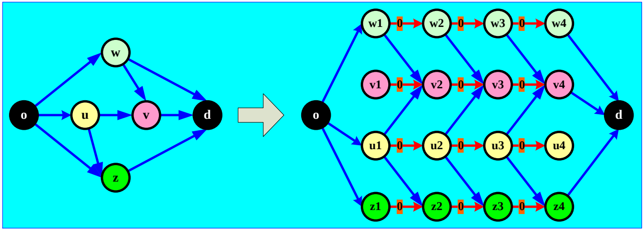

Assure non-negativity of arc-lengths in PSP: For the sequence $\langle \gamma_1, \gamma_2, \ldots, \gamma_N \rangle$ of **breakpoints** (BPs) wrt $L[o,d]$, shift arc lengths by $\max\{0, -L_{\min}\}$, $\boxed{L_{\min} = \min_{\gamma \in [\gamma_1, \gamma_N], a \in A(G)} \{L[a](\gamma)\}}$.

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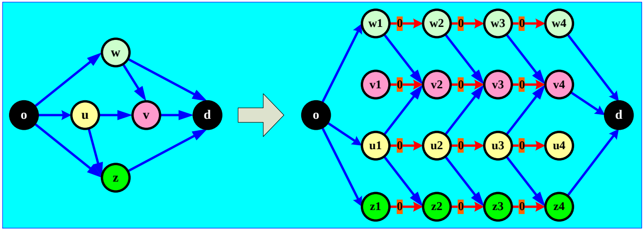

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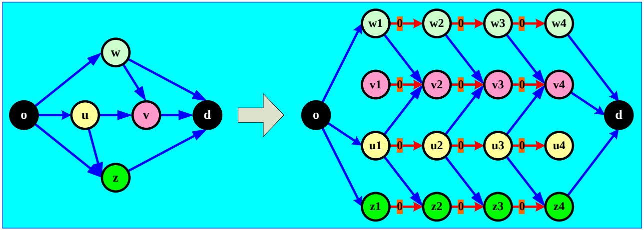

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3. For the TDSP resulting from the scaled PSP when considering γ as departure time, prove that $\forall j \in \{1,\ldots,N-1\}$, at **"time"** $\bar{\gamma}_j \equiv \frac{\gamma_j+\gamma_{j+1}}{2}$ both instances return *the same* shortest $od-$path p_j.

How it works: At given $j \in \{1, \ldots, N-1\}$:

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- **Departure-time perturbations:** Small-enough so as to cause *not too large arc-delay perturbations*: $\forall a \in A,\ \forall \delta_a \in (0,\delta^*]$,

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$\therefore$ Small time-perturbations guarantee sufficiently small **arc-delay perturbations**, and thus, optimality of p_j:

$$\boxed{\begin{array}{rcl} D[p_j](\bar{\gamma}_j) & \leq & \mu \cdot \bar{L}_j + \mu \cdot \frac{(n-1)\Delta_{\min}}{2n} \\ & < & \mu \cdot L'_j - \mu \frac{(n-1)\Delta_{\min}}{2n} \leq D[q](\bar{\gamma}_j), \ \forall q \neq p_j \end{array}}$$

QED

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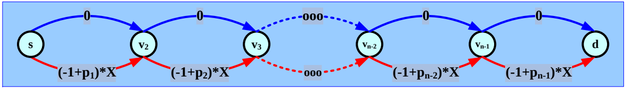

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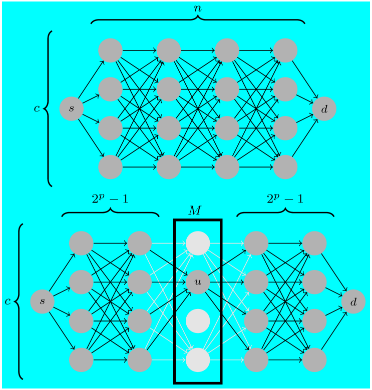

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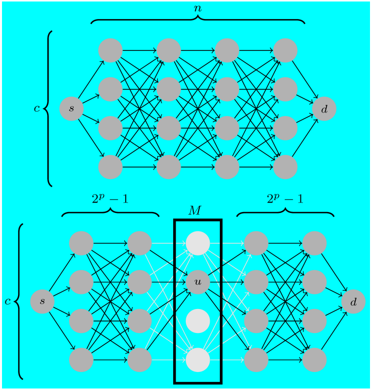

THM4.4 (FHS11)

$|BP(Arr_{\mathrm{lin}}[o,d])| = n^{O(\log n)}$ in any graph G and pair of nodes $o, d \in V(G)$.

(Exact) Output Sensitive Algorithm for Earliest-Arrival Functions

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3. Interesting to discover whether the complexity of the earliest-arrival functions is indeed so bad **in real (e.g., road) networks**.

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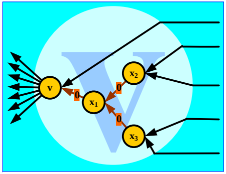
- **ASSUMPTION:** The in-degree of every node in the graph is at most 2.



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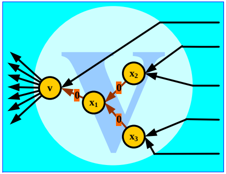

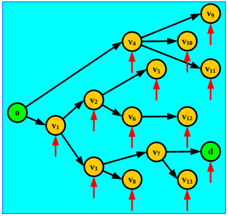

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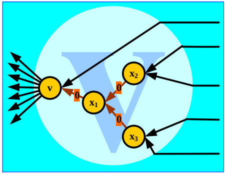

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- Discover *until when* the TDSP tree is **valid**.
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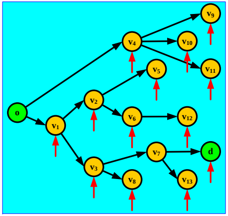

  - **Minimization (vertex) Certificate** $t_{fail}[v]$: Earliest departure time from o at which the two alternatives of v become **equivalent**.
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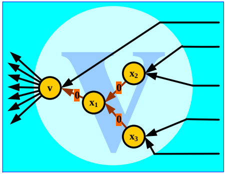

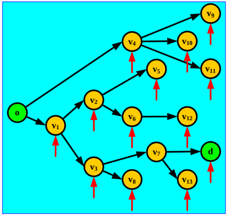

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- All $(m + n)$ certificates temporarily stored in a *priority queue*.

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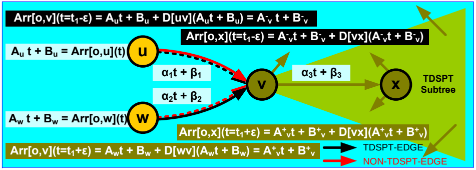
When current time $t_1 > t_0$ matches the earliest failure-time of a certificate in the queue:

if minimization-certificate failure, at node $v \in V$:



then (1) Update shortest ov–path

/* **ONE-BIT** change in combinatorial structure */

(2) Update $Arr[o,x]$ and $t_{fail}[x]$, $\forall x \in T_v$.

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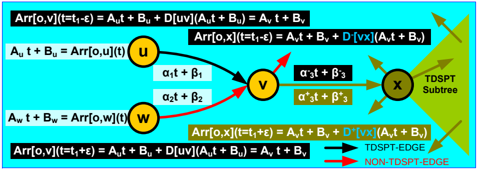

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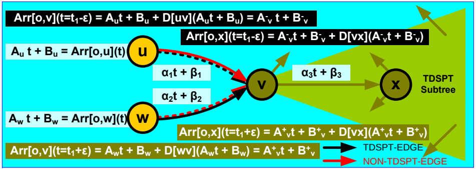

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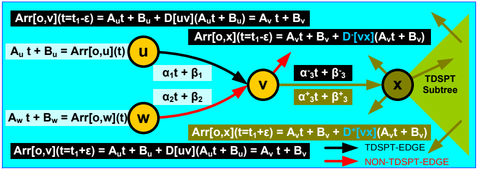

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The Output-Sensitive Algorithm (III)

- What to keep in memory:
  - Breakpoint triples for earliest-arrival functions, plus ONE bit (indicating the parent).
  - Advanced search structures, if number of BPs is large.
  - Only temporarily store certificates in a priority queue.

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  - In the *original graph* (in worst-case): $\mathrm{O}\left(m \times \log^2 n\right)$. Second logarithmic factor is due to **updates of tournament trees** implementing the MIN operator at a particular node, upon emergence of a single certificate failure.

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- *Worst-case time-complexity* of output-sensitive algorithm:

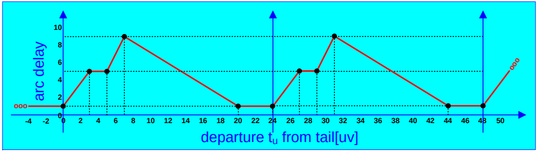
$$\mathrm{O}\left(m \times \log^2 n \times (\mathrm{PRIMBPs} + \mathrm{MINBPs})\right)$$

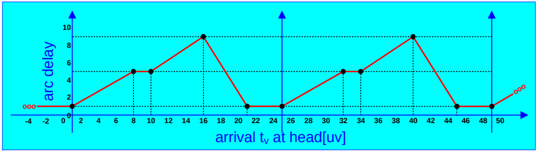
Poly-time Approximation Algorithms

$(1+\varepsilon)-$approximation of $D[o,d]$: Preliminaries

- Why focus on **shortest-travel-time** (delays) functions, and not on **earliest-arrival-time** functions?

- Arc/Path Delay Reversal: Easy task!!!

