BACKWARD DIFFERENCE FORMULAE FOR KURAMOTO–SIVASHINSKY TYPE EQUATIONS*

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Abstract. We analyze the discretization of the periodic initial value problem for Kuramoto–Sivashinsky type equations with Burgers nonlinearity by implicit–explicit backward difference formula (BDF) methods, establish stability and derive optimal order error estimates. We also study discretization in space by spectral methods.

1. Introduction

We construct and analyze efficient numerical methods for periodic initial value problems for evolution equations with Burgers nonlinearity uu_x, of the form

$$u_t + uu_x + \mathcal{P}u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

with $\mathcal{P}$ a linear pseudo-differential operator, and a given initial value $u(\cdot, 0) = u^0$. The solution is required to be L-periodic, $u(x + L, t) = u(x, t)$.

The linear pseudo-differential operator $\mathcal{P}$ is defined by its symbol in Fourier space,

$$(\widehat{\mathcal{P}v})_\ell = \lambda_\ell\, \hat{v}_\ell, \quad \ell \in \mathbb{Z}, \tag{1.2}$$

whenever $v(x) = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell\, \mathrm{e}^{\mathrm{i}\omega\ell x}$, where $\omega := 2\pi/L$. In other words, the functions $\varphi_\ell(x) := \mathrm{e}^{\mathrm{i}\omega\ell x}$ are the eigenfunctions of the operator $\mathcal{P}$, corresponding to the eigenvalues $\lambda_\ell, \ell \in \mathbb{Z}$. We are interested in operators with eigenvalues satisfying

$$\operatorname{Re} \lambda_\ell \geq c_1|\ell|^p, \quad \text{for all } |\ell| \geq \ell_1, \tag{1.3}$$

and

$$|\lambda_\ell| \leq c_2 + c_3|\ell|^p, \quad \text{for all } \ell \in \mathbb{Z}, \tag{1.4}$$

with c_1, c_2, c_3 and p positive constants, and ℓ_1 a positive integer. Condition (1.3) allows finitely many eigenvalues to have negative real parts.

In Section 2 we illustrate the applicability of assumptions (1.2)–(1.4) to five concrete examples: the Kuramoto–Sivashinsky (KS) equation and two dispersively modified variations of it, to an equation introduced by Goodman, and to Otto's model.

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For the discretization in time we shall use a combination of the implicit q-step BDF method (α,β) and the explicit q-step method (α,γ), for $q=1,\dots,6$, described by the polynomials α,β and γ,

$$\tag{1.5} \begin{cases} \alpha(\zeta)=\sum_{j=1}^{q}\frac{1}{j}\zeta^{q-j}(\zeta-1)^j=\sum_{i=0}^{q}\alpha_i\zeta^i, \quad \beta(\zeta)=\zeta^q,\\ \gamma(\zeta)=\zeta^q-(\zeta-1)^q=\sum_{i=0}^{q-1}\gamma_i\zeta^i. \end{cases}$$

The order of the implicit q-step BDF method (α,β) is q. For a given α, the scheme (α,γ) is the unique explicit q-step scheme of order q; the order of all other explicit q-step schemes $(\alpha,\tilde\gamma)$ is at most $q-1$.

Let T be positive, $N\in\mathbb{N}, N\geq q$, and consider a uniform partition $t^n:=nk, n=0,\dots,N$, of the bounded interval $[0,T]$, with time step $k:=T/N$. Assuming we are given starting approximations $U^0,\dots,U^{q-1}$, we discretize in time the periodic initial value problem for equation (1.1) in the time interval $[0,T]$, with initial value u^0, by the implicit–explicit q-step (α,β,γ)-scheme, i.e., we define approximations U^m to the nodal values $u^m:=u(\cdot,t^m)$ of the exact solution as follows

$$\tag{1.6} \sum_{i=0}^{q}\alpha_iU^{n+i}+k\mathcal{P}U^{n+q}=k\sum_{i=0}^{q-1}\gamma_iB(U^{n+i}),$$

$n=0,\dots,N-q$, with $B(v):=-vv_x$. The scheme (1.6) is referred to as the implicit–explicit q-step BDF method; the linear part $\mathcal{P}u$ of the equation is discretized by the implicit BDF method and the nonlinear part $B(u)$ by its explicit counterpart. As a result, the unknown U^{n+q} appears only on the left-hand side of (1.6); therefore, to advance in time, we only need to solve one linear equation of the form $\alpha_qU^{n+q}+k\mathcal{P}U^{n+q}=w$, with given w, which reduces to a linear system if we discretize also in space, at each time level. The computational cost per time step is essentially independent of q; thus, high-order implicit–explicit BDF methods are very efficient.

Implicit–explicit multistep methods, and in particular implicit–explicit BDF schemes, were introduced and analyzed for non-autonomous linear parabolic equations in [13]. The analysis was subsequently extended to nonlinear parabolic equations; see, e.g., [3, 6, 2, 5].

It follows easily from (1.3) that, for a sufficiently large, nonnegative constant $\tilde c$, the eigenvalues $\tilde\lambda_\ell=\tilde c+\lambda_\ell$ of the shifted operator $\tilde{\mathcal{P}}:=\mathcal{P}+\tilde cI$ satisfy the coercivity assumption

$$\tag{1.7} \operatorname{Re}\tilde\lambda_\ell\geq c_0+c_1|\ell|^p, \quad \text{for all } \ \ell\in\mathbb{Z},$$

with c_1,p as in (1.3) and c_0 a positive constant. Now, for a fixed $\tilde c$, let

$$\tag{1.8} \lambda:=\sup_{\ell\in\mathbb{Z}}\frac{|\tilde\lambda_\ell|}{\operatorname{Re}\tilde\lambda_\ell}$$

and

$$\hat\eta_1 = \hat\eta_2 = 0, \quad \hat\eta_3 = \frac{1}{13}, \quad \hat\eta_4 = 0.287806557, \quad \hat\eta_5 = 0.8097337459. \tag{1.9}$$

Assuming $p > 1$ and $\hat\eta_q \lambda < 1$, and following the approach of [6, 2, 5], we establish stability of the q-step scheme (1.6), for $q = 1, \dots, 5$, by energy techniques and, for sufficiently smooth solution u and sufficiently accurate starting approximations $U^0, \dots, U^{q-1}$, derive optimal order error estimates. We also extend the analysis to the fully discrete case; we use the spectral method for the discretization in space.

The stability analysis in [3] concerns a wider class of implicit–explicit multistep methods and more general nonlinearities, is restricted to the case of self-adjoint operators $\mathcal{P}$, is based on spectral and Fourier techniques, and led to sharp stability conditions. Our analysis here uses the energy technique and is based on the Nevanlinna–Odeh multipliers for BDF methods of order up to 5; see the auxiliary Lemma 4.1; this Lemma was recently used first in [22] for the analysis of implicit BDF methods for a class of linear parabolic equations on evolving surfaces and subsequently in [6, 2] both for BDF methods and some computationally less expensive variants for quasi-linear and nonlinear parabolic equations, respectively. The more favorable multipliers of [5] for the three- and five-step BDF schemes allow us to relax the stability condition for the implicit–explicit three- and five-step BDF methods. Implicit–explicit one- and two-step BDF methods for equations similar to the ones considered in this paper are analyzed in [10] by energy techniques.

The accuracy and efficiency of the implicit–explicit BDF methods (1.6) was investigated by extensive numerical experiments in [9, 7, 8, 4] with very satisfactory results. More precisely, implicit–explicit BDF methods were used for the discretization in time of the KS equation in [9], of a nonlinear parabolic system arising in two-phase flows in [7], of a general class of dispersively modified KS equations arising in multiphase hydrodynamics in [8], and of two-dimensional active partial differential equations such as the Topper–Kawahara equation, which is a two-dimensional extension of the dispersively modified KS equation, found in falling film hydrodynamics in [4].

The paper is organized as follows: In Section 2 we discuss five concrete KS-type equations to which our analysis applies, and introduce the function spaces as well as the theoretical preliminaries required for the analysis of the numerical methods (1.6). In Section 3 we prove consistency of the implicit–explicit BDF schemes. Section 4 is devoted to the local stability of the numerical methods. Optimal order error estimates are established in Section 5. In Section 6 we discuss two extensions of the analysis, the first to equations of the form (1.1) with time dependent operators $\mathcal{P}(t)$, and the second to fully discrete schemes; spectral methods are used for the discretization in space. In order to avoid overloading the main text, certain proofs are given in the Appendix.

2. KS-type equations, function spaces and theoretical setting

In this section we discuss five examples of KS-type equations of the form (1.1), to which our analysis applies. We also introduce suitable function spaces and show that the linear operators $\mathcal{P}$ are bounded and satisfy a Gårding inequality, and that the nonlinear operator $B(v) = -vv_x$ satisfies a local Lipschitz condition; see (2.16) below. These properties will play an essential role in our stability analysis in section 4.

