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A posteriori error estimates for Radau IIA methods via maximal parabolic regularity

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Abstract We consider the discretization of differential equations satisfying the maximal parabolic L^p-regularity property in Banach spaces by Radau IIA methods. We establish a posteriori error estimators via the maximal parabolic regularity of the differential equation. To complete the picture, we utilize the maximal parabolic regularity of the numerical methods to prove that the estimators are of optimal order.

Keywords A posteriori error estimates · maximal parabolic regularity · discrete maximal parabolic regularity · Radau IIA methods

Mathematics Subject Classification (2000) 65M15 · 65M12

1 Introduction

We consider the discretization of differential equations satisfying the maximal parabolic L^p-regularity property in Banach spaces by Radau IIA methods. We utilize the collocation approximation in combination with the maximal regularity of the differential equation to establish a posteriori error estimates. To complete the picture, using the maximal regularity of the methods, recently established by Kovács, Li, and Lubich, [11], and pointwise formulations of the numerical methods, we prove that the a posteriori estimators are of asymptotic optimal order of convergence. We are not aware of any previous a posteriori error analysis via maximal parabolic L^p-regularity.

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1.1 An initial value problem

We consider an initial value problem for a linear parabolic equation,

$$\begin{cases} u'(t) + Au(t) = f(t), & 0 < t < T, \\ u(0) = 0, \end{cases} \tag{1.1}$$

in a Banach space X. Our structural assumption is that the operator A is the generator of an analytic semigroup on X having *maximal L^p-regularity*, i.e., the solution u of (1.1) satisfies the stability estimate

$$\|u'\|_{L^p((0,T);X)} + \|Au\|_{L^p((0,T);X)} \leqslant c_{p,X}\|f\|_{L^p((0,T);X)} \quad \forall f \in L^p((0,T);X) \tag{1.2}$$

for some, or, as it turns out, for all $p \in (1,\infty)$, with a constant $c_{p,X}$ independent of T, depending only on p and X. In other words, u' and Au are well defined and have the same regularity as their sum $u' + Au$, that is, the given forcing term f.

It is known that every generator of a bounded analytic semigroup on a Hilbert space has maximal L^p-regularity and that a Banach space with an unconditional basis satisfying this property is a Hilbert space; see [8] and [10], respectively. We refer to [21] for a fundamental characterization of the maximal L^p-regularity property on $X = L^s(\Omega)$, with arbitrary $1 < s < \infty$ and Ω a domain in $\mathbb{R}^d$, and, more generally, on unconditional martingale differences (UMD) spaces, and to the lecture notes [13] for an excellent account of the theory. Coercive elliptic differential operators on $L^s(\Omega), 1 < s < \infty$, with general boundary conditions possess the maximal L^p-regularity property; see [13] and references therein. Throughout the paper, X is a UMD space.

Notice that an initial value problem with not necessarily vanishing initial value $v_0 \in \mathscr{D}(A) := \{v \in X : Av \in X\}$,

$$\begin{cases} v'(t) + Av(t) = g(t), & 0 < t < T, \\ v(0) = v_0, \end{cases} \tag{1.3}$$

can be reduced to the form (1.1) with $u := v - v_0$ and $f := g - Av_0$.

1.2 The numerical methods

Let $N \in \mathbb{N}, k = T/N$ be the constant time step, $t_n := nk, n = 0, \dots, N$, be a uniform partition of the time interval $[0,T]$, and $J_n := (t_n, t_{n+1}]$. For $q \in \mathbb{N}$, with $0 < c_1 < \cdots < c_q = 1$ the Radau nodes in the interval $[0,1]$, let $t_{ni} := t_n + c_ik, i = 1, \dots, q$, be the intermediate nodes; we shall also use the notation $t_{n0} := t_n$.

The q-stage Radau IIA method is specified by the coefficients

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)\,\mathrm{d}\tau, \quad b_i = \int_0^1 \ell_i(\tau)\,\mathrm{d}\tau \ (= a_{qi}), \quad i,j = 1,\dots,q; \tag{1.4}$$

here, $\ell_1, \dots, \ell_q \in \mathbb{P}_{q-1}$ are the Lagrange polynomials for the Radau nodes $c_1, \dots, c_q$, $\ell_i(c_j) = \delta_{ij}$.