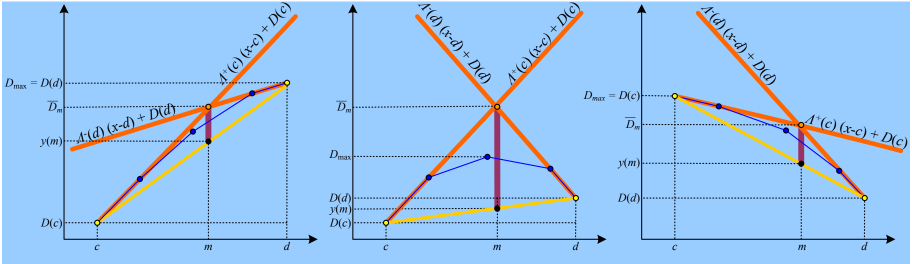




- $t_o = \overleftarrow{Arr}[o,v](t_v) = t_v - \overleftarrow{D}[o,v](t_v)$: **Latest-departure-time** from o to v, as a function of the arrival time t_v at v.

Approximating $D[o,d]$: Quality

- **Maximum Absolute Error:** A crucial quantity both for the time-complexity and for the space-complexity of the algorithm:



(a) $\Lambda^+(c) > \Lambda^-(d) \geq 0$ (b) $\Lambda^+(c) > 0 > \Lambda^-(d)$ (c) $0 \geq \Lambda^+(c) > \Lambda^-(d)$

Approximating $D[o, d]$: Quality

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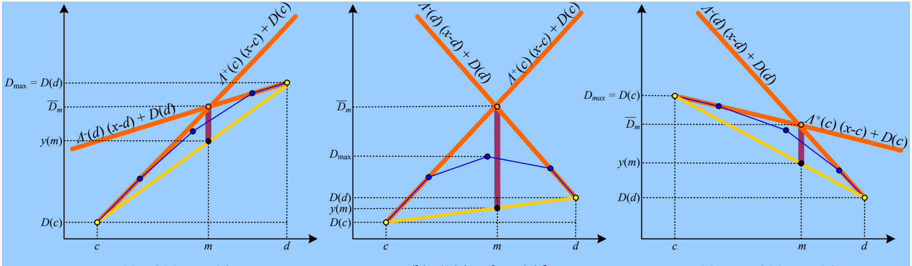

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LEMMA: Closed Form of Maximum Absolute Error (Kontogianis-Zaroliagis (2014))

$$MAE(c, d) = (\Lambda^+(c) - \Lambda^-(d)) \cdot \frac{(m-c)\cdot(d-m)}{L} \leq \frac{L\cdot(\Lambda^+(c) - \Lambda^-(d))}{4}$$

Approximating $D[o, d]$: Basic Idea (I)

- Approximations of $D[o, d]$: For given $\varepsilon > 0$, and $\forall t \in [0, T)$,

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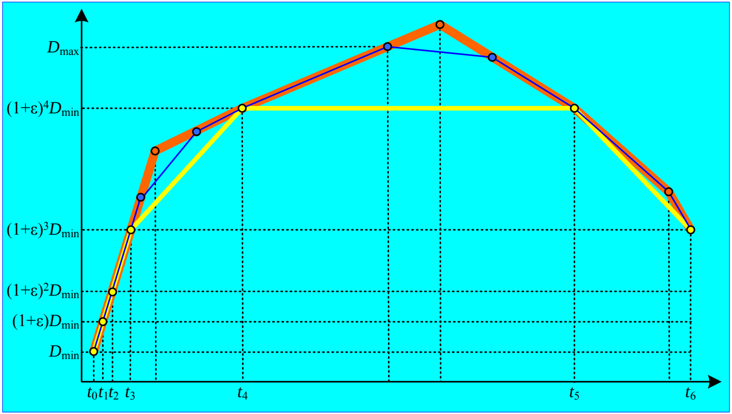
- FOCUS: **Linear** arc-delays. Later extend to pwl arc-delays.
- $D[o,d]$ lies entirely in a **bounding box** that we can easily determine, with only 3 TD-Djikstra probes.

Approximating $D[o,d]$: Basic Idea (II)



- Make the sampling so that $\forall t \in [0, T],\ \overline{D}[o,d](t) \leq (1+\varepsilon) \cdot \underline{D}[o,d](t)$.
- Keep sampling always the **fastest-growing axis** wrt to $D[o,d]$.

One-To-One Approximation: PHASE-1

(Foschini-Hershberger-Suri (2011))

while slope of $D[o,d] \geq 1$ **do**

$$t_0 \xrightarrow{\text{Forward Dijkstra}} t_0 + D[o,d](t_0)$$

$$\downarrow$$

$$t_1 \xleftarrow{\text{Backward Dijkstra}} t_0 + (1+\varepsilon)^{1/2} D[o,d](t_0) = t_1 + D[o,d](t_1)$$

$$\downarrow$$

$$t_2 \xleftarrow{\text{Backward Dijkstra}} t_1 + (1+\varepsilon)^{1/2} D[o,d](t_1) = t_2 + D[o,d](t_2)$$

$$\vdots$$

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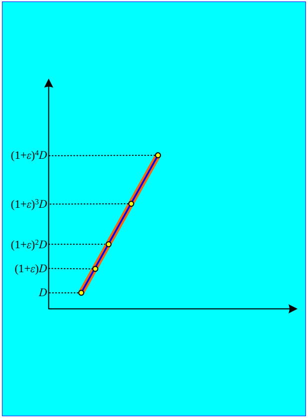
Bad Case for (Foschini-Hersberger-Suri (2011)) :



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$t_0 \xrightarrow{\text{Forward Dijkstra}} t_0 + D[o,d](t_0)$

$t_{1,1} \xleftarrow{\text{Backward Dijkstra}} \mathrm{MAX}\{\, t_{fail},\ t_0 + (1+\varepsilon)\, D[o,d](t_0) \,\}$ *MaxAbsError* $< \varepsilon\, D[o,d](t_0)$

o
o
o

$t_1 = t_{1,k} \xleftarrow{\text{Backward Dijkstra}} t_0 + (1+\varepsilon)^k D[o,d](t_0) = t_1 + D[o,d](t_1)$ *MaxAbsError* $< \varepsilon\, D[o,d](t_0)$

$t_{1,k+1} \xleftarrow{\text{Backward Dijkstra}} t_0 + (1+\varepsilon)^{k+1} D[o,d](t_0)$ *MaxAbsError* $> \varepsilon\, D[o,d](t_0)$

$t_{2,1} \xleftarrow{\text{Backward Dijkstra}} \mathrm{MAX}\{\, t_{fail},\ t_1 + (1+\varepsilon)\, D[o,d](t_1) \,\}$ *MaxAbsError* $< \varepsilon\, D[o,d](t_1)$

o
o
o

One-To-One Approximation: PHASE-2

(Foschini-Hershberger-Suri (2011))

Slope of $D[o, d] \leq 1$:

repeat

Apply **BISECTION** to the remaining time-interval(s)

until desired approximation guarantee (wrt **Max Absolute Error**) is achieved.

One-To-All Approximaton via Bisection (I)

(Kontogiannis-Zaroliagis (2013))

ASSUMPTION 1: Concavity of arc-delays. /* to be removed later */

- Implies concavity of the *unknown* function $D[o, d]$.

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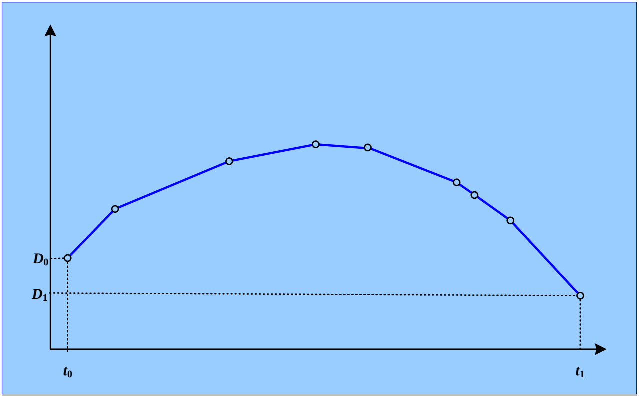
ASSUMPTION 2: Bounded Travel-Time Slopes. Small slopes of the (pwl) arc-delay functions.

- **Verified** by TD-traffic data for road network of Berlin (TomTom (February 2013)) that all arc-delay slopes are in $[-0.5, 0.5]$.
- Slopes of *shortest-travel-time* function $D[o, d]$ from $[-\Lambda_{min}, \Lambda_{max}]$, for some constants $\Lambda_{max} > 0,\ \Lambda_{min} \in [0, 1)$.

Under ASSUMPTIONS 1-2: Execute `Bisection` to *sample simultaneously* all distance values from *o*, at mid-points of time intervals, until required approximation guarantee is achieved *for each destination node*.