2.1. **KS-type equations.** Under assumption (1.3), global existence of solutions of (1.1) has been established for $p > 3/2$ (see [26]); when $p \geq 2$, it can be deduced from [14] that equation (1.1) possesses a global attractor, compact in every Sobolev norm. Analyticity of solutions of (1.1) for $p > 2$ is established in [17].

Otto's model, also referred to as generalized Kuramoto–Sivashinsky (gKS) equation [25],

$$u_t + uu_x - |\partial_x|^a u + |\partial_x|^b u = 0, \tag{2.1}$$

where $b > a \geq 0$ and $|\partial_x|^\sigma$ is the pseudo-differential operator defined as

$$|\partial_x|^\sigma v(x) \;=\; \sum_{\ell\in\mathbb{Z}} |\omega\ell|^\sigma \hat{v}_\ell \, \mathrm{e}^{\mathrm{i}\ell\omega x},$$

for $v(x) = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\omega x}$, is a special case of (1.1) with $\mathcal{P}v = -|\partial_x|^a v + |\partial_x|^b v$. Now, obviously,

$$\mathcal{P}v \;=\; \sum_{\ell\in\mathbb{Z}} \big(|\omega\ell|^b - |\omega\ell|^a\big)\hat{v}_\ell \, \mathrm{e}^{\mathrm{i}\ell\omega x},$$

whence

$$\lambda_\ell = |\omega\ell|^b - |\omega\ell|^a, \quad \ell \in \mathbb{Z}; \tag{2.2}$$

cf. (1.2). In this case, both (1.3) and (1.4) are satisfied. Indeed, first, according to Young's inequality,

$$|\omega\ell|^a \leq \frac{a}{b}|\omega\ell|^b + \frac{b-a}{b},$$

whence

$$\lambda_\ell \geq \frac{b-a}{b}\big(|\omega\ell|^b - 1\big), \quad \ell \in \mathbb{Z}; \tag{2.3}$$

thus, (1.3) is satisfied with $p = b$. Furthermore,

$$|\lambda_\ell| \leq |\omega\ell|^b + |\omega\ell|^a \leq \big(1 + \frac{a}{b}\big)|\omega\ell|^b + \frac{b-a}{b};$$

therefore, (1.4) is also satisfied.

Concerning the ratio λ, cf. (1.8), shifting the eigenvalues $\lambda_\ell \in \mathbb{R}, \ell \in \mathbb{Z}$, see (2.2), by a suitable positive constant $\tilde{c}$, $\widetilde{\lambda}_\ell := \lambda_\ell + \tilde{c}$, cf. (1.7), we see that (1.8) is satisfied with $\lambda = 1$.

Special cases of equation (2.1), and, in particular, of equation (1.1), are the Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{2.4}$$

corresponding to the parameters $a = 2$ and $b = 4$, and the Burgers–Sivashinsky equation

$$u_t + uu_x - u - u_{xx} = 0, \tag{2.5}$$

corresponding to the case $a = 0$ and $b = 2$; the latter equation was introduced by Jonathan Goodman [15].

KS is a simple partial differential equation, which exhibits a particularly complex dynamical behavior as the period L grows. It arises in a variety of applications, for example, in concentration waves in chemically reacting systems [21], in flame propagation and reaction combustion [24], in free surface film-flows of viscous liquids, and in the dynamics of interfaces in two-phase flows in cylindrical geometries [19]. It is one of the simplest PDEs with a convective nonlinearity and a band of unstable modes, in its linearized version (around zero), and thus it has served as a typical example on which the general notions of inertial manifold theory are applied. This means that the long time dynamic behavior of KS is captured well by a finite dimensional dynamical system, the number of degrees of freedom of which is at least as large as the number of linearly unstable Fourier frequencies [12].

We shall also consider dispersively modified variations of the KS equation of the form

$$u_t + uu_x + u_{xx} + u_{xxxx} + \mathcal{D}u = 0. \tag{2.6}$$

Special cases of (2.6) are

$$u_t + uu_x + u_{xx} + du_{xxx} + u_{xxxx} = 0, \tag{2.7}$$

derived in falling flows and known as the Kawahara equation [20], and an equation derived in the context of interfacial hydrodynamics (see [19, 18]), in which the dispersive pseudo-differential operator $\mathcal{D}$ is defined as

$$(\widehat{\mathcal{D}v})_\ell = \mathrm{i}\, d_\ell\, \hat{v}_\ell, \quad d_\ell = \frac{(\omega\ell)^2 I_1^2(\omega\ell)}{\omega\ell I_1^2(\omega\ell) - \omega\ell I_0^2(\omega\ell) + 2I_0(\omega\ell)I_1(\omega\ell)}, \quad \ell \in \mathbb{Z}, \tag{2.8}$$

with $I_\nu(\xi)$ the modified Bessel function of the first kind of order ν. As

$$I_\nu(x) = \frac{\mathrm{e}^x}{\sqrt{2\pi x}}\left(1 + \frac{1-4\nu^2}{8x} + \mathcal{O}\Big(\frac{1}{x^2}\Big)\right),$$

for large x (see [1, p. 377, §9.7]), it follows that $d_\ell = (\omega\ell)^2 + \mathcal{O}(\ell)$, for large $|\ell|$.

In the cases of dispersively modified KS equations, the operators are not self-adjoint; however, since the dispersive terms in both (2.7) and (2.6)–(2.8) are of lower order than the dominating term, we can achieve $\lambda \le 1 + \varepsilon$ (see (1.8)) for any positive ε, for a suitable positive constants $\tilde c$. Indeed, for equations (2.7) and (2.6)–(2.8), we have

$$\lambda_\ell = -(\omega\ell)^2 - \mathrm{i}d(\omega\ell)^3 + (\omega\ell)^4 \quad \text{and} \quad \lambda_\ell = -(\omega\ell)^2 + (\omega\ell)^4 + \mathrm{i}d_\ell, \tag{2.9}$$

$\ell \in \mathbb{Z}$, respectively. Therefore, for the shifted eigenvalues $\tilde\lambda_\ell := \lambda_\ell + \tilde c$, we have

$$\frac{|\tilde\lambda_\ell|}{\operatorname{Re}\tilde\lambda_\ell} \le 1 + d^2 \frac{|\omega\ell|^3}{\tilde c - (\omega\ell)^2 + (\omega\ell)^4} \quad \text{and} \quad \frac{|\tilde\lambda_\ell|}{\operatorname{Re}\tilde\lambda_\ell} \le 1 + \frac{|d_\ell|}{\tilde c - (\omega\ell)^2 + (\omega\ell)^4},$$

respectively. The fractions on the right-hand sides tend to zero as $|\ell|$ tends to infinity, and can be made arbitrarily small for $|\ell| \le \tilde\ell$, with a fixed $\tilde\ell$, by taking $\tilde c$ sufficiently large; this is straight-forward for the first fraction, while for the second fraction it follows from the fact that $d_\ell = (\omega\ell)^2 + \mathcal{O}(\ell)$, for large $|\ell|$.

2.2. **Two essential properties of the linear operator.** For $s \in \mathbb{R}$, we denote by H^s_{per} the Sobolev space of order s, consisting of the L-periodic elements of $H^s_{\text{loc}}(\mathbb{R})$, with norm[1]

$$\|v\|_{H^s} := \Big(\sum_{\ell\in\mathbb{Z}} (1+\omega^2\ell^2)^s\, |\hat v_\ell|^2\Big)^{1/2}.$$

Clearly, H^s_{per} is a Hilbert space, for every $s \in \mathbb{R}$. Let $H := H^0_{\text{per}} = \mathrm{L}^2_{\text{per}}$. Then the norm on H, which we shall be denoting by $|\cdot|$, is induced by the inner product

$$(u,v) = \frac{1}{L}\int_0^L u(x)\,\overline{v}(x)\,dx = \sum_{\ell\in\mathbb{Z}} \hat u_\ell\, \overline{\hat v}_\ell.$$

Condition (1.7) is equivalent to

$$\operatorname{Re}\tilde\lambda_\ell \ge \kappa\big(1+\omega^2\ell^2\big)^d, \quad \text{for all } \ell \in \mathbb{Z}, \tag{1.7$'$}$$

with $d = p/2$ and κ a suitable positive constant.

Now, with $d := p/2$, we introduce in $V := H^d_{\text{per}}$ and $V' = H^{-d}_{\text{per}}$ the operator dependent norms $\|\cdot\|$ and $\|\cdot\|_\star$, respectively, by

$$\|v\| := \Big(\sum_{\ell\in\mathbb{Z}} \operatorname{Re}\tilde\lambda_\ell\, |\hat v_\ell|^2\Big)^{1/2}, \quad \|v\|_\star := \Big(\sum_{\ell\in\mathbb{Z}} (\operatorname{Re}\tilde\lambda_\ell)^{-1} |\hat v_\ell|^2\Big)^{1/2}. \tag{2.10}$$

Notice that the norm $\|\cdot\|$ is induced by the inner product

$$\langle u, v\rangle = (\mathcal{S}u, v),$$

with $\mathcal{S} := \frac12(\tilde{\mathcal{P}} + \tilde{\mathcal{P}}^\star)$, the self-adjoint part of the operator $\tilde{\mathcal{P}}$, and in view of (1.7) (see also (1.7$'$)) and (1.4), the norms $\|\cdot\|$ and $\|\cdot\|_{H^d}$ are equivalent.