Relations (1.4) reflect the fact that the Radau IIA method is of collocation type, i.e., its *stage order* is q. It is well known that the *order* p of the q-stage Radau IIA method is $2q-1, p = 2q-1$, the weights $b_1, \dots, b_q$ are positive, and the $q \times q$

symmetric matrix M with entries $m_{ij} := b_i a_{ij} + b_j a_{ji} - b_i b_j, i, j = 1, \dots, q$, is positive semidefinite. In particular, the Radau IIA methods are *algebraically stable.* These methods are also *strongly* A-stable; more precisely, the stability function r,

$$r(z) := 1 + zb^\top (I - z\mathcal{O})^{-1}\mathbb{1} \quad \text{with} \quad \mathbb{1} := (1, \dots, 1)^\top \in \mathbb{R}^q,$$

with the invertible coefficient matrix $\mathcal{O} = (a_{ij})_{i,j=1,\dots,q} \in \mathbb{R}^{q,q}$, of the q-stage Radau IIA method vanishes at infinity, $r(\infty) = 1 - b^T \mathcal{O}^{-1}\mathbb{1} = 0$. The first member of this family, for $q = 1$, is the implicit Euler method.

With starting value $U_0 = 0$, we consider the discretization of the initial value problem (1.1) by the q-stage Radau IIA method: we recursively define approximations $U_\ell \in \mathscr{D}(A)$ to the nodal values $u(t_\ell)$, as well as internal approximations $U_{\ell i} \in \mathscr{D}(A)$ to the intermediate values $u(t_{\ell i})$, by

$$\begin{cases} U_{ni} = U_n - k\displaystyle\sum_{j=1}^{q} a_{ij}\big(AU_{nj} - f(t_{nj})\big), \quad i = 1, \dots, q, \\ U_{n+1} = U_n - k\displaystyle\sum_{i=1}^{q} b_i\big(AU_{ni} - f(t_{ni})\big), \end{cases} \tag{1.5}$$

$n = 0, \dots, N-1$. Notice that, as a consequence of the fact that $a_{qi} = b_i, i = 1, \dots, q$, we have $U_{n+1} = U_{nq}$. Here, we assumed that $f(t) \in X$ for $t \in (0, T]$.

Notice also that adding v_0 to both sides of (1.5) and replacing f by $g - Av_0$, we see that the Radau IIA approximations V_{ni} and V_n for the initial value problem (1.3) are $V_{ni} = U_{ni} + v_0$ and $V_n = U_n + v_0$, which is the discrete analogue of $v = u + v_0$. Therefore, without loss of generality, we may consider the discretization of (1.1).

For $s \in \mathbb{N}_0$, we denote by $\mathbb{P}(s)$ the space of polynomials of degree at most s with coefficients in $\mathscr{D}(A)$, i.e., the elements g of $\mathbb{P}(s)$ are of the form

$$g(t) = \sum_{j=0}^{s} t^j w_j, \quad w_j \in \mathscr{D}(A), \quad j = 0, \dots, s.$$

With this notation, let $\mathcal{V}_k^{\mathrm{c}}(s)$ and $\mathcal{V}_k^{\mathrm{d}}(s)$ be the spaces of continuous and possibly discontinuous, respectively, piecewise elements of $\mathbb{P}(s)$,

$$\mathcal{V}_k^{\mathrm{c}}(s) := \{v \in C\big([0, T]; \mathscr{D}(A)\big) : v|_{J_n} \in \mathbb{P}(s),\ n = 0, \dots, N-1\},$$
$$\mathcal{V}_k^{\mathrm{d}}(s) := \{v : [0, T] \to \mathscr{D}(A),\ v|_{J_n} \in \mathbb{P}(s),\ n = 0, \dots, N-1\}.$$

The spaces $\mathcal{X}_k^{\mathrm{c}}(s)$ and $\mathcal{X}_k^{\mathrm{d}}(s)$ are defined analogously, with coefficients $w_j \in X$.

Since its stage order is at least q, it is known that the q-stage Radau IIA method is equivalent to the collocation method with the Radau nodes $c_1, \dots, c_q$ in the following sense: Seek a function $\widehat{U} \in \mathcal{V}_k^{\mathrm{c}}(q)$ satisfying the initial condition $\widehat{U}(0) = 0$ as well as the collocation conditions

$$\widehat{U}'(t_{ni}) + A\widehat{U}(t_{ni}) = f(t_{ni}), \quad i = 1, \dots, q, \quad n = 0, \dots, N-1. \tag{1.6}$$