Example of Bisection Execution : INPUT = **UNKNOWN BLUE** function

One-To-All Approximaton via Bisection (II)

(Kontogiannis-Zaroliagis (2013))

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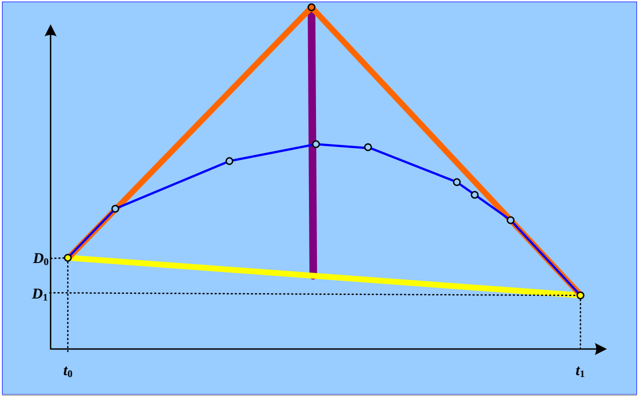

Example of Bisection Execution : **ORANGE** = Upper Bound, **YELLOW** = Lower Bound

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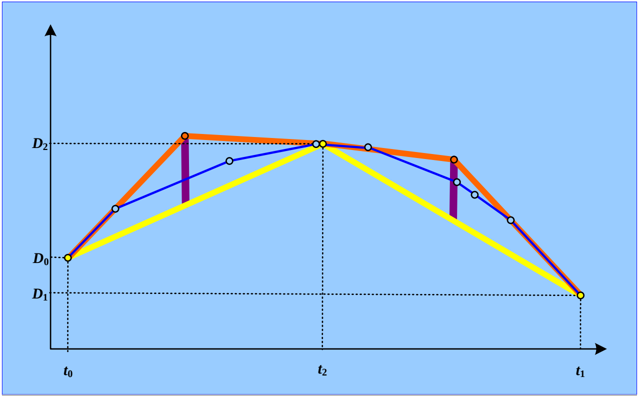

Example of Bisection Execution : Level-1 Recursion

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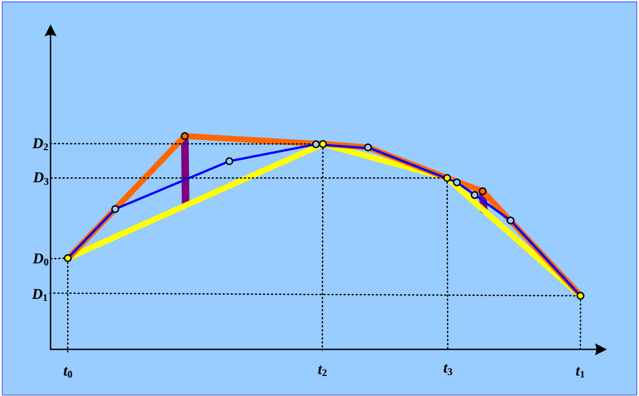


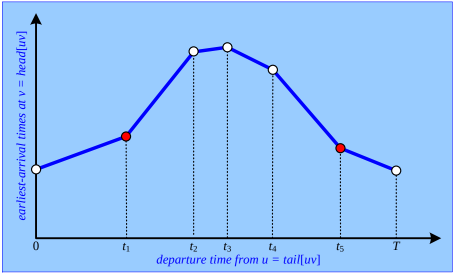
Example of Bisection Execution : Level-2 Recursion

One-To-All Approximaton via Bisection (III)

(Kontogiannis-Zaroliagis (2013))

Only under ASSUMPTION 2: For continuous, pwl arc-delays.

1. Call `Reverse TD-Dijkstra` to project each concavity-spoiling PB to a PI of the origin o.
2. For each pair of *consecutive PIs* at o, run `Bisection` for the corresponding departure-times interval.



3. Return the *concatenation* of approximate distance summaries.

Approximating $D[o,d]$: Space/Time Complexity

THEOREM: Space Complexity (KZ (2014))

Let K^* be the total number of concavity-spoiling BPs among all the arc-delay functions in the instance.

Space Complexity: For a *given orign* $o \in V$ and *all* possible destinations $d \in V$, the following complexity bounds hold for creating all the approximation functions $\overline{D}[o,\star] = (\overline{D}[o,d])_{d\in V}$:

1. $\mathrm{O}\left(\frac{K^*}{\mathrm{e}} \log\left(\frac{D_{\max}[o,\star](0,T)}{D_{\min}[o,\star](0,T)}\right)\right)$
2. In each interval of *consecutive* PIs,
   $|UBP[o,d]| \leq 4 \cdot$ (minimum #BPs for any $(1+\varepsilon)$−approximation.

Time Complexity: The number of *shortest-path probes* executed for the computation of the approximate distance functions is:

$$TDSP[o,d] \in \mathrm{O}\left(\log\left(\frac{T}{\varepsilon \cdot D_{\min}[o,d]}\right) \cdot \frac{K^*}{\varepsilon} \log\left(\frac{D_{\max}[o,\star](0,T)}{D_{\min}[o,\star](0,T)}\right)\right)$$

Implementation Issues wrt One-To-All Bisection

- `One-To-All Bisection` of (KZ (2014)) is a **label-setting** approximation method that provably works *space/time optimally* (within constant factors) wrt **concave** continuous pwl arc-delay functions.

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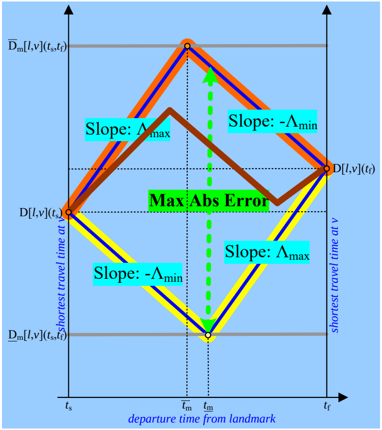
- A novel **one-to-all** (again **label-setting**) approximation technique, called the `Trapezoidal` method ((KWZ (2016))) avoids entirely the dependence of the required space from the network structure (and, of course, the degree of disconcavity).

The Trapezoidal One-To-All Approximation Method

- **Sample travel-times** to all destinations, from coarser to finer departure-times from the (common) origin.
- Between consecutive samples of the same resolution, the unknown function is *bounded within a given trapezoidal*.
- ``Freeze'' destinations within intervals with satisfactory approximation guarantee.



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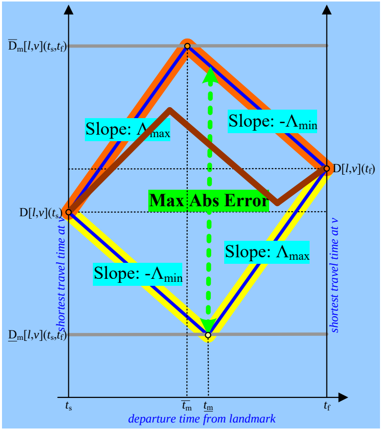

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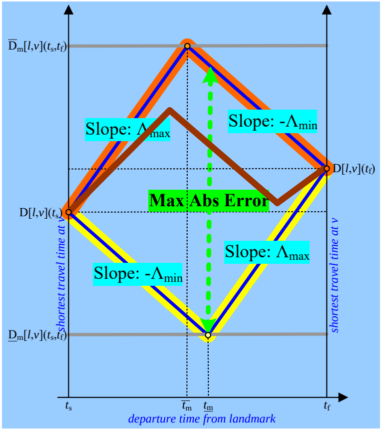

☺ Avoids dependence on concavity-spoiling BPs of the metric.

☹ Cannot provide good approximations for ``nearby'' destinations around the origin.

Time-Dependent Oracles

Distance Oracles

- Extremely successful theme in static graphs.
  - In theory:
    - P-Space: Subquadratic (sometimes quasi-linear).
    - Q-Time: Constant.
    - Stretch: Small (sometimes PTAS).
  - In practice:
    - P-Space: A few GBs (sometimes less than 1 GB).
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FOR THE REST OF THE TALK

The focus is on **time-dependent oracles**, with **provably good** preprocessing-space / query-time / stretch tradeoffs.

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  - $O(n+m+K)$ size (K = total number of PBs of arc-delays).
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Distance Oracles

Is it a Success Story in Time-Dependent Graphs?

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Is there a smooth tradeoff among space / query time / stretch?

FLAT TD-Oracle

Landmark Selection Policy

- Rationale: Identify a few ``important'' vertices (**landmarks**) in the network, which are assumed to be crucial for almost all shortest paths. Then compute approximate **travel-time summaries** (functions) $\Delta[\ell, v](t)$, $\forall(\ell, v) \in L \times V,\ \forall t \in [0, T)$ s.t.:

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Preprocessing of FLAT

(KZ (2014), KMPPWZ2015, KMPPWZ2016)

- Each landmark is **informed** about all destinations.
- **Subquadratic** preprocessing space/time.
- Query time **sublinear** in the network size.
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When the landmark set $L \subset V$ is chosen **uniformly at random**:

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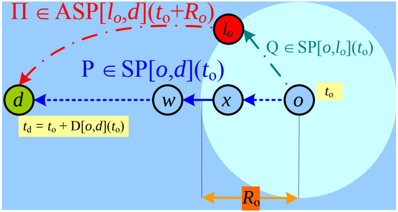
(KWZ-2016) If each vertex becomes a landmark with probability $\rho = n^{-\delta}$, BIS is used for $F = \sqrt{n}$ *``nearby''* destinations and TRAP is used for the rest *``faraway''* destinations from each landmark, then the preprocessing space and time are $\mathrm{O}\left(n^{2-\delta} \cdot \mathrm{polylog}(n)\right)$.