[1]Note that, if s is a nonnegative integer, then $\|\cdot\|_{H^s}$ is equivalent to the norm defined by

$$\|u\|_s = \Big(\sum_{j=0}^s \int_0^L |u^{(j)}(x)|^2\,dx\Big)^{1/2}.$$

Denoting by $(\cdot,\cdot)$ also the duality pairing between $H^{-d}_{\rm per}$ and $H^d_{\rm per}$, which reduces to the inner product $(\cdot,\cdot)$ in $H\times V$, for $v\in H^d_{\rm per}$ we obviously have

$$\operatorname{Re}(\tilde{\mathcal P}v,v)=\operatorname{Re}\sum_{\ell\in\mathbb Z}\lambda_\ell\,|\hat v_\ell|^2;$$

in particular, the operator $\tilde{\mathcal P}$ is *coercive*; more precisely,

$$\operatorname{Re}(\tilde{\mathcal P}v,v)=\|v\|^2,\quad \text{for all } v\in V=H^d_{\rm per}. \tag{2.11}$$

Therefore, since $(\tilde{\mathcal P}v,v)=(\mathcal Pv,v)+\tilde c(v,v)$, the operator $\mathcal P$ satisfies the Gårding inequality

$$\operatorname{Re}(\mathcal Pv,v)\ge\|v\|^2-\tilde c|v|^2,\quad \text{for all } v\in V=H^d_{\rm per}. \tag{2.12}$$

Moreover, $\mathcal P:V\to V'$ is *bounded*; more precisely,

$$\|\mathcal Pv\|_\star\le\lambda\|v\|+\tilde c\|v\|_\star,\quad \text{for all } v\in V=H^d_{\rm per}, \tag{2.13}$$

with the bound λ given by (1.8). Indeed, for $v,w\in H^d_{\rm per}$, we have

$$(\tilde{\mathcal P}v,w)=\sum_{\ell\in\mathbb Z}\tilde\lambda_\ell\hat v_\ell\,\overline{\widehat w}_\ell,$$

whence, in view of (1.8),

$$|(\tilde{\mathcal P}v,w)|\le\sum_{\ell\in\mathbb Z}|\tilde\lambda_\ell|\,|\hat v_\ell|\,|\widehat w_\ell|\le\lambda\sum_{\ell\in\mathbb Z}\operatorname{Re}\tilde\lambda_\ell|\hat v_\ell|\,|\widehat w_\ell|\le\lambda\|v\|\,\|w\|;$$

thus,

$$\|\tilde{\mathcal P}v\|_\star\le\lambda\|v\| \tag{2.14}$$

and (2.13) follows. It is also obvious that λ is the norm of the operator $\tilde{\mathcal P}$ as a linear mapping from the space V to the space V', endowed with the specific norms $\|\cdot\|$ and $\|\cdot\|_\star$, respectively.

Notice that, in view of (1.7) and (1.4), the constant λ in (1.8) is finite. For our analysis, the important consequence from (1.7), (1.4), and (1.8) is that all eigenvalues of the operator $\tilde{\mathcal P}$ are contained in the sector S_ϑ,

$$S_\vartheta:=\{z\in\mathbb C: z=\rho\,\mathrm e^{\mathrm i\varphi},\rho\ge0,|\varphi|\le\vartheta\},$$

with $\vartheta\in[0,\pi/2)$ such that $\cos\vartheta=1/\lambda$.

2.2.1. *Existence and uniqueness of the approximations.* For given $U^n,\dots,U^{n+q-1}\in V$, the scheme (1.6) is of the form

$$\alpha_q v+k\mathcal Pv=w, \tag{2.15}$$

with given $w\in V'$ and unknown v. The bilinear form $a:V\times V\to\mathbb R, a(v,\tilde v):=\alpha_q(v,\tilde v)+k(\mathcal Pv,\tilde v)$, is bounded and, for $k\le\tilde c/\alpha_q$, coercive. Indeed, in view of (2.13), we have

$$|a(v,\tilde v)|\le\alpha_q|v|\,|\tilde v|+k\big(\lambda\|v\|+\tilde c\|v\|_\star\big)\|\tilde v\|,$$

and the boundedness of a follows easily from the fact that the norms $|\cdot|$ and $\|\cdot\|_\star$ are dominated by $\|\cdot\|$. Furthermore,

$$\operatorname{Re} a(v,v) = (\alpha_q - \tilde{c}k)|v|^2 + k\operatorname{Re}(\tilde{\mathcal{P}}v, v);$$

thus, or $k \le \alpha_q/\tilde{c}$, the coercivity of a is a consequence of the coercivity of $\tilde{\mathcal{P}}$; see (2.11). Now, existence and uniqueness of the solution $v \in V$ of (2.15) follow easily from the Lax–Milgram lemma.

We infer that, for $k \le \alpha_q/\tilde{c}$, the approximations $U^q, \dots, U^N \in V$ are well defined.

2.3. **Local Lipschitz continuity of the nonlinear operator.** Assume that $p > 1$, i.e., $d > 1/2$, and let $d' \in (1/2, d), d' \le 1$. Then, for any positive μ, there exists a constant ν, depending on μ, such that the operator

$$B : V \to V', \quad B(v) := -vv_x,$$

satisfies the local Lipschitz condition

$$\|B(v) - B(\tilde{v})\|_\star \le \mu\|v - \tilde{v}\| + \nu|v - \tilde{v}| \quad \text{for all } v, \tilde{v} \in T_u, \tag{2.16}$$

in the tube T_u,

$$T_u := \big\{v \in V : \min_t \|v - u(t)\|_{H^{d'}} \le 1\big\}, \tag{2.17}$$

around the solution u, defined in terms of the norm of $H^{d'}$.

Indeed, first, obviously, for $v, \tilde{v}, w \in V$, we have

$$\big(B(v) - B(\tilde{v}), w\big) = -\frac{1}{2}\big((v^2 - \tilde{v}^2)_x, w\big). \tag{2.18}$$

Furthermore,

$$\begin{aligned} |(z_x, w)| &\le \sum_{\ell \in \mathbb{Z}} |\omega\ell|\, |\hat{z}_\ell|\, |\hat{w}_\ell| = \sum_{\substack{\ell \in \mathbb{Z} \\ \ell \ne 0}} |\omega\ell|^{1-d} |\hat{z}_\ell|\, |\omega\ell|^d |\hat{w}_\ell| \\ &\le \Big(\sum_{\substack{\ell \in \mathbb{Z} \\ \ell \ne 0}} |\omega\ell|^{2(1-d)} |\hat{z}_\ell|^2\Big)^{1/2} \Big(\sum_{\substack{\ell \in \mathbb{Z} \\ \ell \ne 0}} |\omega\ell|^{2d} |\hat{w}_\ell|^2\Big)^{1/2}, \end{aligned}$$

whence

$$|(z_x, w)| \le \|z\|_{H^{1-d}}\, \|w\|_{H^d}. \tag{2.19}$$

From (2.18) and (2.19), and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{H^d}$, we infer that

$$\|B(v) - B(\tilde{v})\|_\star \le c\|v^2 - \tilde{v}^2\|_{H^{1-d}}. \tag{2.20}$$

Now, according to Corollary A.2 (see Appendix), with $u = v + \tilde{v}$, we have

$$\|(v + \tilde{v})(v - \tilde{v})\|_{H^{1-d}} \le \|v + \tilde{v}\|_{H^{d'}} \big(c_{\tilde{\varepsilon}}|v - \tilde{v}| + \tilde{\varepsilon}\, \|v - \tilde{v}\|_{H^d}\big),$$

whence, since $v, \tilde{v} \in T_u$,

$$\|v^2 - \tilde{v}^2\|_{H^{1-d}} \le C\big(c_{\tilde{\varepsilon}}|v - \tilde{v}| + \tilde{\varepsilon}\, \|v - \tilde{v}\|\big). \tag{2.21}$$

The desired local Lipschitz condition (2.16) is now an immediate consequence of (2.20) and (2.21).

Remark 2.1 (Alternative forms of the local Lipschitz condition (2.16))**.** The local Lipschitz condition (2.16) is perfectly suitable for the stability analysis as well as for the error analysis of the time stepping methods. However, the appearance of the strong norm $\|\cdot\|$ on its right-hand side results in a reduction by one of the order in the space discretization parameter in the analysis of fully discrete methods.

In the case $p \geq 2$, we can recover the full order in the space discretization parameter, while, in the case $1 < p < 2$, we can partly restore the order reduction, to an extend depending on the value of p. To this end, we shall modify the local Lipschitz condition (2.16). First, for $p \geq 2$, i.e., $d \geq 1$, the norm $\|\cdot\|_{H^{1-d}}$ is weaker than the L^2-norm; thus (2.20) yields

$$\|B(v) - B(\tilde{v})\|_\star \leq C|v^2 - \tilde{v}^2|$$

and we easily infer that

$$\|B(v) - B(\tilde{v})\|_\star \leq c\|v + \tilde{v}\|_{L^\infty}|v - \tilde{v}|, \quad p \geq 2, \tag{2.22}$$

a local Lipschitz condition that will allow us to recover the full order in this case; see (6.10) in the sequel.