Then, $\widehat{U}(t_{ni}) = U_{ni}, i = 1, \dots, q, n = 0, \dots, N-1$; in particular, $\widehat{U}(t_{nq}) = U_{n+1}$. Thus, [3] and [4], if we let $I_{q-1} : C\big([0, T]; X\big) \to \mathcal{X}_k^{\mathrm{d}}(q-1)$ denote the interpolation operator at the collocation nodes $t_{ni}, i = 1,\dots,q, n = 0,\dots,N-1$, and use the fact that $\widehat{U}' \in \mathcal{V}_k^{\mathrm{d}}(q-1)$, we can write (1.6) in *pointwise form* as

$$\widehat{U}'(t) + I_{q-1}A\widehat{U}(t) = I_{q-1}f(t), \quad t \in (t_n, t_{n+1}], \quad n = 0,\dots,N-1. \tag{1.7}$$

The interpolants $U := I_{q-1}\widehat{U}$ and $I_{q-1}f$ are elements of $\mathcal{V}_k^{\mathrm{d}}(q-1)$ and $\mathcal{X}_k^{\mathrm{d}}(q-1)$, respectively, and thus, in general, discontinuous at the nodes $t_0,\dots,t_{N-1}$. The pointwise form (1.7) of the numerical method is crucial; it will allow us to prove optimality of the a posteriori error estimator.

To give a Galerkin in time formulation of the method, let X' be the dual of X and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X'. Then, the variational formulation of the Radau IIA method (1.5), cf. the pointwise form (1.7), is: seek $\widehat{\mathcal{U}} \in \mathcal{V}_k^{\mathrm{c}}(q)$ such that

$$\int_{J_n} \big(\langle\widehat{U}', v\rangle + \langle AU, v\rangle\big)\,\mathrm{d}t = \int_{J_n} \langle I_{q-1}f, v\rangle\,\mathrm{d}t \quad \forall v \in \mathbb{P}_{X'}(q-1), \tag{1.8}$$

$n = 0,\dots,N-1$, with $U = I_{q-1}\widehat{U}$; the elements g of the test space $\mathbb{P}_{X'}(q-1)$ are polynomials of degree at most $q-1$ in time with coefficients in X', $g(t) = w_0 + tw_1 + \dots + t^{q-1}w_{q-1}, w_j \in X'$. The formulation (1.8) is one of the alternative ways to connect Radau IIA methods and discontinuous or perturbed continuous Galerkin in time discetizations; see [19] and [4].

1.3 Main results

We denote by $R \in L^p((0,T);X)$ the *residual* of the approximate solution $\widehat{U}$ of the q-stage Radau IIA method,

$$R(t) := \widehat{U}'(t) + A\widehat{U}(t) - f(t), \quad t \in (t_n, t_{n+1}], \quad n = 0,\dots,N-1, \tag{1.9}$$

i.e., the amount by which $\widehat{U}$ misses being exact solution of the differential equation in (1.1). Then, due to the fact that the evolution operator is applicable to $\widehat{U}$, the error $e := u - \widehat{U}$ satisfies the *error equation*

$$e'(t) + Ae(t) = -R(t), \quad t \in (t_n, t_{n+1}], \quad n = 0,\dots,N-1. \tag{1.10}$$

Now, the maximal L^p-regularity of the operator A and the triangle inequality, respectively, applied to the error equation (1.10) yield the upper and lower a posteriori error bounds

$$|R\|_{L^p((0,t);X)} \leqslant \|e'\|_{L^p((0,t);X)} + \|Ae\|_{L^p((0,t);X)} \leqslant c_{p,X}\|R\|_{L^p((0,t);X)}, \tag{1.11}$$

for all $0 < t \leqslant T$, for any $p \in (1,\infty)$, with a constant $c_{p,X}$ depending only on p and X; see (1.2). Notice that the residual R is a computable quantity, depending only on the numerical solution $\widehat{U}$ and the given forcing term f.

Our main task in the following is to establish that (1.11) is sharp is the sense that $\|R\|_{L^p((0,t);X)}$ has the same asymptotic order of convergence as the optimal convergence rate of the error in the discrete maximal regularity framework, [11, 12]. To achieve this goal, we study in detail the behavior of $\|R\|_{L^p((0,t);X)}$ using the following ingredients: (i) an explicit representation of R in terms of $\widehat{U}, U$, and the interpolation error involving the right-hand side f, (ii) a detailed consistency analysis involving the exact solution u, (iii) the discrete maximal parabolic regularity property of the Radau IIA methods, recently established in [11]. Our main result is stated in Theorem 3.1. Furthermore, we extend the a posteriori analysis to the case of nonautonomous equations assuming that the operator $A = A(t)$ satisfies a Lipschitz condition with respect to t; see (4.3).