Forward Constant Approximation (FCA)

1. Grow TD-Dijkstra ball $B(o, t_o)$ until the closest landmark ℓ_o, or d, is settled
2. **return** $sol_o = D[o, \ell_o](t_o) + \Delta[\ell_o, d](t_o + D[o, \ell_o](t_o))$



FCA: A Constant-Approximation Query Algorithm (KZ (2014))

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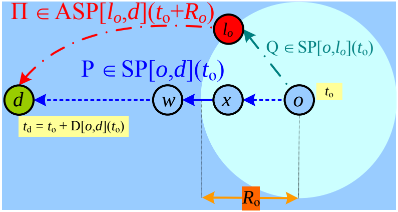
2. **return** $sol_o = D[o, \ell_o](t_o) + \Delta[\ell_o, d](t_o + D[o, \ell_o](t_o))$



Complexity of FCA for random landmarks

- Constant approximation guarantee: $sol_o \le (1 + \epsilon + \psi) \cdot D[o, d](t_o)$, for $\psi = 1 + \Lambda_{\max}(1 + \epsilon)(1 + 2\zeta + \Lambda_{\max}\zeta) + (1 + \epsilon)\zeta \in O(1)$.
- Sublinear Query-time: $O\left(\frac{1}{\rho} \cdot \ln\left(\frac{1}{\rho}\right) \log\log(K_{\max})\right)$

FCA+: A natural extension of FCA

(KMPPWZ (2015-2016))

Extended Forward Constant Approximation (FCA+)

1. Grow TD-Dijkstra ball $B(o, t_o)$ until the N closest landmarks $\ell_o, \ldots, \ell_{N-1}$ (or d) are settled.
2. **return** $\min_{i \in \{0,1,\ldots,N-1\}} \{ sol_i = D[o, \ell_i](t_o) + \Delta[\ell_i, d](t_i + D[o, \ell_i](t_o)) \}$

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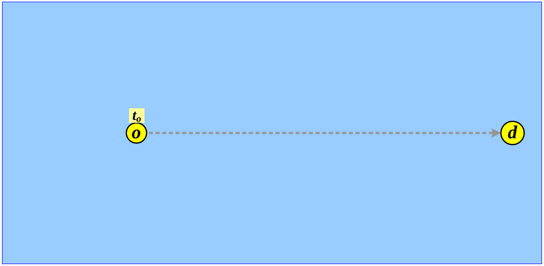

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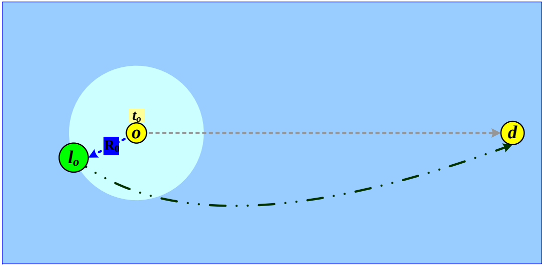

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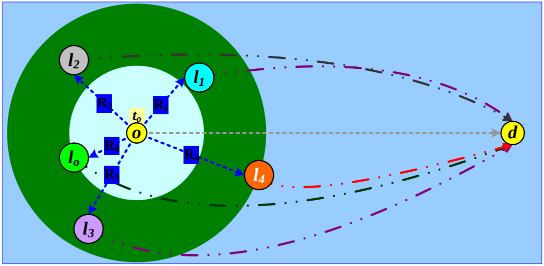

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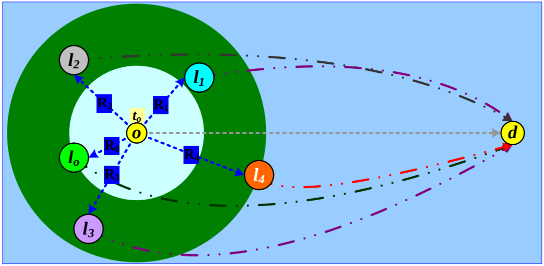

Performance of FCA+ for random landmarks

- In theory: Analogous to that of FCA.

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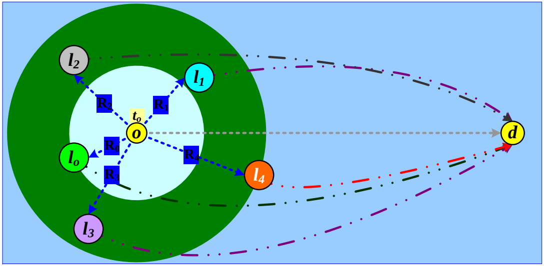

Performance of FCA+ for random landmarks

- In theory: Analogous to that of FCA.
- In practice: Performance analogous to (indeed, better than) that of RQA.

RQA: A Sublinear-Time Approximation Scheme (KZ (2014))

Recursive Query Approximation (RQA)

while recursion budget R not exhausted **do**

  Grow TD-Dijkstra ball $B(w_i, t_i)$ until closest landmark ℓ_i is settled

  $sol_i = D[o, w_i](t_o) + D[w_i, \ell_i](t_i) + \Delta[\ell_i, d](t_i + D[w_i, \ell_i](t_i))$

  Run RQA at *each boundary node* of $B(w_i, t_i)$ with budget $R - 1$

end while

return best solution found

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5. **end while**
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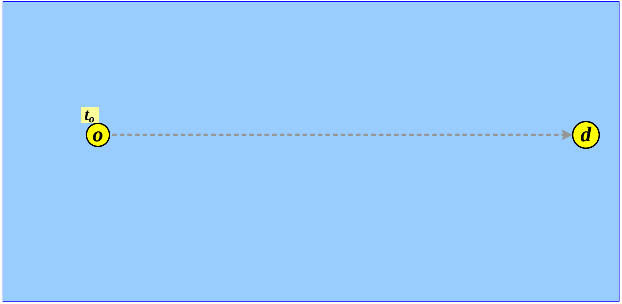

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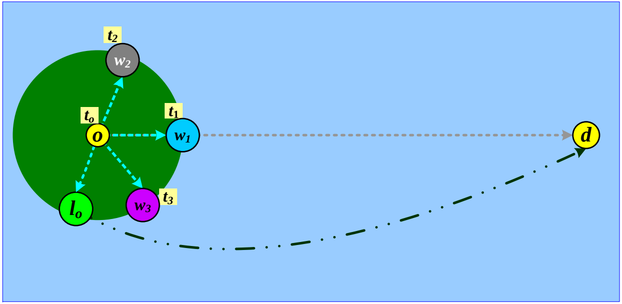

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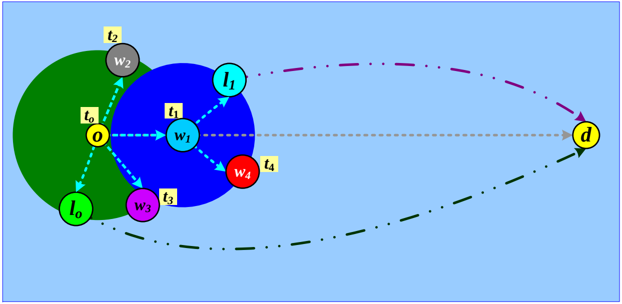

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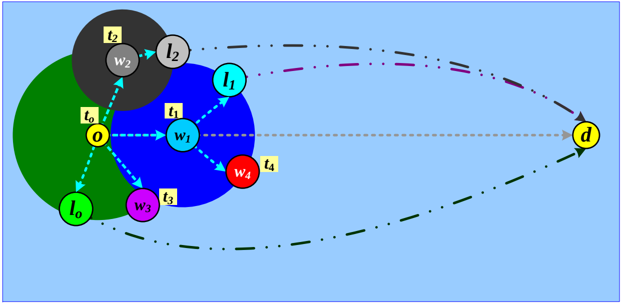

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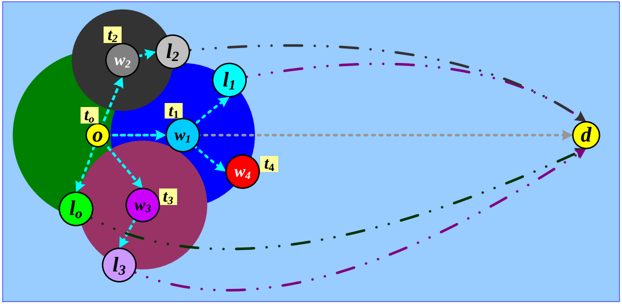

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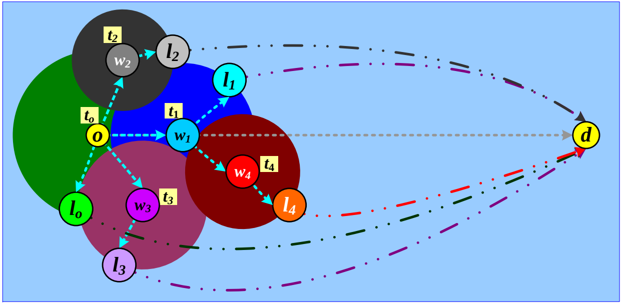

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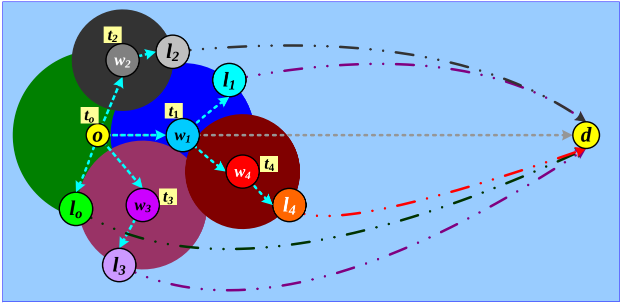