In the case $1 < p < 2$, i.e., $1/2 < d < 1$, we shall use the interpolation inequality

$$\|w\|_{H^{1-d}} \leq \|w\|^d_{\mathrm{L}^2}\|w\|^{1-d}_{H^1}. \tag{2.23}$$

In general, for $r < s < t$, Hölder's inequality yields

$$\|w\|^{t-r}_{H^s} \leq \|w\|^{t-s}_{H^r}\|w\|^{s-r}_{H^t},$$

and (2.23) is a special case of this inequality for $r = 0, s = 1 - d$ and $t = 1$. Now, from (2.20) and (2.23), we easily infer that

$$\|B(v) - B(\tilde{v})\|_\star \leq c\|v + \tilde{v}\|_{W^{1,\infty}}|v - \tilde{v}|^d\,\|v - \tilde{v}\|^{1-d}, \quad 1 < p < 2, \tag{2.24}$$

with $d = p/2$, a local Lipschitz condition that will allow us to partly restore the order reduction in this case; see (6.11) in the sequel. □

3. Consistency

In this section we prove consistency of the implicit–explicit BDF scheme (1.6) for the solution u of the periodic initial value problem for equation (1.1).

The order of the q-step methods (α, β) and (α, γ) is q, i.e.,

$$\sum_{i=0}^{q} i^\ell \alpha_i = \ell q^{\ell-1} = \ell \sum_{i=0}^{q-1} i^{\ell-1}\gamma_i, \quad \ell = 0, 1, \dots, q. \tag{3.1}$$

The consistency errors E^n of the scheme (1.6) for the solution u of the periodic initial value problem for equation (1.1), i.e., the amount by which the exact solution misses satisfying (1.6), is given by

$$kE^n = \sum_{i=0}^{q} \alpha_i u^{n+i} + k\mathcal{P}u^{n+q} - k\sum_{i=0}^{q-1} \gamma_i B(u^{n+i}), \tag{3.2}$$

$n = 0, \dots, N-q$. Here, $u^{n+i} := u(\cdot, t^{n+i})$ denote the nodal values of the exact solution $u(\cdot, t)$.

Lemma 3.1 (Consistency of implicit–explicit BDF schemes)**.** *The consistency error* (3.2) *of the scheme* (1.6) *is bounded by*

$$\max_{0\le n\le N-q} \|E^n\|_\star \le \tilde{C}k^q, \tag{3.3}$$

provided that the solution u is sufficiently regular.

Proof. This short and elementary proof proceeds along the lines of analogous proofs in, e.g., [3, 6, 2, 5]; it is included here for the convenience of the reader. Letting

$$E_1^n := \sum_{i=0}^{q} \alpha_i u^{n+i} - ku'(t^{n+q}), \quad E_2^n := kB(u^{n+q}) - k\sum_{i=0}^{q-1} \gamma_i B(u^{n+i}),$$

and using the differential equation in (1.1), we infer that

$$kE^n = E_1^n + E_2^n. \tag{3.4}$$

Furthermore, by Taylor expanding about t^n and using the order conditions of the implicit (α, β)-scheme, i.e., the first equality in (3.1), and the second equality in (3.1), respectively, leading terms of order up to $q-1$ cancel, and we obtain

$$E_1^n = \frac{1}{q!}\Big[\sum_{i=0}^{q} \alpha_i \int_{t^n}^{t^{n+i}} (t^{n+i}-s)^q \partial_t^{q+1} u(\cdot, s)ds - qk\int_{t^n}^{t^{n+q}} (t^{n+q}-s)^{q-1} \partial_t^{q+1} u(\cdot, s)ds\Big]$$

and

$$E_2^n = \frac{k}{(q-1)!}\Big[\int_{t^n}^{t^{n+q}} (t^{n+q}-s)^{q-1} \partial_t^q \widetilde{B}(\cdot, s)ds - \sum_{i=0}^{q} \gamma_i \int_{t^n}^{t^{n+i}} (t^{n+i}-s)^{q-1} \partial_t^q \widetilde{B}(\cdot, s)ds\Big],$$

with $\widetilde{B}(\cdot, t) := B(u(\cdot, t)), t \in [0, T]$, respectively. Thus, under the pertinent regularity requirements, we obtain the desired consistency estimate (3.3). □

4. Stability

In this section, following the approach of [6, 2, 5], we establish local stability of the implicit–explicit BDF scheme (1.6), under a suitable sufficient stability condition, by the energy technique.

The (implicit) BDF methods are A-stable for $q = 1$ and $q = 2$, i.e., $A(\vartheta_q)$-stable with $\vartheta_1 = \vartheta_2 = 90°$, and $A(\vartheta_q)$-stable for $q = 3, \dots, 6$ with $\vartheta_3 = 86.03°$, $\vartheta_4 = 73.35°$, $\vartheta_5 = 51.84°$ and $\vartheta_6 = 17.84°$; see [16, Section V.2].

4.1. **The Nevanlinna–Odeh multipliers for BDF methods.** Based on Dahlquist's G-stability theory, Nevanlinna and Odeh [23] proved the following result for BDF methods of order up to five; this result allows us to establish stability by the energy method.

Lemma 4.1 (Multipliers for BDF methods, [23]). *Let $\alpha \in \mathbb{P}_q, q \le 5$, be the generating polynomial of the q-step BDF method; see* (1.5). *Let $(\cdot,\cdot)$ be an inner product with associated norm $|\cdot|$. Then, there exist a multiplier $\eta_q, 0 \le \eta_q < 1$, a positive definite symmetric matrix $G = (g_{ij}) \in \mathbb{R}^{q,q}$ and reals $\delta_0, \dots, \delta_q$ such that for $v^0, \dots, v^q$ in the inner product space,*

$$\operatorname{Re}\Big(\sum_{i=0}^{q}\alpha_i v^i, v^q - \eta_q v^{q-1}\Big) = \sum_{i,j=1}^{q} g_{ij}(v^i, v^j) - \sum_{i,j=1}^{q} g_{ij}(v^{i-1}, v^{j-1}) + \Big|\sum_{i=0}^{q}\delta_i v^i\Big|^2.$$

The smallest possible values of the multipliers η_q are

$$\eta_1 = \eta_2 = 0,\ \eta_3 = 0.0836,\ \eta_4 = 0.2878,\ \eta_5 = 0.8160. \qquad \square$$

4.2. **Stability.** Since the differential equation (1.1) is nonlinear, besides the approximations $U^n \in T_u$ satisfying (1.6), we consider implicit–explicit BDF approximations $V^n \in T_u$ such that

$$\sum_{i=0}^{q}\alpha_i V^{n+i} + k\mathcal{P}V^{n+q} = k\sum_{i=0}^{q-1}\gamma_i B(V^{n+i}), \tag{4.1}$$

$n = 0, \dots, N-q$.

Theorem 4.1 (Local stability of implicit–explicit BDF schemes). *Assume that the stability constant λ in* (1.8) *is small enough, such that*

$$\eta_q\lambda < 1. \tag{4.2}$$

Then the implicit–explicit BDF method (1.6) *is locally stable, in the sense that, with $\vartheta^m := U^m - V^m$, for k sufficiently small,*

$$|\vartheta^n|^2 + k\sum_{\ell=q}^{n}\|\vartheta^\ell\|^2 \le C\sum_{j=0}^{q-1}\big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big), \tag{4.3}$$

for $n = q, \dots, N$, with a constant C independent of k, n and the approximations.