One of the reasons why maximal parabolic regularity is an interesting framework for stability, is that it allows, in combination with the variation of constants formula and fixed point arguments, efficient treatment of a large class of nonlinear evolution equations, [17,13]. Our present work is a first necessary step towards treating time discretizations of nonlinear parabolic equations in the maximal L^p-regularity framework, where a posteriori error control is particularly relevant. We refer to [6] for a posteriori error analysis for low order time discretizations for nonlinear equations via $C^{1,\alpha}$ maximal parabolic regularity.

The main results of [11] as well as of this article are transferred to discontinuous Galerkin (dG) time-discrete methods via a suitable interpretation of dG methods as modified Radau IIA schemes in [1].

Although we consider time-discrete schemes only, the extension of our approach to both space and time discretizations of parabolic equations is important; see Remark 3.3. Notice that partial results can be obtained by applying the time-discrete analysis of the present work to space discrete evolution equations of the form (1.1) with space discrete operators A_h resulting from conforming finite element discretizations of coercive, selfadjoint, second-order elliptic operators A. Such space discrete operators, on quasi-uniform triangulations of a bounded domain Ω, are known to inherit the maximal L^p-regularity property of A on $L^s(\Omega), 1 < s < \infty$; see [14,15] and references therein.

An outline of the paper is as follows. For the reader's convenience we present the analysis for the first member of the Radau IIA family, namely the implicit Euler method, separately in Section 2, and treat high-order Radau IIA methods in Section 3. We extend both the maximal regularity property and the a posteriori error analysis to nonautonomous equations in Section 4.

2 The implicit Euler method

This section is devoted to both the a priori and a posteriori error analysis of the implicit Euler method for the initial value problem (1.1). We present a complete analysis; in particular, we show that the a posteriori estimator is of optimal order. The a priori error analysis is based on the *discrete* maximal parabolic regularity property of the implicit Euler method; cf. (2.6). In contrast, the a posteriori error analysis is based on the *continuous* maximal parabolic regularity property (1.2). We combine both maximal regularity properties to show that the a posteriori error estimator is of optimal order.

Let $k = T/N$ be a constant time step and $t_n := nk, n = 0, 1, \dots, N$, be the nodes of a uniform partition of the time interval $[0, T]$. We consider the discretization of the initial value problem (1.1) by the implicit Euler method, i.e., we define approximations $U_\ell \in \mathscr{D}(A)$ to the nodal values $u_\ell := u(t_\ell)$ of the solution u as follows

$$\partial U_{n+1} + AU_{n+1} = f(t_{n+1}), \quad n = 0, \dots, N-1, \tag{2.1}$$

with ∂v_{n+1} the backward difference quotient, $\partial v_{n+1} := (v_{n+1} - v_n)/k$, and starting value $U_0 = 0$.

2.1 Residual and a posteriori error estimates

It is well known that the implicit Euler scheme can be viewed as collocation method with node $c_1 = 1$. Then, the approximate solution $\widehat{U} : [0,T] \to \mathscr{D}(A)$ is the piecewise linear interpolant of the nodal approximations U_ℓ,

$$\widehat{U}(t) := U_{n+1} + (t - t_{n+1})\partial U_{n+1}, \quad t \in J_n, \quad n = 0, \dots, N-1, \tag{2.2}$$

with $J_n := (t_n, t_{n+1}]$. The *residual* R of $\widehat{U}$ is the amount by which the collocation approximate solution $\widehat{U}$ misses satisfying the parabolic equation in (1.1),

$$R(t) := \widehat{U}'(t) + A\widehat{U}(t) - f(t), \quad t \in J_n, \quad n = 0, \dots, N-1. \tag{2.3}$$

Notice that the residual R is a computable a posteriori quantity.

We consider the error $e := u - \widehat{U}$. Subtracting (2.3) from the differential equation in (1.1), we obtain the *error equation*,

$$e'(t) + Ae(t) = -R(t), \quad t \in (0,T]. \tag{2.4}$$

Since $e(0)$ vanishes, the maximal L^p-regularity (1.2) for the error equation (2.4) yields the desired a posteriori error estimate

$$\|R\|_{L^p((0,T);X)} \leqslant \|e'\|_{L^p((0,T);X)} + \|Ae\|_{L^p((0,T);X)} \leqslant c_{p,X}\|R\|_{L^p((0,T);X)}. \tag{2.5}$$

2.2 Discrete maximal parabolic regularity and a priori error estimates

Fundamental results concerning the discrete maximal parabolic regularity for A-stable Runge–Kutta methods with invertible coefficient matrices and, under natural additional conditions, for backward difference formula (BDF) methods were recently established in [11]; we refer to [11] also for an overview of previous work on this topic.