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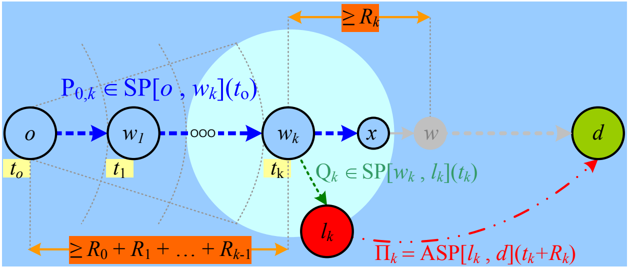
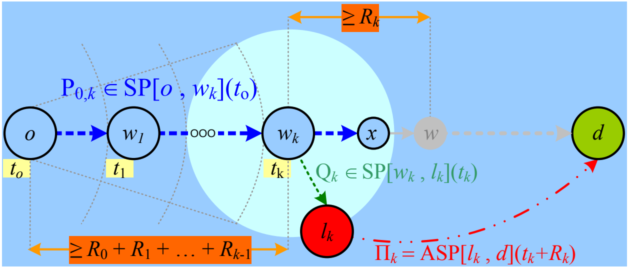
Complexity of RQA for random landmarks

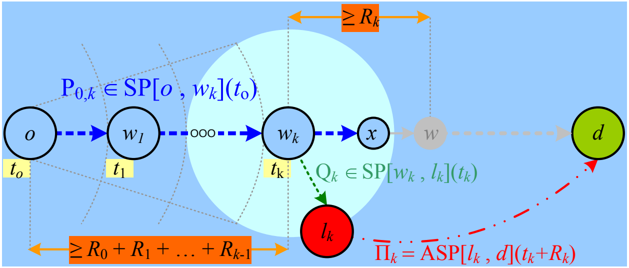
- PTAS: $sol \leq (1 + \sigma) \cdot D[o, d](t_o)$, for $\sigma = \varepsilon \cdot \frac{(1+\varepsilon/\psi)^{R+1}}{(1+\varepsilon/\psi)^{R+1}-1}$ and $R \in \mathrm{O}(1)$.
- Sublinear Query-time: $\mathrm{O}\left(\left(\frac{1}{\rho}\right)^{R+1} \cdot \ln\left(\frac{1}{\rho}\right) \log\log(K_{\max})\right)$


$\geq R_k$
$P_{0,k} \in SP[o , w_k](t_o)$
o
w_1
w_k
x
w
d
t_o
t_1
t_k
$Q_k \in SP[w_k , l_k](t_k)$
l_k
$\geq R_0 + R_1 + \ldots + R_{k-1}$
$\Pi_k = ASP[l_k , d](t_k+R_k)$

RQA: Boosting The Approximation Guarantee Of FCA (KZ (2014))



1. One of the discovered approximate *od*−paths has all its ball centers at *nodes of the (unknown) shortest od-path*.



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$$\boxed{\forall \beta > 1,\ \forall \lambda \in (0,1),\ \lambda \cdot OPT + (1-\lambda)\cdot \beta \cdot OPT < \beta \cdot OPT}$$

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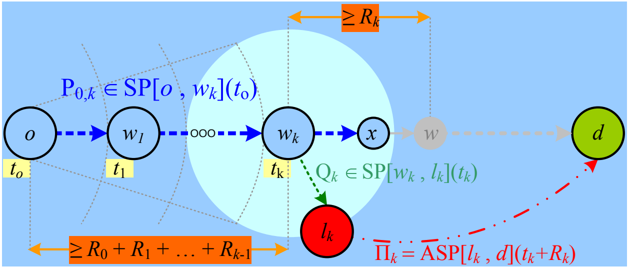

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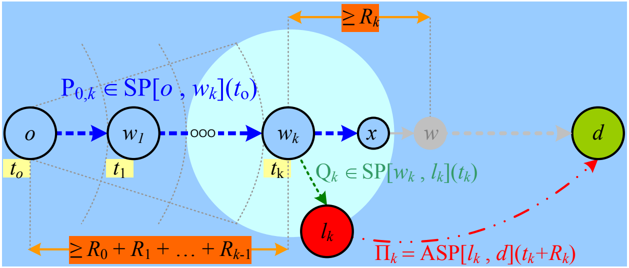

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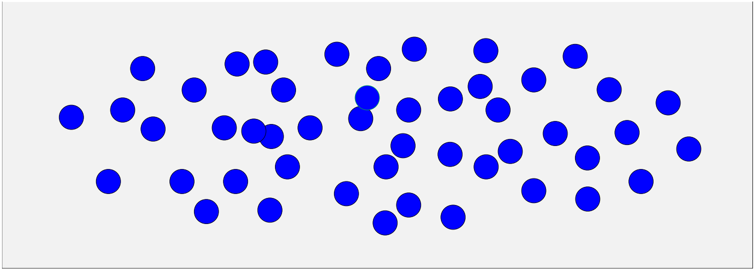
3. Approximation guarantee for **suffix subpath** to destination depends on *last ball radius*.
4. $R = O(1)$ recursion budget suffices to ensure guarantee close to $1+\varepsilon$.

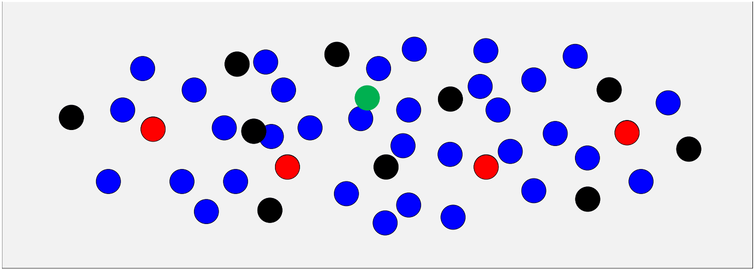
HORN Oracle

HORN Preprocessing: Hierarchy of Landmarks (I) (KWZ (2016))



HORN Preprocessing: Hierarchy of Landmarks (I) (KWZ (2016))

- Selection of landmark sets (colors indicate sizes of coverages).



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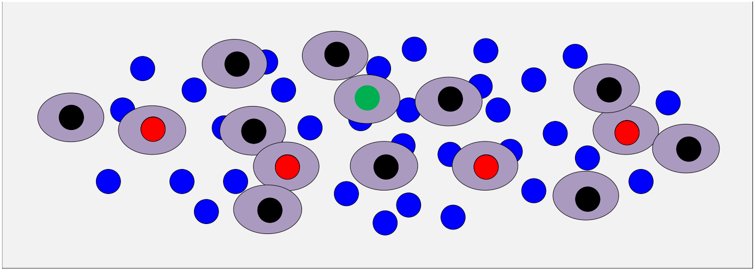

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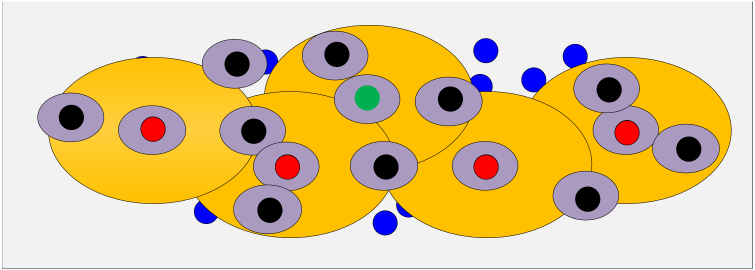

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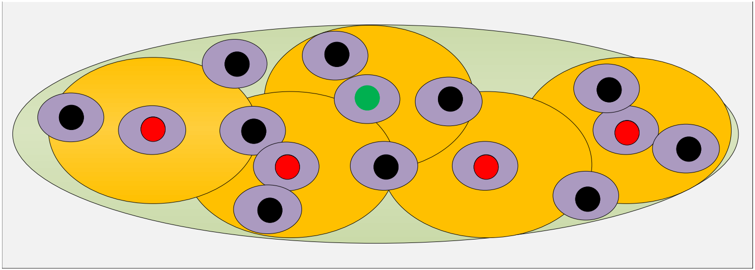
 . . .

- **Global-coverage** landmarks ``learn´´ travel-time functions to their (up to long-range) destinations.



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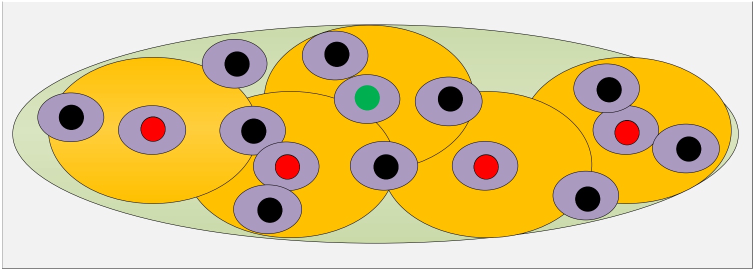

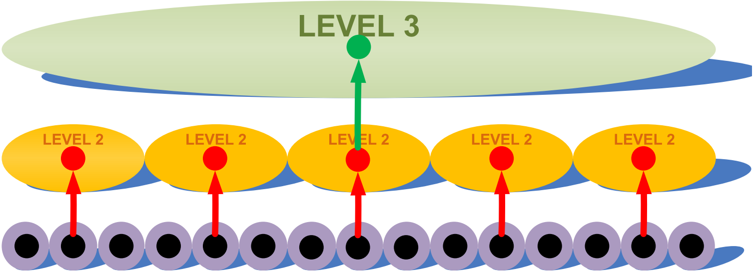

Creating Distance Summaries From Landmarks (KWZ (2016))

Preprocessing of HORN

- Depending on its level, each landmark has its own **coverage**, a given-size set of surrounding vertices for which it is *informed*.
- Exponentially decreasing sequence of *landmark set sizes*.
- exponentially increasing sequence of *coverages per landmark*

$\therefore$ $\boxed{O(\log\log(n)) \text{ levels}}$ $\Rightarrow$ **Subquadratic** preprocessing space/time.