Proof. This proof proceeds along the lines of analogous proofs in, e.g., [6, 2, 5]; it is included here for the convenience of the reader. Letting $b^m := B(U^m) - B(V^m)$ and subtracting (4.1) from (1.6), we obtain

$$\sum_{i=0}^{q}\alpha_i\vartheta^{n+i} + k\mathcal{P}\vartheta^{n+q} = k\sum_{i=0}^{q-1}\gamma_i b^{n+i}, \tag{4.4}$$

$n=0,\dots,N-q$. We take in (4.4) the inner product with $\vartheta^{n+q}-\eta_q\vartheta^{n+q-1}$, and then real parts and get

$$\operatorname{Re}\Big(\sum_{i=0}^{q}\alpha_i\vartheta^{n+i},\vartheta^{n+q}-\eta_q\vartheta^{n+q-1}\Big)+kI_{n+q}=kJ_{n+q} \tag{4.5}$$

with $I_{n+q}:=\operatorname{Re}\big(\mathcal{P}\vartheta^{n+q},\vartheta^{n+q}-\eta_q\vartheta^{n+q-1}\big)$, whence

$$I_{n+q}=\operatorname{Re}\big(\tilde{\mathcal{P}}\vartheta^{n+q},\vartheta^{n+q}-\eta_q\vartheta^{n+q-1}\big)-\tilde{c}\big(\vartheta^{n+q},\vartheta^{n+q}-\eta_q\vartheta^{n+q-1}\big), \tag{4.6}$$

and

$$J_{n+q}:=\operatorname{Re}\Big(\sum_{i=0}^{q-1}\gamma_i b^{n+i},\vartheta^{n+q}-\eta_q\vartheta^{n+q-1}\Big). \tag{4.7}$$

With the notation $\Theta^n:=(\vartheta^{n-q+1},\dots,\vartheta^n)^T$ and the norm $|\Theta^n|_G$ given by

$$|\Theta^n|_G^2=\sum_{i,j=1}^{q}g_{ij}(\vartheta^{n-q+i},\vartheta^{n-q+j}),$$

in view of Lemma 4.1, relation (4.5) yields the estimate

$$|\Theta^{n+q}|_G^2-|\Theta^{n+q-1}|_G^2+kI_{n+q}\le kJ_{n+q}. \tag{4.8}$$

Furthermore, in view of the coercivity condition (2.11) and the boundedness condition (2.14), for the operator $\tilde{\mathcal{P}}$, we have

$$I_{n+q}\ge\|\vartheta^{n+q}\|^2-\eta_q\lambda\|\vartheta^{n+q}\|\,\|\vartheta^{n+q-1}\|-\tilde{c}\big(|\vartheta^{n+q}|^2+\eta_q|\vartheta^{n+q}|\,|\vartheta^{n+q-1}|\big),$$

and hence

$$\begin{aligned}I_{n+q}\ge{}&\Big(1-\frac{\lambda}{2}\eta_q\Big)\|\vartheta^{n+q}\|^2-\frac{\lambda}{2}\eta_q\|\vartheta^{n+q-1}\|^2\\&-\tilde{c}\Big(1+\frac{1}{2}\eta_q\Big)|\vartheta^{n+q}|^2-\frac{\tilde{c}}{2}\eta_q|\vartheta^{n+q-1}|^2.\end{aligned} \tag{4.9}$$

We shall now estimate J_{n+q}. First, we have

$$J_{n+q}\le\sum_{i=0}^{q-1}|\gamma_i|\,\|b^{n+i}\|_\star\big(\|\vartheta^{n+q}\|+\eta_q\|\vartheta^{n+q-1}\|\big),$$

whence, in view of the local Lipschitz condition (2.16), for any positive ε,

$$\begin{aligned}
&J_{n+q} \le \sum_{i=0}^{q-1} |\gamma_i| \big(\mu\|\vartheta^{n+i}\| + \nu|\vartheta^{n+i}|\big)\big(\|\vartheta^{n+q}\| + \eta_q\|\vartheta^{n+q-1}\|\big)\\
&\le \frac{1}{2}\sum_{i=0}^{q-1}|\gamma_i|\Big[(\mu+\varepsilon)\big(\|\vartheta^{n+q}\|^2+\eta_q\|\vartheta^{n+q-1}\|^2\big) + (1+\eta_q)\big(\mu\|\vartheta^{n+i}\|^2 + \frac{\nu}{\varepsilon}|\vartheta^{n+i}|^2\big)\Big]\\
&= \frac{\mu+\varepsilon}{2}\Big(\sum_{i=0}^{q-1}|\gamma_i|\Big)\big(\|\vartheta^{n+q}\|^2+\eta_q\|\vartheta^{n+q-1}\|^2\big)\\
&\quad + \frac{1+\eta_q}{2}\sum_{i=0}^{q-1}|\gamma_i|\big(\mu\|\vartheta^{n+i}\|^2 + \frac{\nu}{\varepsilon}|\vartheta^{n+i}|^2\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_{n+q} \le \frac{\mu+\varepsilon}{2}&(2^q-1)\big(\|\vartheta^{n+q}\|^2+\eta_q\|\vartheta^{n+q-1}\|^2\big)\\
&+ \frac{1+\eta_q}{2}\sum_{i=0}^{q-1}|\gamma_i|\big(\mu\|\vartheta^{n+i}\|^2 + \frac{\nu}{\varepsilon}|\vartheta^{n+i}|^2\big).
\end{aligned} \tag{4.10}$$

From (4.8), (4.9) and (4.10), we obtain

$$\begin{aligned}
&|\Theta^{n+q}|_G^2 - |\Theta^{n+q-1}|_G^2 + k\big(1-\frac{\lambda}{2}\eta_q\big)\|\vartheta^{n+q}\|^2 - k\frac{\lambda}{2}\eta_q\|\vartheta^{n+q-1}\|^2\\
&- k\tilde c\Big(1+\frac{1}{2}\eta_q\Big)|\vartheta^{n+q}|^2 - k\frac{\tilde c}{2}\eta_q|\vartheta^{n+q-1}|^2\\
&\le \frac{\mu+\varepsilon}{2}(2^q-1)k\big(\|\vartheta^{n+q}\|^2+\eta_q\|\vartheta^{n+q-1}\|^2\big)\\
&+ \frac{1+\eta_q}{2}k\sum_{i=0}^{q-1}|\gamma_i|\big(\mu\|\vartheta^{n+i}\|^2 + \frac{\nu}{\varepsilon}|\vartheta^{n+i}|^2\big).
\end{aligned} \tag{4.11}$$

Now, in view of the stability condition (4.2), we can choose μ and ε small enough, such that

$$1-\lambda\eta_q-(2^q-1)(1+\eta_q)(\mu+\varepsilon)\ge\rho>0.$$

Then, from (4.11) we get

$$\begin{aligned}
&|\Theta^{n+q}|_G^2 - |\Theta^{n+q-1}|_G^2 + \rho k\|\vartheta^{n+q}\|^2 + \frac{1+\eta_q}{2}(2^q-1)(\mu+\varepsilon)k\|\vartheta^{n+q}\|^2\\
&+ \frac{\eta_q}{2}[\lambda+(2^q-1)(\mu+\varepsilon)]k\big(\|\vartheta^{n+q}\|^2 - \|\vartheta^{n+q-1}\|^2\big)\\
&\le \tilde c(1+\eta_q)k\big(|\vartheta^{n+q}|^2+|\vartheta^{n+q-1}|^2\big) + \frac{1+\eta_q}{2}k\sum_{i=0}^{q-1}|\gamma_i|\big(\mu\|\vartheta^{n+i}\|^2 + \frac{\nu}{\varepsilon}|\vartheta^{n+i}|^2\big).
\end{aligned} \tag{4.12}$$

Summing here from $n=0$ to $n=m-q$, and using the fact that $|\gamma_0|+\dots+|\gamma_{q-1}|=2^q-1$, we obtain

$$|\Theta^m|_G^2-|\Theta^{q-1}|_G^2+\rho k\sum_{\ell=q}^m\|\vartheta^\ell\|^2+\frac{1+\eta_q}{2}(2^q-1)(\mu+\varepsilon)k\sum_{\ell=q}^m\|\vartheta^\ell\|^2$$
$$+\frac{\eta_q}{2}[\lambda+(2^q-1)(\mu+\varepsilon)]k\big(\|\vartheta^m\|^2-\|\vartheta^{q-1}\|^2\big)\le 2\tilde c(1+\eta_q)k\sum_{\ell=q-1}^m|\vartheta^\ell|^2$$
$$+\frac{1+\eta_q}{2}(2^q-1)\mu k\sum_{\ell=q}^{m-1}\|\vartheta^\ell\|^2+ck\sum_{j=0}^{q-1}\|\vartheta^j\|^2+\frac{\nu}{\varepsilon}\frac{1+\eta_q}{2}(2^q-1)\mu k\sum_{\ell=0}^{m-1}|\vartheta^\ell|^2.$$

Now, the first term on the right-hand side is absorbed by the fourth term on the left-hand side, and we easily get

$$\begin{aligned}|\Theta^m|_G^2+\rho k\sum_{\ell=q}^m\|\vartheta^\ell\|^2&\le|\Theta^{q-1}|_G^2+ck\sum_{j=0}^{q-1}\|\vartheta^j\|^2\\&+2\tilde c(1+\eta_q)k\sum_{\ell=q-1}^m|\vartheta^\ell|^2+\frac{\nu}{\varepsilon}\frac{1+\eta_q}{2}(2^q-1)\mu k\sum_{\ell=0}^{m-1}|\vartheta^\ell|^2.\end{aligned}\tag{4.13}$$

If we now use the lower bound $|\Theta^m|_G^2\ge c_q|\vartheta^m|^2$, with c_q the smallest eigenvalue of the matrix G, as well as the fact that

$$|\Theta^{q-1}|_G^2\le\widehat C\big(|\vartheta^0|^2+\dots+|\vartheta^{q-1}|^2\big),$$

with $\widehat C$ he largest eigenvalue of G, and apply the discrete Gronwall inequality, we obtain the desired stability estimate (4.3), provided k is sufficiently small. □