In particular, the implicit Euler method preserves the maximal parabolic regularity,

$$\|(\partial U_n)_{n=1}^M\|_{\ell^p(X)} + \|(AU_n)_{n=1}^M\|_{\ell^p(X)} \leqslant C_{p,X}\|(f(t_n))_{n=1}^M\|_{\ell^p(X)}, \tag{2.6}$$

$M = 1, \dots, N$, with a method-dependent constant $C_{p,X}$, independent of M, T, and the time step k. Here, for a sequence $(v_n)_{n=0,\dots,N} \subset X$ and $M \leqslant N$, we used the notation

$$\|(v_n)_{n=1}^M\|_{\ell^p(X)} := \Big(k\sum_{n=1}^M \|v_n\|_X^p\Big)^{1/p}.$$

Notice that $\|(v_n)_{n=1}^M\|_{\ell^p(X)}$ is the $L^p((0,t_M);X)$ norm of the piecewise constant function v taking the values $v(t) = v_{n+1}, t_n < t < t_{n+1}$. For the discrete maximal parabolic regularity (2.6) of the implicit Euler method, see [5, Remark 5.2] and [11, Theorem 3.1].

The a priori error estimate is an easy consequence of the stability estimate (2.6); for the reader's convenience, we recall the details; we shall use the result to show that the a posteriori error estimator in (2.5) is of optimal order. For consistency with the notation for high order methods, we let the *consistency error* $E_\ell \in X, \ell = 1, \dots, N$, of the implicit Euler method for the solution u of (1.1) be given by

$$E_{n+1} := k\big[\partial u_{n+1} + Au_{n+1} - f(t_{n+1})\big], \quad n = 0, \dots, N-1; \tag{2.7}$$

notice that the consistency error is the amount by which u misses satisfying the implicit Euler method. Using the differential equation in (1.1), we can write E_{n+1} in the form

$$E_{n+1} = k\big[\partial u_{n+1} - u'(t_{n+1})\big],$$

and, under obvious regularity assumptions, easily infer by the Taylor theorem that

$$E_{n+1} = -\int_{t_n}^{t_{n+1}} (t - t_n) u''(t)\, \mathrm{d}t.$$

Then, Hölder's inequality yields the desired consistency estimate

$$\|E_{n+1}\|_X \leqslant \frac{k^{1+\frac{1}{s}}}{(s+1)^{1/s}} \Big(\int_{t_n}^{t_{n+1}} \|u''(t)\|_X^p \, \mathrm{d}t \Big)^{1/p}, \tag{2.8}$$

with s the dual exponent of p, that is, $\frac{1}{p} + \frac{1}{s} = 1$. Therefore,

$$\begin{aligned} \|(E_n)_{n=1}^M\|_{\ell^p(X)}^p &= k \sum_{n=1}^M \|E_n\|_X^p \leqslant k \Big(\frac{k^{1+\frac{1}{s}}}{(s+1)^{1/s}} \Big)^p \sum_{n=0}^{M-1} \int_{t_n}^{t_{n+1}} \|u''(t)\|_X^p \, \mathrm{d}t \\ &= \Big(\frac{k^2}{(s+1)^{1/s}} \Big)^p \int_0^{t_M} \|u''(t)\|_X^p \, \mathrm{d}t, \end{aligned}$$

whence

$$\|(E_n)_{n=1}^M\|_{\ell^p(X)} \leqslant \frac{k^2}{(s+1)^{1/s}} \|u''\|_{L^p((0,t_M);X)}, \quad M = 1, \dots, N. \tag{2.9}$$

Let $e_\ell := u_\ell - U_\ell, \ell = 0, 1, \dots, N$. Subtracting the implicit Euler method (2.1) from the consistency relation (2.7), we obtain the error equation

$$\partial e_{n+1} + Ae_{n+1} = \frac{1}{k} E_{n+1}, \quad n = 0, \dots, N-1. \tag{2.10}$$

Combining the discrete maximal parabolic regularity stability estimate (2.6) for the error equation (2.10) with the consistency estimate (2.9), we obtain the desired a priori error estimate,

$$\|(\partial e_n)_{n=1}^M\|_{\ell^p(X)} + \|(Ae_n)_{n=1}^M\|_{\ell^p(X)} \leqslant C_{p,X} \frac{k}{(s+1)^{1/s}} \|u''\|_{L^p((0,t_M);X)}, \tag{2.11}$$

$M = 1, \dots, N$, with the constant $C_{p,X}$ of (2.6).