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Preprocessing Complexity of HORN (KWZ (2016))

An appropriate construction of the hierarchy assures preprocessing space and time $O\left(n^{2-\frac{\delta}{R+1}} \cdot \text{polylog}(n)\right)$, i.e., **subquadratic**. R is the *recursion budget* (depth), and $\delta \in (0,1)$ is the targeted exponent of sublinearity, for the query algorithm to be used (see next slides).

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NEXT: Query algorithm with constant approximation, or even **PTAS**, and query-time **sublinear** in the **Dijkstra Rank** of the query at hand.

HQA: The Query Algorithm of HORN

(KWZ (2016))

Rationale of the hierarchy...

level	targeted DR	Q-time	coverage	TRAP	Ring
1	$N_1 = n^{(\gamma-1)/\gamma}$	N_1^{δ}	$c_1 = N_1 \cdot n^{\xi_1}$	$\sqrt{c_1}$	$N_1^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, \ln(n)\right]$
2	$N_2 = n^{(\gamma^2-1)/\gamma^2}$	N_2^{δ}	$c_2 = N_2 \cdot n^{\xi_2}$	$\sqrt{c_2}$	$N_2^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, \ln(n)\right]$
$\vdots$					
k	$N_k = n^{(\gamma^k-1)/\gamma^k}$	N_k^{δ}	$c_k = N_k \cdot n^{\xi_k}$	$\sqrt{c_k}$	$N_k^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, \ln(n)\right]$
k+1	$N_{k+1} = n$	n^{δ}	$c_{k+1} = n$	$\sqrt{n}$	$\left(N_k^{\delta/(R+1)} \cdot \ln(n), n\right]$

1. Mimic FLAT in each level i: All level-i landmarks are informed about c_i destinations around them.

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1. Mimic FLAT in each level i: All level-i landmarks are informed about c_i destinations around them.
2. The *density* of level-i landmarks is such that ALL queries of Dijkstra rank $\leq N_i$ can be answered by using ONLY level-i landmarks.

HQA: The Query Algorithm of HORN

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k+1	$N_{k+1} = n$	n^δ	$c_{k+1} = n$	$\sqrt{n}$	$\left(N_k^{\delta/(R+1)} \cdot \ln(n), n\right]$

1. Mimic FLAT in each level i: All level-i landmarks are informed about c_i destinations around them.
2. The *density* of level-i landmarks is such that ALL queries of Dijkstra rank $\leq N_i$ can be answered by using ONLY level-i landmarks.
3. **FACT:** Running RQA at the **appropriate level** of the hierarchy would yield a good approximation.

Rationale of the hierarchy...

level	targeted DR	Q-time	coverage	TRAP	Ring
1	$N_1 = n^{(\gamma-1)/\gamma}$	N_1^δ	$c_1 = N_1 \cdot n^{\xi_1}$	$\sqrt{c_1}$	$N_1^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, \ln(n)\right]$
2	$N_2 = n^{(\gamma^2-1)/\gamma^2}$	N_2^δ	$c_2 = N_2 \cdot n^{\xi_2}$	$\sqrt{c_2}$	$N_2^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, \ln(n)\right]$
			$\vdots$		
k	$N_k = n^{(\gamma^k-1)/\gamma^k}$	N_k^δ	$c_k = N_k \cdot n^{\xi_k}$	$\sqrt{c_k}$	$N_k^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, \ln(n)\right]$
k+1	$N_{k+1} = n$	n^δ	$c_{k+1} = n$	$\sqrt{n}$	$\left(N_k^{\delta/(R+1)} \cdot \ln(n), n\right]$

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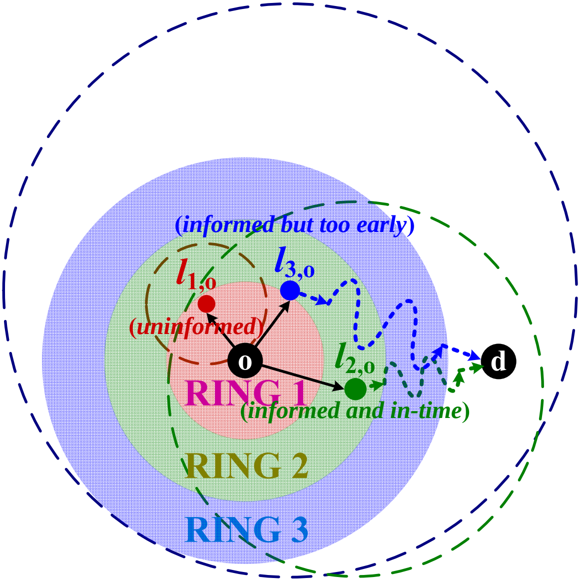
2. The *density* of level-i landmarks is such that ALL queries of Dijkstra rank $\leq N_i$ can be answered by using ONLY level-i landmarks.
3. **FACT:** Running RQA at the **appropriate level** of the hierarchy would yield a good approximation.
4. **CHALLENGE:** *``Guess´´* the appropriate level, **whp** . Then, sublinearity in N_i (rather than n) can be achieved.

Guessing the appropriate level in the hierarchy...

- level-1 landmark $\ell_{1,o}$ is **uninformed**.
- level-3 landmark $\ell_{3,o}$, although informed, came **too early**.
- level-2 landmark $\ell_{2,o}$ is **informed** and within the **right distance**.



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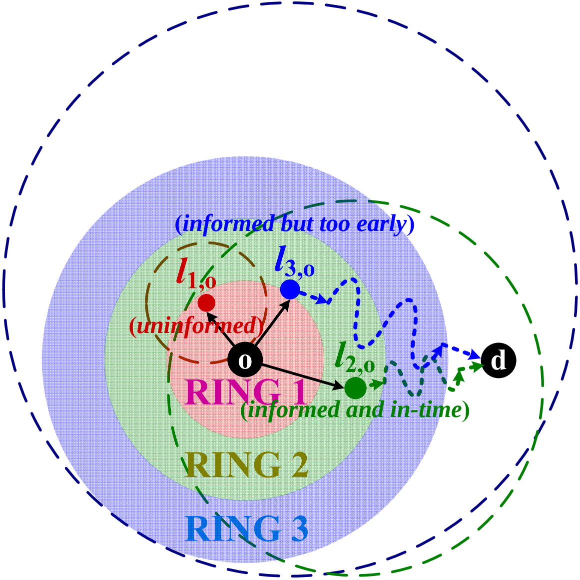
∴ RQA will use only level-(≥ 2) landmarks from now on.



Hierarchical Query Algorithm (HQA)

1. Grow a unique TD-ball from (o, t_o), until the first **informed landmark** ℓ_o discovered *at the right distance* (not too close, not too far) from o.
2. (ESC) Interrupt the process if an informed landmark is discovered very close to the origin (already a good approximation).
3. (ALH) Execute an appropriate variant of RQA, using only landmarks of level at least as high as that of ℓ_o.
4. Return the best approximation, via all discovered informed landmarks.

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Performance of HQA for random landmarks

HORN can be fine-tuned so that it achieves **subquadratic** preprocessing space and time, and query-response time $\tilde{O}(N_i^\delta)$, i.e., **sublinear** in N_i, when $N_{i-1} < DR[o, d](t_o) \leq N_i$, with probability $1 - \mathrm{O}\left(\frac{1}{n}\right)$. The approximation guarantee is $1 + \varepsilon \cdot \frac{(1+\varepsilon/\psi)^{R+1}}{(1+\varepsilon/\psi)^{R+1}-1}$, where $R \leq \frac{2\delta}{\alpha} - 1$ is the recursion budget.

- ☹ Despite using only landmarks of the appropriate level (and above), RQA may **fail to provide approximate paths** via every landmark that it settles (some of them may be ``uninformed'').

HQA: The Query Algorithm of HORN (KWZ (2016))

Approximation guarantee of RQA (in FLAT) also holds for HQA...

- ☹ Despite using only landmarks of the appropriate level (and above), RQA may **fail to provide approximate paths** via every landmark that it settles (some of them may be ``uninformed'').

- ☺ By defining the **appropriate level** i according to the first landmark that is *both ``informed'' and at the ``right'' distance*, we can guarantee that the closest level-i landmark to subsequent ball centers along the unknown shortest path are **always informed**.

HQA: The Query Algorithm of HORN

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- By defining the **appropriate level** *i* according to the first landmark that is *both ``informed'' and at the ``right'' distance*, we can guarantee that the closest level-*i* landmark to subsequent ball centers along the unknown shortest path are **always informed**.

- Analysis of RQA's approximation guarantee still works, because it is based on the via-landmark paths corresponding only to balls centered at vertices of the unknown shortest *od*-path.

- The quality of approximation provided via an informed landmark is dependent on the *landmark's relative distance* from the origin.
- For the first *informed* level-i landmark, the probability of its distance from o NOT belonging to the $N_i^{\delta/(R+1)} \cdot \left(\frac{1}{\ln(n)}, ln(n)\right]$ is $\mathrm{O}\left(\frac{1}{n}\right)$, where i is the appropriate level for (o, d, t_o).

∴ **Success of (ALH)** criterion, which happens **whp** , reveals *asymptotic bounds*, for the (unknown) distance (and Dijkstra rank) from o to d.

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∴ **Success of (ALH)** criterion, which happens **whp** , reveals *asymptotic bounds*, for the (unknown) distance (and Dijkstra rank) from o to d.

- Given that (ESC) did not occur (which could only improve the performance), and that (ALH) succeeds in its ``guess'' of the appropriate level, the corresponding variant of RQA works fine.
- Level-$(k+1)$ landmarks would *always provide a solution*, in time $\mathrm{o}(n)$.