Remark 4.1 (Relaxation of the stability condition (4.2) for the three- and five-step methods)**.** For the implicit–explicit three- and five-step BDF methods the sufficient stability condition (4.2) can be directly relaxed to

$$\hat\eta_q\lambda<1,\tag{4.14}$$

with $\hat\eta_3$ and $\hat\eta_5$ as in (1.9), using the multipliers of [5, Lemma 2.3] and [5, (4.8)–(4.9)], respectively. □

4.3. **Sufficient and necessary stability conditions.** The sufficient stability condition (4.2) for the local stability of the implicit–explicit BDF method (1.6) is void, and, in particular, optimal in the case of the implicit–explicit one- and two-step BDF methods, since $\eta_1=\eta_2=0$. This is due to the fact that the corresponding implicit methods are A-stable. In other words, in these two cases, condition (1.4) on the behavior of the absolute values of the eigenvalues of the operator $\mathcal P$ is not needed. It is noteworthy that, without a condition of the form (1.4) the domain V of the operator $\mathcal P$ is a subspace of H^d_{per}, and the norm $\|\cdot\|$ is stronger that the norm $\|\cdot\|_{H^d}, d=p/2$. The norms $\|\cdot\|$ and $\|\cdot\|_{H^d}, d=p/2$, are equivalent, if we replace (1.4) by a corresponding assumption on the behavior of the real parts of the eigenvalues of $\mathcal{P}$, namely

$$\operatorname{Re}\lambda_\ell \le c_2 + c_3|\ell|^p, \quad \text{for all } \ell \in \mathbb{Z}. \tag{4.15}$$

Sufficient stability conditions. The implicit–explicit three-, four-, and five-step BDF methods (1.6) are, according to (4.2) and (4.14), locally stable, provided

$$\lambda < \frac{1}{\hat\eta_3} = 13, \quad \lambda < \frac{1}{\eta_4} = 3.47463516, \quad \lambda < \frac{1}{\hat\eta_5} = 1.23497392, \tag{4.16}$$

respectively.

Necessary stability conditions. According to the von Neumann criterion, necessary stability conditions even for the stability of the implicit three-, four-, and five-step BDF methods for the linear parabolic equation $u_t + \mathcal{P}u = 0$ are $\lambda \le 1/\cos\vartheta_q, q = 3, 4, 5$, respectively, that is

$$\lambda \le 14.45087, \quad \lambda \le 3.4904014, \quad \lambda \le 1.62892979, \tag{4.17}$$

respectively. Indeed, otherwise, the eigenvalues of the "rotated" operator $\mathcal{P} = \mathrm{e}^{\mathrm{i}\varphi}A$, with A a positive definite self-adjoint operator and $\vartheta_q < |\varphi| \le \pi/2$, lie on the ray $z = \rho\mathrm{e}^{\mathrm{i}\varphi}, \rho > 0$, which is outside the stability sector S_{ϑ_q} of the implicit q-step BDF method, and the method cannot be unconditionally stable; cf. also [2, 5].

5. Error estimates

Combining local stability and consistency of the implicit–explicit BDF scheme (1.6), we derive here optimal order error estimates.

Theorem 5.1 (Optimal order error estimates)**.** *Let the solution u of the periodic initial value problem for equation* (1.1) *be sufficiently smooth, such that the consistency estimate* (3.3) *holds true, the constant λ in* (1.8) *be such that the sufficient stability condition* (4.2) *be satisfied, and the starting approximations $U^0,\dots,U^{q-1} \in V$ be such that*

$$|u^j - U^j|^2 + k\|u^j - U^j\|^2 \le c_1 k^{2q}, \quad j = 0,\dots,q-1. \tag{5.1}$$

Let $U^q,\dots,U^N \in V$ be recursively defined by (1.6)*, and $e^n := u^n - U^n, n = 0,\dots,N$. Then, there exists a constant C, independent of k and m, such that, for k sufficiently small,*

$$|e^m|^2 + k\sum_{\ell=0}^{m}\|e^\ell\|^2 \le C\Big\{\sum_{j=0}^{q-1}\big(|e^j|^2 + k\|e^j\|^2\big) + k\sum_{\ell=0}^{m-q}\|E^\ell\|_\star^2\Big\}, \tag{5.2}$$

$m = q-1,\dots,N$, and

$$\max_{0\le n\le N}|u(t^n) - U^n| \le Ck^q. \tag{5.3}$$

Proof. This proof proceeds along the lines of analogous proofs in, e.g., [6, 2, 5]; it is included here for the convenience of the reader. According to the consistency estimate (3.3) and our assumption (5.1) on the starting approximations, there exists a constant $C_\star$ such that the right-hand side of (5.2) can be estimated by $C_\star^2 k^{2q}$,

$$C\Big\{\sum_{j=0}^{q-1}\big(|e^j|^2+k\|e^j\|^2\big)+k\sum_{\ell=0}^{m-q}\|E^\ell\|_\star^2\Big\}\le C_\star^2k^{2q}. \tag{5.4}$$

Now, since (5.3) is a consequence of (5.2) and (5.4), it remains to prove (5.2). Subtracting (1.6) from (3.2), we obtain

$$\sum_{i=0}^{q}\alpha_i e^{n+i}+k\mathcal{P}e^{n+q}=k\sum_{i=0}^{q-1}\gamma_i\big[B(u^{n+i})-B(U^{n+i})\big]+kE^n. \tag{5.5}$$

If we take here the inner product with $e^{n+q}-\eta_q e^{n+q-1}$, proceed exactly as in the proof of Theorem 4.1, and assume for the time being that $U^j\in T_u, j=0,\dots,n+q-1$, we easily arrive at

$$c_q|e^{n+q}|^2+\frac12\rho k\sum_{\ell=q}^{n+q}\|e^\ell\|^2\le C\sum_{j=0}^{q-1}\big(|e^j|^2+k\|e^j\|^2\big)+k\sum_{\ell=0}^{n}\operatorname{Re}(E^\ell,e^{\ell+q}-\eta_q e^{\ell+q-1});$$

cf. (4.3). Now, bounding

$$\operatorname{Re}(E^\ell,e^{\ell+q}-\eta_q e^{\ell+q-1})\le\frac{\rho}{4(1+\eta_q)}\|e^{\ell+q}\|^2+\frac{\rho\eta_q}{4(1+\eta_q)}\|e^{\ell+q-1}\|^2+2\frac{1+\eta_q}{\rho}\|E^\ell\|_\star^2$$

and summing up, we obtain

$$c_q|e^{n+q}|^2+\frac{\rho k}{4}\sum_{\ell=q}^{n+q}\|e^\ell\|^2\le C\sum_{j=0}^{q-1}\big(|e^j|^2+k\|e^j\|^2\big)+2\frac{1+\eta_q}{\rho}k\sum_{\ell=0}^{n}\|E^\ell\|_\star^2,$$

and infer that (5.2) holds true for $m=n+q$.

Clearly, the estimate (5.2) is valid for $m=q-1$. Assume that it holds for $m=q-1,\dots,n+q-1, 0\le n\le N-q$. Then, according to (5.4) and the induction hypothesis, we have, for k small enough,

$$\max_{0\le j\le n+q-1}\|e^j\|\le C_\star k^{q-1/2}\le 1, \tag{5.6}$$

and thus $U^j\in T_u, j=0,\dots,n+q-1$. Therefore, as we proved above, (5.2) holds indeed for $m=n+q$ as well, and the proof is complete. □

6. Extensions: Fully discrete methods, time-dependent operators

In this section we extend our analysis to fully discrete schemes, with spectral methods used for the discretization in space, and comment on the extension to the case of time dependent operators $\mathcal{P}$.

6.1. **Time-dependent operators and norms.** Our analysis can be easily extended to the more general case of time-dependent operators $\mathcal{P}$. In that case, it is advantageous to let the quantity λ in (1.8) depend on t as well.

More precisely, with $\lambda_\ell(t)$ the eigenvalues of $\mathcal{P}(t)$ and $\tilde\lambda_\ell(t) = \lambda_\ell(t) + \tilde c$ the eigenvalues of the shifted operators $\tilde{\mathcal{P}}(t) := \mathcal{P}(t)+\tilde c I$, we can introduce in $V := H^d_{\text{per}}$ and $V' = H^{-d}_{\text{per}}$ the time-dependent norms $\|\cdot\|_t$ and $\|\cdot\|_{\star,t}$, respectively, by

$$\|v\|_t := \Big(\sum_{\ell\in\mathbb{Z}} \operatorname{Re}\tilde\lambda_\ell(t)\,|\hat v_\ell|^2\Big)^{1/2}, \quad \|v\|_{\star,t} := \Big(\sum_{\ell\in\mathbb{Z}} \big(\operatorname{Re}\tilde\lambda_\ell(t)\big)^{-1}|\hat v_\ell|^2\Big)^{1/2}. \tag{6.1}$$

Then, we have

$$\operatorname{Re}(\tilde{\mathcal{P}}(t)v,v) = \|v\|_t^2, \quad \text{for all } v\in V = H^d_{\text{per}}, \tag{6.2}$$

and

$$\|\tilde{\mathcal{P}}(t)v\|_{\star,t} \le \lambda(t)\|v\|_t, \quad \text{for all } v\in V = H^d_{\text{per}}, \tag{6.3}$$

with the bound $\lambda(t)$ given by

$$\lambda(t) := \sup_{\ell\in\mathbb{Z}} \frac{|\tilde\lambda_\ell(t)|}{\operatorname{Re}\tilde\lambda_\ell(t)}; \tag{6.4}$$

cf. (1.8).