Remark 2.1 ($\ell^\infty(X)$ estimate) In view of $e_0 = 0$, we have $e_m = k\sum_{n=1}^m \partial e_n$, and thus

$$\|e_m\|_X \leqslant k\sum_{n=1}^m \|\partial e_n\|_X = \sum_{n=1}^m k^{1/s}\big(k^{1/p}\|\partial e_n\|_X\big).$$

Therefore, the discrete Hölder inequality yields

$$\|e_m\|_X \leqslant (t_m)^{1/s}\|(\partial e_n)_{n=1}^m\|_{\ell^p(X)}. \tag{2.12}$$

From (2.12) and (2.11) we obtain the $\ell^\infty(X)$ estimate

$$\|e_m\|_X \leqslant C_{p,X}(t_m)^{1/s}\frac{k}{(s+1)^{1/s}}\|u''\|_{L^p((0,t_m);X)}, \quad m = 1,\dots,N, \tag{2.13}$$

with a mildly growing factor $(t_m)^{1/s}$.

2.3 Optimality of the a posteriori error estimate (2.5)

Using (2.2), for $t \in J_n$, we have

$$\widehat{U}'(t) + A\widehat{U}(t) = \partial U_{n+1} + AU_{n+1} + (t - t_{n+1})A\partial U_{n+1},$$

and, in view of (2.1), infer that the residual can also be written in the form

$$R(t) = (t - t_{n+1})A\partial U_{n+1} + [f(t_{n+1}) - f(t)], \quad t \in J_n, \quad n = 0,\dots,N-1. \tag{2.14}$$

Therefore,

$$R(t) = (t - t_{n+1})A\partial u_{n+1} - (t - t_{n+1})A\partial e_{n+1} + [f(t_{n+1}) - f(t)], \quad t \in J_n, \tag{2.15}$$

$n = 0,\dots,N-1$.

Let us denote by $R_1(t), R_2(t)$ and $R_3(t)$ the first, second and third terms on the right-hand side of (2.15), respectively. We shall estimate each one of these terms separately. First, for R_1, we have

$$R_1(t) = (t - t_{n+1})A\partial u_{n+1} = \frac{t - t_{n+1}}{k}\int_{t_n}^{t_{n+1}} Au'(\tau)\,\mathrm{d}\tau, \quad t \in J_n.$$

Therefore,

$$\|R_1(t)\|_X \leqslant \int_{t_n}^{t_{n+1}} \|Au'(\tau)\|_X\,\mathrm{d}\tau, \quad t \in J_n,$$

whence

$$\int_{t_n}^{t_{n+1}} \|R_1(t)\|_X^p\,\mathrm{d}t \leqslant k\Big(\int_{t_n}^{t_{n+1}} \|Au'(\tau)\|_X\,\mathrm{d}\tau\Big)^p \leqslant kk^{p/s}\int_{t_n}^{t_{n+1}} \|Au'(\tau)\|_X^p\,\mathrm{d}\tau,$$

i.e.,

$$\int_{t_n}^{t_{n+1}} \|R_1(t)\|_X^p\,\mathrm{d}t \leqslant k^p\int_{t_n}^{t_{n+1}} \|Au'(\tau)\|_X^p\,\mathrm{d}\tau.$$

This yields the desired estimate for R_1,

$$\|R_1\|_{L^p((0,T);X)} \leqslant k\|Au'\|_{L^p((0,T);X)}. \tag{2.16}$$

Similarly, for R_3, we have

$$R_3(t) = \int_t^{t_{n+1}} f'(\tau)\,\mathrm{d}\tau, \quad t \in J_n,$$

and thus

$$\|R_3(t)\|_X \leqslant \int_{t_n}^{t_{n+1}} \|f'(\tau)\|_X\,\mathrm{d}\tau, \quad t \in J_n.$$

Proceeding as in the case of R_1, we arrive at the desired estimate for R_3,

$$\|R_3\|_{L^p((0,T);X)} \leqslant k\|f'\|_{L^p((0,T);X)}. \tag{2.17}$$

Next, we shall use the a priori error estimate (2.11) to bound R_2. We have

$$R_2(t) = -(t-t_{n+1})A\partial e_{n+1} = -\frac{t-t_{n+1}}{k}(Ae_{n+1}-Ae_n), \quad t \in J_n,$$

and thus

$$\|R_2(t)\|_X \leqslant \|Ae_{n+1}\|_X + \|Ae_n\|_X, \quad t \in J_n.$$

Therefore,

$$\int_{t_n}^{t_{n+1}} \|R_2(t)\|_X^p\,dt \leqslant k\big(\|Ae_{n+1}\|_X + \|Ae_n\|_X\big)^p,$$

whence

$$\|R_2\|_{L^p((0,T);X)} \leqslant \Big(k\sum_{n=0}^{N-1}\big(\|Ae_{n+1}\|_X + \|Ae_n\|_X\big)^p\Big)^{1/p}.$$