∴ **Failure-of-(ALH)** contribution to the expectation of the query-time is negligible.

Experimental Evaluation

Experimental Evaluation

Identities of Instances

PARAMETER \ INSTANCE	Berlin (TomTom)	Germany (PTV AG)
#Nodes	473,253	4,692,091
#Edges	1,126,468	11,183,060
Time Period	24h (Tue)	24h (Tue-Wed-Thu)
$\lambda_{\max}$	0.017	0.130
$-\lambda_{\min}$	-0.013	-0.130
#Arcs with constant traversal-times	924,254	10,310,234
#Arcs with non-constant traversal-times	20,2214	872,826
Min #Breakpoints	4	5
Avg #Breakpoints	10.4	16.3
Max #Breakpoints	125	52
Total #Breakpoints	3,234,213	25,424,506

(A) Three variants of **random selection** method:

- **RANDOM (R)**: Independent and uniform random selections.
- **IMPORTANT RANDOM (IR)**: Move each selection of (R) to the most important node within a small ball from the selection.
- **SPARSE RANDOM (SR)**: Sequential random selection. Each selected landmark excludes a small neighborhood around it from future selections.

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(B) Partition-dependent selections: Given a graph partition, consider as candidate landmarks only the boundary nodes of the partition.

- **METIS (M)** / **KAHIP (K)**: Start from a METIS / KaHIP partition.
- **SPARSE KAHIP (SK)**: Start from a finer KaHIP partition. Choose randomly, assuring sparsity, landmarks from the boundary nodes.
- **HYBRID (H)**: In a KaHIP partition, half landmarks chosen randomly (and sparsely) from boundary nodes. Remaining nodes equi-distributed randomly in the cells.

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(C) **BETWEENESS CENTRALITY (BC)**: Choose landmarks sequentially, assuring sparsity, according to an approximate BC order.

- Preprocessing of FLAT @ BERLIN:

	BERLIN		GERMANY	
Parallelism	1 thread	6 threads	1 thread	6 threads
Time per landmark	69.5sec	11.5sec	481sec	80.2sec
Space per landmark	13.8MB		25.7MB	

- Responsiveness to **live-traffic reporting**: Averaging $1,000$ random disruptions of 15-min duration.

	BERLIN		GERMANY	
	#Affected Landmarks	Update Time (sec)	#Affected Landmarks	Update Time (sec)
SR_{2000}	32	21.4	3	37.2
SK_{2000}	36	28.8	4	39.1

Experimental Evaluation

(KMPPWZ (2016))

Query-Time Performance: Speedup $> 1,146$ for Berlin and > 902 for Germany.

Berlin: $n = 473,253$ vertices, $m = 1,126,468$ arcs.

Germany: $n = 4,692,091$ vertices, $m = 11,183,060$ arcs.

- **BERLIN:** 1.32sec resolution and $10,000$ *random* queries.

	TDD		FCA		FCA$^+$(6)		RQA	
	Time (msec)	Rel.Error %	Time (msec)	Rel.Error %	Time (msec)	Rel.Error %	Time (msec)	Rel.Error %
R_{2000}	92.906	0	0.100	0.969	0.527	0.405	0.519	0.679
K_{2000}			0.115	1.089	0.321	0.405	0.376	0.523
H_{2000}			0.102	0.886	0.523	0.332	0.445	0.602
IR_{2000}			0.086	0.923	0.489	0.379	0.473	0.604
SR_{2000}			0.081	0.771	0.586	0.317	0.443	0.611
SK_{2000}			0.083	0.781	0.616	0.227	0.397	0.464
R_{541}			0.326	1.854	1.887	0.693	1.904	1.610
SR_{541}			0.451	1.638	3.252	0.614	2.856	1.531
R_{270}			0.639	2.583	3.707	0.881	3.842	2.482
SR_{270}			0.730	2.198	4.491	0.745	4.271	2.336

- **GERMANY:** 8.82sec resolution and 10,000 *random* queries.

	TDD		FCA		FCA$^+$(6)		RQA	
	Time (msec)	Rel.Error %	Time (msec)	Rel.Error %	Time (msec)	Rel.Error %	Time (msec)	Rel.Error %
R_{2000}	$1,145.060$	0	1.532	1.567	8.529	0.742	9.219	1.502
K_{2000}			10.455	2.515	15.209	1.708	30.577	2.343
SR_{2000}			1.275	1.444	9.952	0.662	9.011	1.412
SK_{2000}			1.269	1.534	9.689	0.676	7.653	1.475

Experimental Evaluation

(KMPPWZ (2016))

Dijkstra-Rank Performance: Speedup $> 1,570$ for Berlin and $> 1,531$ for Germany.

Berlin: $n = 473,253$ vertices, $m = 1,126,468$ arcs.

Germany: $n = 4,692,091$ vertices, $m = 11,183,060$ arcs.

- **BERLIN:** 1.32sec resolution and 10, 000 *random* queries.

	TDD		FCA		FCA$^+$(6)		RQA	
	Rank	Speedup	Rank	Speedup	Rank	Speedup	Rank	Speedup
R_{2000}	146, 022	1	150	973.480	877	166.502	925	157.862
K_{2000}			190	768.537	866	168.616	670	217.943
H_{2000}			154	948.195	851	171.589	777	187.931
IR_{2000}			135	1, 081.644	823	177.426	839	174.043
SR_{2000}			119	1, 227.075	952	153.384	776	188.173
SK_{2000}			93	1, 570.129	755	193.406	501	291.461
R_{541}			545	267.930	3, 178	45.947	3, 406	42.872
SR_{541}			638	228.874	3, 684	39.637	3, 950	36.967
R_{270}			1, 075	135.834	6, 198	23.559	6, 702	21.788
SR_{270}			1, 195	122.194	7, 362	19.835	7, 398	19.738

- **GERMANY:** 8.82sec resolution and 10, 000 *random* queries.

	TDD		FCA		FCA$^+$(6)		RQA	
	Rank	Speedup	Rank	Speedup	Rank	Speedup	Rank	Speedup
R_{2000}	1, 717, 793	1	1, 659	1, 035.439	10, 159	169.091	11, 045	155.527
K_{2000}			9, 302	184.669	15, 373	111.741	30, 137	56.999
SR_{2000}			1, 277	1, 345.178	9, 943	172.764	9, 182	187.082
SK_{2000}			1, 122	1, 531.010	9, 000	190.866	7, 975	215.397

- Landmark hierarchies for HORN, with HR and HSR landmark sets:

Level	Size of Levels		Area of coverage	Excluded Ball Size (for *HSR*)	
	$\lvert L\rvert = 10,256$	$\lvert L\rvert = 20,513$		$\lvert L\rvert = 10,256$	$\lvert L\rvert = 20,513$
L_1	7,685	15,370	1,274	35	15
L_2	1,604	3,208	29,243	150	80
L_3	697	1,394	154,847	350	180
L_4	270	541	292,356	800	400

- Performance of HQA at 2.64sec resolution and 10,000 *random* queries:

	TDD				HQA			
	Time (msec)	Rel.Error %	Rank	Speedup	Time (msec)	Rel.Error %	Rank	Speedup
HR_{10256}	92.906	0	146,022	1	0.354	1.499	636	229.594
HSR_{10256}					0.436	1.409	721	202.527
HR_{20513}					0.217	1.051	324	450.685
HSR_{20513}					0.314	0.919	378	386.302

- HQA vs. FLAT/FCA in Berlin:

	Improvement in			Deterioration in
	Query Times (%)	Worst-case Relative Error (%)	Dijkstra Ranks (%)	Space (times)
R_{270} vs HR_{10256}	44.60	41.96	40.83	6.089
SR_{270} vs HSR_{10256}	40.27	35.89	39.66	6.407
R_{541} vs HR_{20513}	33.43	43.31	40.55	6.195
SR_{541} vs HSR_{20513}	30.37	43.89	40.75	6.438

Work In Progress (I)

(KPPWZ (2017))

- CFLAT -- A combinatorial oracle that:
  - Preprocesses and stores only **time-varying shortest-path trees**, rather than travel-time functions: Each vertex has a *time-dependent parent*, per landmark.
  - Avoids **duplicates** in preprocessed data, by storing common departure-time sequences only once and having all the relevant landmark-vertex pairs index them.

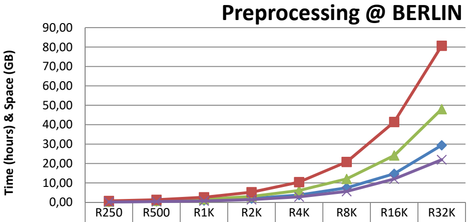
- CFCA -- A novel query algorithm that:
  1. Computes, in reverse order, many candidate paths from each discovered landmark to the destination.
  2. Runs TD-Dijkstra in the subgraph induced by the edges of these paths.

Work In Progress (II)

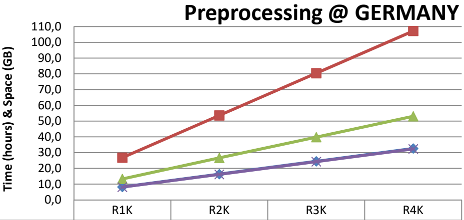
(KPPWZ (2017))

- Experimental Evaluation for CFLAT:
  - More detailed average-case statistics (50, 000 random queries).
  - Significant preprocessing space/time requirements.
  - Comparable query times with FLAT/FCA+, but now including the path reconstruction in the measurements.
  - Improved approximation guarantees.
  - Study the tails of the statistics (existence of outliers).