Besides the conditions $\hat\eta_q\lambda(t) < 1, t\in[0,T]$, on $\lambda(t)$, cf. (4.2) and (4.14), in this case we need also to relate the time-dependent norms $\|\cdot\|_t$, for different values of t. To this end, it suffices to impose a mild Lipschitz condition on the operators $\mathcal{P}(t)$ with respect to t, namely

$$\big\|\big(\mathcal{P}(t)-\mathcal{P}(\tilde t)\big)v\big\|_{H^{-d}} \le L|t-\tilde t|\,\|v\|_{H^d}, \tag{6.5}$$

for $t,\tilde t\in[0,T]$ and $v\in V = H^d_{\text{per}}$; we refer to [2] for details.

6.2. **Fully discrete methods.** Let $M\in\mathbb{N}$ and $S_M := \operatorname{span}\{\varphi_{-M},\dots,\varphi_M\}$ with $\varphi_\ell(x) := \mathrm{e}^{\mathrm{i}\omega\ell x}$. Let $P_M : V'\to S_M$ denote the orthogonal L^2-projection operator onto S_M, i.e., $(v-P_Mv,\chi)=0, \chi\in S_M$. If we expand $v\in L^2_{\text{per}}$ in a Fourier series,

$$v = \sum_{\ell\in\mathbb{Z}} \hat v_\ell\,\varphi_\ell,$$

then P_Mv corresponds to the partial sum

$$P_Mv = \sum_{\ell=-M}^{M} \hat v_\ell\,\varphi_\ell.$$

This projection has the following approximation property: *There exists a constant* c, *independent of* v *and* M, *such that, for* $v\in H^m_{\text{per}}$,

$$\|v-P_Mv\|_{H^\ell} \le cM^{\ell-m}\|v^{(m)}\|_{L^2}, \quad \ell = 0,\dots,m; \tag{6.6}$$

(cf. [11, (9.1.10)]). Clearly, P_M commutes with differentiation in time as well as with $\mathcal{P}, P_M\mathcal{P} = \mathcal{P}P_M$; see (1.2). Furthermore, we define the discrete operator $B_M : H^2_{\text{per}}\to S_M, B_M := P_MB$.

In the *semidiscrete* problem corresponding to the periodic initial value problem for (1.1), we seek a function $u_M, u_M(\cdot,t)\in S_M$, satisfying

$$\begin{cases} \partial_t u_M(\cdot,t)+\mathcal{P}u_M(\cdot,t)=B_M(u_M(\cdot,t)), & 0<t<T,\\ u_M(\cdot,0)=u_M^0, \end{cases} \tag{6.7}$$

with $u_M^0\in S_M$ a given approximation to the initial value u^0.

We recursively define a sequence of approximations $U^\ell\in S_M$ to $u(\cdot,t^\ell)$ by

$$\sum_{i=0}^{q}\alpha_i U^{n+i}+k\mathcal{P}U^{n+q}=k\sum_{i=0}^{q-1}\gamma_i B_M(U^{n+i}), \tag{6.8}$$

$n=0,\dots,N-q$; cf. (1.6). Let $W(\cdot,t)\in S_M$ denote the L^2-projection of $u(\cdot,t)$ in $S_M, W(\cdot,t)=P_M u(\cdot,t), t\in[0,T]$.

Let $E_M(t)\in S_M$ denote the *consistency error* of the semidiscrete equation (6.7) for W,

$$E_M(t):=W_t(\cdot,t)+\mathcal{P}W(\cdot,t)-B_M(W(\cdot,t)),\quad t\in[0,T]. \tag{6.9}$$

Obviously

$$E_M(t)=W_t(\cdot,t)+P_M\mathcal{P}u(\cdot,t)-P_MB(W(\cdot,t)),$$

whence, in view of (1.1),

$$E_M(t)=P_M\big[B(u(\cdot,t))-B(W(\cdot,t))\big].$$

In the case $p\ge 2$, the Lipschitz condition (2.22) and the approximation property (6.6) yield, under obvious regularity assumptions, the following optimal order estimate for the consistency error E_M,

$$\max_{0\le t\le T}\|E_M(t)\|_\star\le C(u)\,M^{-m}. \tag{6.10}$$

In the case $1<p<2$, we use instead the Lipschitz condition (2.24) and the approximation property (6.6), and obtain, under obvious regularity assumptions, the following estimate for the consistency error E_M,

$$\max_{0\le t\le T}\|E_M(t)\|_\star\le C(u)\,M^{-m+1-\frac{p}{2}}. \tag{6.11}$$

We can now derive an error estimate for the fully discrete approximations:

Theorem 6.1 (Error estimates for fully discrete methods)**.** *Assume that we are given starting approximations $U^0, U^1,\dots,U^{q-1}\in S_M$ to $u(\cdot,t^0),\dots,u(\cdot,t^{q-1})$ such that*

$$|u(\cdot,t^j)-U^j|^2+k\|u(\cdot,t^j)-U^j\|^2\le\begin{cases} c_1(k^q+M^{-m}) & \textit{for}\ \ p\ge 2,\\ c_1(k^q+M^{-m+1-\frac{p}{2}}) & \textit{for}\ \ 1<p<2, \end{cases} \tag{6.12}$$

$j=0,\dots,q-1$. *Let $U^n\in S_M, n=q,\dots,N$, be recursively defined by* (6.8). *Then, if the solution u of the periodic initial value problem for* (1.1) *is sufficiently smooth,* *there exists a constant C, independent of k and M, such that, for k sufficiently small and M sufficiently large,*

$$\max_{0\le n\le N}|u(\cdot,t^n)-U^n|\le\begin{cases}C(k^q+M^{-m}) & \text{for } p\ge 2,\\ C(k^q+M^{-m+1-\frac{p}{2}}) & \text{for } 1<p<2.\end{cases}\tag{6.13}$$

Proof. Let $\tilde W^j:=W(\cdot,t^j), j=0,\dots,q-1$, and define $\tilde W^n\in S_M, n=q,\dots,N$, by applying the time stepping scheme to equation (6.9), i.e., by

$$\sum_{i=0}^{q}\alpha_i\tilde W^{n+i}+k\mathcal{P}\tilde W^{n+q}=k\sum_{i=0}^{q-1}\gamma_i\big[B_M(\tilde W^{n+i})+E_M(t^{n+i})\big].\tag{6.14}$$

Then, it follows easily from Theorem 5.1 that

$$\max_{0\le n\le N}|W(\cdot,t^n)-\tilde W^n|\le Ck^q.\tag{6.15}$$

In view of (6.6) and (6.15), it remains to estimate $\vartheta^n:=\tilde W^n-U^n$. Subtracting (6.8) from (6.14), we obtain

$$\sum_{i=0}^{q}\alpha_i\vartheta^{n+i}+k\mathcal{P}\vartheta^{n+q}=k\sum_{i=0}^{q-1}\gamma_i\big[B_M(\tilde W^{n+i})-B_M(U^{n+i})\big]+k\sum_{i=0}^{q-1}\gamma_iE_M(t^{n+i});$$

compare with (5.5). Therefore, in analogy to (5.2), there exists a constant C, independent of k and m, such that, for k sufficiently small,

$$|\vartheta^m|^2+k\sum_{\ell=0}^{m}\|\vartheta^\ell\|^2\le C\Big\{\sum_{j=0}^{q-1}\big(|\vartheta^j|^2+k\|\vartheta^j\|^2\big)+k\sum_{\ell=0}^{m-q}\|E_M(t^\ell)\|_\star^2\Big\},\tag{6.16}$$

$m=q-1,\dots,N,$

From this estimate, (6.12) and (6.10), for $p\ge 2$, or (6.11), for $1<p<2$, we easily infer, for k sufficiently small and M sufficiently large, that

$$\max_{0\le n\le N}|\tilde W^n-U^n|\le\begin{cases}C(k^q+M^{-m}) & \text{for } p\ge 2,\\ C(k^q+M^{-m+1-\frac{p}{2}}) & \text{for } 1<p<2,\end{cases}\tag{6.17}$$

respectively. From (6.17), (6.15) and (6.6) the desired estimate (6.13) follows and the proof is complete. □

Appendix

Lemma A.1. *If $s\ge 0$, then there exists a positive constant c, such that*

$$\|uv\|_{H^s}\le c\big(\|u\|_{H^s}\|v\|_{L^\infty}+\|u\|_{L^\infty}\|v\|_{H^s}\big),$$

for every $u,v\in H^s_{\text{per}}\cap L^\infty$.