Using here the Minkowski inequality for the ℓ^p norm on $\mathbb{R}^N$, we infer that

$$\|R_2\|_{L^p((0,T);X)} \leqslant 2\Big(k\sum_{n=1}^{N}\|Ae_n\|_X^p\Big)^{1/p} = 2\|(Ae_n)_{n=1}^N\|_{\ell^p(X)}.$$

In view of the a priori error estimate (2.11), this yields the desired estimate for R_2,

$$\|R_2\|_{L^p((0,T);X)} \leqslant 2c_{p,X}\frac{k}{(s+1)^{1/s}}\|u''\|_{L^p((0,T);X)}. \tag{2.18}$$

From (2.16), (2.17) and (2.18) we infer that the a posteriori error estimator $\|R\|_{L^p((0,T);X)}$ in (2.5) is of optimal order $O(k)$ in any fixed interval $[0,T]$, provided that $f', Au', u'' \in L^p((0,T);\,X)$; of course, if two of these functions are elements of $L^p((0,T);X)$, then the third belongs to the same space as a linear combination of the other two. Notice also that if $f(0)$ vanishes, then $u'(0)$ vanishes as well, and in view of $u''+Au'=f'$, we actually only need to assume that $f' \in L^p((0,T);X)$; then, $u'', Au' \in L^p((0,T);X)$ by maximal parabolic L^p-regularity.

Let us emphasize that the residual estimates (2.16)–(2.18) and (2.5) yield optimal order a priori error estimates for the implicit Euler method in the continuous $L^p((0,T);X)$-norm; this complements the corresponding a priori error estimate (2.11) in the discrete $\ell^p(X)$-norm.

3 High-order Radau IIA methods

In this section we present our main results. We establish a posteriori as well as a priori error estimates for Radau IIA methods. Furthermore, we show that the a posteriori error estimator is of optimal order.

As already mentioned, the maximal L^p-regularity of the operator A applied to the error equation (1.10) yields the a posteriori error estimate (1.11). Our goal here is to show that the estimator on the right-hand side of (1.11) is of optimal order.

We first derive an explicit representation of the residual R and subsequently show that the a posteriori estimator is of optimal order via a priori error analysis.

We refer also to [20,19,2,3,4,16] and [6] for a posteriori error analyses for parabolic equations in Hilbert and Banach spaces, respectively.

3.1 Explicit representation of the residual

The residual R of the collocation approximate solution $\widehat{U}$ of the q-stage Radau IIA method, given in (1.9), is, obviously, a computable quantity. Let us give here an explicit representation of it; compare to [3, Theorem 2.2].

Notice that, in view of the pointwise form (1.7) of the numerical method, the residual can also be written in the form

$$R(t) = A\big[\widehat{U}(t) - I_{q-1}\widehat{U}(t)\big] - \big[f(t) - I_{q-1}f(t)\big], \quad t \in (t_n, t_{n+1}], \tag{3.1}$$

$n = 0, \dots, N-1$.

The residual R of (3.1) seems suitable for a posteriori error estimates in this case; this is due to the fact that the corresponding a priori error estimates, see (3.18) and (3.19) in the sequel, are of order $O(k^q)$. This is in contrast to [3] and [4], where it was advantageous to introduce a suitable higher-order *reconstruction*, an element of $\mathcal{V}_k^c(q+1)$, of the collocation approximation $\widehat{U} \in \mathcal{V}_k^c(q)$.

Any form of the polynomial interpolation remainder not relying on the mean value theorem, since our functions are vector-valued, can be used for the representation of the interpolation errors $\widehat{U}(t) - I_{q-1}\widehat{U}(t)$ and $f(t) - I_{q-1}f(t)$; for instance, the Kowalewski remainder representation [7, Ex. 1, pp. 71–72] leads to

$$\begin{cases} \widehat{U}(t) - I_{q-1}\widehat{U}(t) = k^q \Phi_q\big(\frac{t-t_n}{k}\big)\widehat{U}^{(q)}, \\ f(t) - I_{q-1}f(t) = \dfrac{1}{(q-1)!}\displaystyle\sum_{i=1}^{q} \ell_{ni}(t) \int_{t_{ni}}^{t} (t_{ni} - \tau)^{q-1} f^{(q)}(\tau)\,\mathrm{d}\tau, \end{cases} \tag{3.2}$$

for $t \in J_n$, with

$$\Phi_q(\tau) := \frac{1}{q!}\prod_{i=1}^{q}(\tau - c_i), \quad \tau \in [0,1], \quad \ell_{ni}(\tau) = \prod_{\substack{j=1\\ j\neq i}}^{q} \frac{\tau - t_{nj}}{t_{ni} - t_{nj}}, \quad \tau \in J_n.$$