Preprocessing of CFLAT (RANDOM landmarks) (KPPWZ (2017))

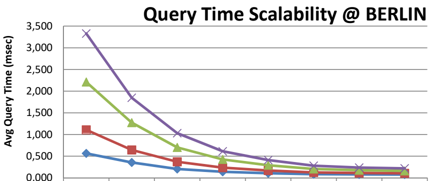


	R250	R500	R1K	R2K	R4K	R8K	R16K	R32K
Time (12 threads)	0,23	0,46	0,92	1,95	3,73	7,45	14,70	29,38
Time (6 threads)	0,38	0,75	1,53	3,02	6,08	12,05	24,12	48,02
Space (uncompr.)	0,70	1,30	2,60	5,20	10,40	20,80	41,40	80,66
Space (compr.)	0,17	0,34	0,69	1,40	2,80	5,60	12,00	21,94

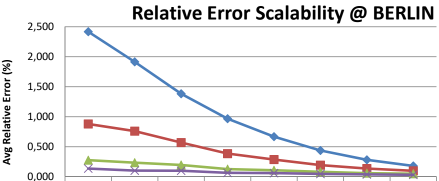


	R1K	R2K	R3K	R4K
Time (12 threads)	8,1	16,3	24,4	32,6
Time (6 threads)	13,3	26,6	39,8	53,0
Space (uncompr.)	26,8	53,6	80,4	107,2
Space (compr.)	8,1	16,1	24,2	32,3

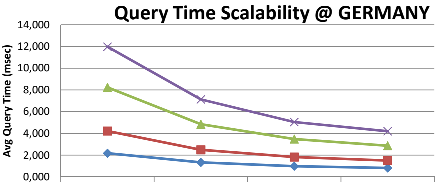
Query-time / Error Scalability CFCA (KPPWZ (2017))



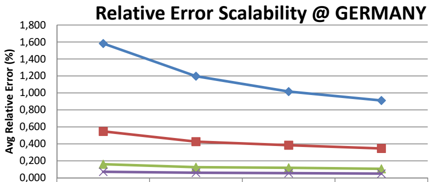
	R250	R500	R1K	R2K	R4K	R8K	R16K	R32K
CFCA(1)	0,565	0,356	0,205	0,140	0,106	0,085	0,079	0,077
CFCA(2)	1,109	0,644	0,371	0,236	0,169	0,125	0,112	0,105
CFCA(4)	2,212	1,274	0,702	0,425	0,291	0,204	0,177	0,161
CFCA(6)	3,330	1,848	1,031	0,613	0,411	0,281	0,237	0,219



	R250	R500	R1K	R2K	R4K	R8K	R16K	R32K
CFCA(1)	2,418	1,915	1,383	0,967	0,668	0,438	0,282	0,180
CFCA(2)	0,880	0,760	0,570	0,385	0,287	0,193	0,136	0,098
CFCA(4)	0,276	0,234	0,196	0,127	0,108	0,082	0,059	0,050
CFCA(6)	0,136	0,102	0,100	0,064	0,060	0,047	0,038	0,032



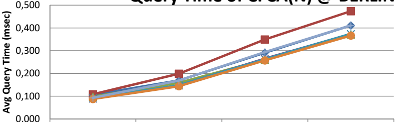
	R1K	R2K	R3K	R4K
CFCA(1)	2,175	1,333	0,981	0,819
CFCA(2)	4,219	2,492	1,824	1,503
CFCA(4)	8,238	4,845	3,477	2,864
CFCA(6)	11,974	7,130	5,045	4,201



	R1K	R2K	R3K	R4K
CFCA(1)	1,582	1,197	1,016	0,911
CFCA(2)	0,547	0,426	0,384	0,346
CFCA(4)	0,160	0,125	0,118	0,106
CFCA(6)	0,071	0,060	0,055	0,049

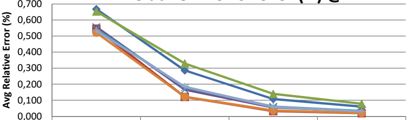
Query-time / Error of CFCA w.r.t. Landmark Sets (KPPWZ (2017))

Query Time of CFCA(N) @ BERLIN



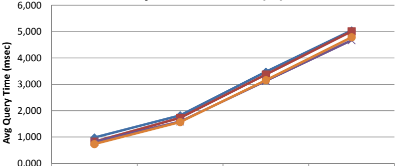
	N=1	N=2	N=4	N=6
R4K	0,106	0,169	0,291	0,411
SR4K	0,108	0,199	0,349	0,474
IR4K	0,098	0,159	0,261	0,371
SK4K	0,089	0,147	0,265	0,373
KC4K	0,091	0,148	0,262	0,374
BC4K	0,088	0,144	0,257	0,367
KB4K	0,093	0,169	0,294	0,411

Relative Error of CFCA(N) @ BERLIN



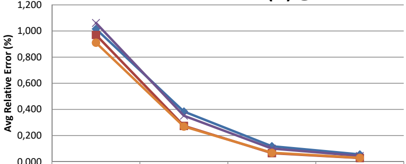
	N=1	N=2	N=4	N=6
R4K	0,668	0,287	0,108	0,060
SR4K	0,546	0,121	0,033	0,019
IR4K	0,653	0,329	0,140	0,078
SK4K	0,557	0,166	0,055	0,033
KC4K	0,544	0,181	0,060	0,033
BC4K	0,521	0,121	0,036	0,021
KB4K	0,534	0,184	0,058	0,031

Query Time of CFCA(N) @ GERMANY



	N=1	N=2	N=4	N=6
R3K	0,981	1,824	3,477	5,045
SR3K	0,825	1,727	3,373	5,010
SK3K	0,819	1,587	3,129	4,669
BC3K	0,733	1,568	3,158	4,787

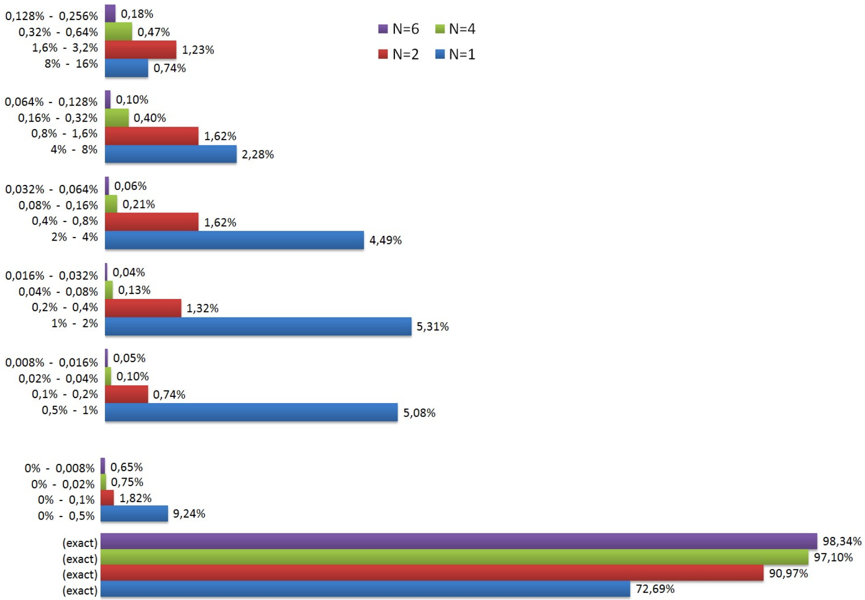
Relative Error of CFCA(N) @ GERMANY



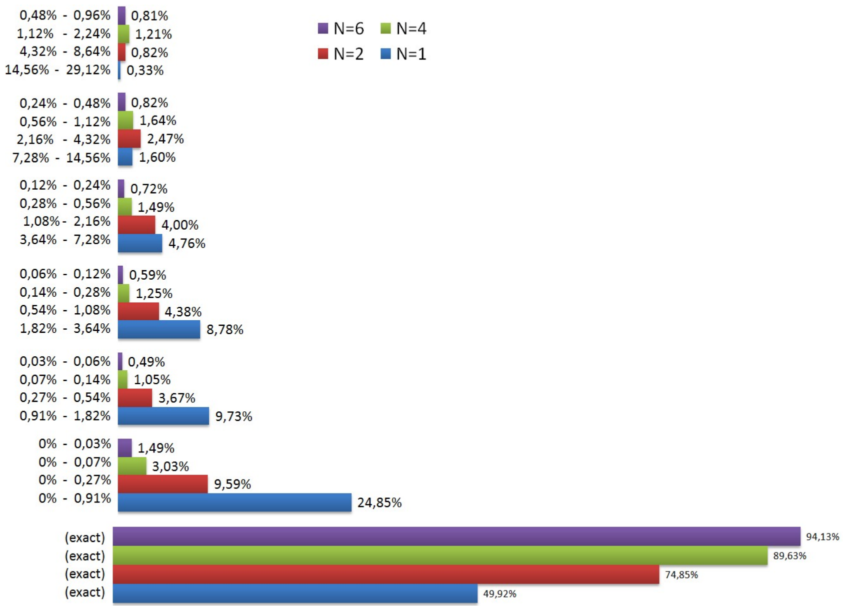
	N=1	N=2	N=4	N=6
R3K	1,016	0,384	0,118	0,055
SR3K	0,970	0,274	0,065	0,028
SK3K	1,062	0,352	0,100	0,044
BC3K	0,911	0,269	0,068	0,029

Exploring Outliers of Relative Error in BERLIN

(KPPWZ (2017))



Exploring Outliers of Relative Error in GERMANY (KPPWZ (2017))



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