Proof. We first establish the auxiliary inequality

$$\big(1+(x+y)^2\big)^{s/2}\le c_s\big((1+x^2)^{s/2}+(1+y^2)^{s/2}\big),\quad x,y\in\mathbb{R},\ s\ge 0,\tag{A.1}$$

where $c_s = \max\{2^{s-1}, 1\}$. For $s \geq 1$, the convexity of $f(x) = (1+x^2)^{s/2}$ yields that

$$\Big(1+\Big(\frac{x+y}{2}\Big)^2\Big)^{s/2} = f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\big(f(x)+f(y)\big) = \frac{1}{2}\big((1+x^2)^{s/2}+(1+y^2)^{s/2}\big)$$

and hence

$$\big(1+(x+y)^2\big)^{s/2} \leq \big(4+(x+y)^2\big)^{s/2} \leq 2^{s-1}\left((1+x^2)^{s/2}+(1+y^2)^{s/2}\right).$$

For $s\in[0,1]$, and nonnegative x and y, we first note that

$$(x+y)^s \leq x^s + y^s;$$

this is trivial for $s = 0$ and $s = 1$, and, for $s \in (0, 1)$, follows immediately from the fact that the function $\varphi(t) := 1 + t^s - (1 + t)^s$ vanishes at $t = 0$ and is increasing for positive t. Thus

$$1+(x+y)^2 \leq 2^{2/s} + (x^s+y^s)^{2/s} = \|X+Y\|_{2/s}^{2/s}$$

with $X := (1, x^s)^T$ and $Y := (1, y^s)^T$. Now, $\|X\|_{2/s} = (1 + x^2)^{s/2}$, and from the estimate above we obtain

$$1+(x+y)^2 \leq \big(\|X\|_{2/s}+\|Y\|_{2/s}\big)^{2/s} = \big((1+x^2)^{s/2}+(1+y^2)^{s/2}\big)^{2/s},$$

which immediately yields (A.1). Next

$$\begin{aligned}(1+\omega^2k^2)^{s/2}\big|(\widehat{uv})_k\big| \;&=\; (1+\omega^2k^2)^{s/2}\Big|\sum_{\ell\in\mathbb{Z}}\hat{u}_{k-\ell}\hat{v}_\ell\Big|\\ &\leq c_s\sum_{\ell\in\mathbb{Z}}\big((1+\omega^2(k-\ell)^2)^{s/2}+(1+\omega^2\ell^2)^{s/2}\big)|\hat{u}_{k-\ell}||\hat{v}_\ell|\\ &= c_s\sum_{\ell\in\mathbb{Z}}(1+\omega^2(k-\ell)^2)^{s/2}|\hat{u}_{k-\ell}||\hat{v}_\ell| + c_s\sum_{\ell\in\mathbb{Z}}(1+\omega^2\ell^2)^{s/2}|\hat{u}_{k-\ell}||\hat{v}_\ell|.\end{aligned}$$

Therefore

$$\begin{aligned}(1+\omega^2k^2)^s\big|(\widehat{uv})_k\big|^2 \leq 2c_s^2\bigg(&\Big(\sum_{\ell\in\mathbb{Z}}(1+\omega^2(k-\ell)^2)^{s/2}|\hat{u}_{k-\ell}||\hat{v}_\ell|\Big)^2\\ &+\Big(\sum_{\ell\in\mathbb{Z}}(1+\omega^2\ell^2)^{s/2}|\hat{u}_{k-\ell}||\hat{v}_\ell|\Big)^2\bigg).\end{aligned} \tag{A.2}$$

If we set

$$f(x) = \sum_{k\in\mathbb{Z}}|\hat{u}_k|\mathrm{e}^{\mathrm{i}kx}, \quad F(x) = \sum_{k\in\mathbb{Z}}(1+\omega^2k^2)^{s/2}|\hat{u}_k|\mathrm{e}^{\mathrm{i}kx},$$

$$g(x) = \sum_{k\in\mathbb{Z}}|\hat{v}_k|\mathrm{e}^{\mathrm{i}kx}, \quad G(x) = \sum_{k\in\mathbb{Z}}(1+\omega^2k^2)^{s/2}|\hat{u}_k|\mathrm{e}^{\mathrm{i}kx},$$

then

$$\|f\|_{\mathrm{L}^2} = \|u\|_{\mathrm{L}^2},\ \|g\|_{\mathrm{L}^2} = \|v\|_{\mathrm{L}^2},\ \|F\|_{\mathrm{L}^2} = \|u\|_{H^s},\ \text{and}\ \|G\|_{\mathrm{L}^2} = \|v\|_{H^s},$$

and (A.2) provides that

$$(1+\omega^2k^2)^s\big|(\widehat{uv})_k\big|^2 \leq 2c_s^2\Big(\big|(\widehat{fG})_k\big|^2+\big|(\widehat{Fg})_k\big|^2\Big),$$

and hence

$$\begin{aligned}\|uv\|_{H^s}^2 &= \sum_{k\in\mathbb{Z}}(1+\omega^2k^2)^s\big|(\widehat{uv})_k\big|^2 \le 2c_s^2\sum_{k\in\mathbb{Z}}\Big(\big|(\widehat{fG})_k\big|^2+\big|(\widehat{Fg})_k\big|^2\Big)\\ &= 2c_s^2\Big(\|fG\|_{\mathrm{L}^2}^2+\|Fg\|_{L^2}^2\Big) \le 2c_s^2\big(\|f\|_{\mathrm{L}^\infty}^2\|G\|_{L^2}^2+\|F\|_{L^2}^2\|g\|_{\mathrm{L}^\infty}^2\big)\\ &= 2c_s^2\big(\|u\|_{\mathrm{L}^\infty}^2\|v\|_{H^s}^2+\|u\|_{H^s}^2\|v\|_{\mathrm{L}^\infty}^2\big);\end{aligned}$$

thus,

$$\|uv\|_{H^s} \le \sqrt{2}\,c_s\,\big(\|u\|_{\mathrm{L}^\infty}\|v\|_{H^s}+\|u\|_{H^s}\|v\|_{\mathrm{L}^\infty}\big),$$

which concludes the proof. □

Corollary A.1. *If $s>1/2$, then there exists a positive constant C_s such that, for every $u,v\in H^s_{\mathrm{per}}$,*

$$\|uv\|_{H^s} \;\le\; C_s\,\|u\|_{H^s}\|v\|_{H^s}. \tag{A.3}$$

Proof. If $s>1/2$, then $H^s_{\mathrm{per}}\subset \mathrm{L}^\infty$ and for every $u\in H^s_{\mathrm{per}}$,

$$\|u\|_{\mathrm{L}^\infty}\le c\,\|u\|_{H^s},$$

with a positive c depending on s but independent of u. Inequality (A.3) is now a consequence of Lemma A.1. □

We are now ready to prove the following result, which we used in subsection 2.3:

Corollary A.2. *For every $\beta>\beta'>1$, and $\varepsilon>0$, there exists a positive constant $\hat c_\varepsilon$, such that*

$$\|uv\|_{H^{1-\beta/2}} \;\le\; \|u\|_{H^{\beta'/2}}\big(\hat c_\varepsilon\|v\|_{\mathrm{L}^2}+\varepsilon\,\|v\|_{H^{\beta/2}}\big), \tag{A.4}$$

for all $u,v\in H^{\beta/2}_{\mathrm{per}}$.

Proof. As $1-\frac{\beta}{2}<\frac12<\frac{\beta'}{2}<\frac{\beta}{2}$, then due to (A.3)

$$\|uv\|_{H^{1-\beta/2}} \le \|uv\|_{H^{\beta'/2}} \le C_{\beta'/2}\,\|u\|_{H^{\beta'/2}}\|v\|_{H^{\beta'/2}}. \tag{A.5}$$

Let $\hat c_\varepsilon>0$ be such that

$$\hat c_\varepsilon^2=\max\big\{x^{\beta'}-\varepsilon^2x^\beta : x\ge 1\big\}$$

i.e., $x^{\beta'}\le \hat c_\varepsilon^2+\varepsilon^2x^\beta$, for all $x\ge1$. Then

$$\begin{aligned}\|v\|_{H^{\beta'/2}}^2 &= \sum_{k\in\mathbb{Z}}(1+\omega^2k^2)^{\beta'}|\widehat{v}_k|^2 \le \sum_{k\in\mathbb{Z}}\big(\hat c_\varepsilon^2+\varepsilon^2(1+\omega^2k^2)^\beta\big)|\widehat{v}_k|^2\\ &= \hat c_\varepsilon^2\|v\|_{\mathrm{L}^2}^2+\varepsilon^2\|v\|_{H^\beta}^2 \le \Big(\hat c_\varepsilon\|v\|_{\mathrm{L}^2}+\varepsilon\|v\|_{H^\beta}\Big)^2,\end{aligned}$$

and thus

$$\|v\|_{H^{\beta'/2}}\le \hat c_\varepsilon\|v\|_{\mathrm{L}^2}+\varepsilon\|v\|_{H^\beta}. \tag{A.6}$$

Now (A.4) is a consequence of (A.5) and (A.6). □

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