Using (3.2), we can rewrite (3.1) in the form

$$R(t) = k^q \Phi_q\big(\frac{t-t_n}{k}\big) A\widehat{U}^{(q)} - \frac{1}{(q-1)!}\sum_{i=1}^{q} \ell_{ni}(t) \int_{t_{ni}}^{t} (t_{ni} - \tau)^{q-1} f^{(q)}(\tau)\,\mathrm{d}\tau, \tag{3.3}$$

$t \in J_n, n = 0, \ldots, N-1$.

Notice that, in the case of the implicit Euler method $q = 1$, the corresponding derivative of $\widehat{U}$ is $\widehat{U}'$ and since we have estimates for Ae_n, we can show that the residual R is indeed of (optimal) first order in k.

3.2 Optimality of the estimator via a priori error analysis

We first recall in Lemma 3.1 the discrete maximal parabolic regularity property of the q-stage Radau IIA method and then prove consistency estimates. Combining these results we derive a priori error estimates, which will be instrumental to establish sharp asymptotic upper bounds for the estimator in (1.11).

3.2.1 Discrete maximal parabolic regularity

We first recall the maximal parabolic regularity property for Radau IIA methods.

Lemma 3.1 ([11, Corollary 5.2, Theorem 5.1]; maximal regularity of Radau IIA methods) *The Radau IIA approximations $U_0, \ldots, U_N$ are well defined by* (1.5) *and satisfy the maximal parabolic regularity stability estimates*

$$\|(\partial U_n)_{n=1}^N\|_{\ell^p(X)} + \|(AU_n)_{n=1}^N\|_{\ell^p(X)} \leqslant C_{p,X} \sum_{i=1}^q \|(f(t_{ni}))_{n=0}^{N-1}\|_{\ell^p(X)} \tag{3.4}$$

and

$$\sum_{i=1}^q \|(AU_{ni})_{n=0}^{N-1}\|_{\ell^p(X)} \leqslant C_{p,X} \sum_{i=1}^q \|(f(t_{ni}))_{n=0}^{N-1}\|_{\ell^p(X)} \tag{3.5}$$

with a constant $C_{p,X}$ independent of N and T, depending on the method, i.e., on q.

The following maximal regularity property of the collocation approximation $\widehat{U}$ is an easy consequence of Lemma 3.1.

Corollary 3.1 (Maximal regularity of the collocation approximation) *The collocation approximation $\widehat{U}$ satisfies the maximal regularity estimates*

$$\sum_{i=1}^q \|(\widehat{U}'(t_{ni}))_{n=0}^{N-1}\|_{\ell^p(X)} \leqslant C_{p,X,q} \sum_{i=1}^q \|(f(t_{ni}))_{n=0}^{N-1}\|_{\ell^p(X)} \tag{$\star$}$$

and

$$\begin{aligned} &\|\widehat{U}'\|_{L^p((0,T);X)} + \|A\widehat{U}\|_{L^p((0,T);X)} + \|AU\|_{L^p((0,T);X)} \\ &\qquad \leqslant C_{p,X,q} \sum_{i=1}^q \|(f(t_{ni}))_{n=0}^{N-1}\|_{\ell^p(X)} \end{aligned} \tag{$\star\star$}$$

with a constant $C_{p,X,q}$ independent of N and T.

Proof First, according to the collocation equations (1.6), there holds

$$\sum_{i=1}^{q} \|(\widehat{U}'(t_{ni}))_{n=0}^{N-1}\|_{\ell^p(X)} = \sum_{i=1}^{q} \|\big(-AU_{ni} + f(t_{ni})\big)_{n=0}^{N-1}\|_{\ell^p(X)},$$

and the triangle inequality and (3.5) yield ($\star$).

Furthermore, the Lagrange form of $\widehat{U}'$,

$$\widehat{U}'(t) = \sum_{i=1}^{q} \ell_{ni}\widehat{U}'(t_{ni}), \quad t_n < t < t_{n+1},$$

with ℓ_{ni} the Lagrange polynomials ℓ_i shifted to the interval J_n, yields

$$\begin{aligned}\int_{J_n} \|\widehat{U}'\|_X^p \,\mathrm{d}t &= \int_{J_n} \|\sum_{i=1}^{q} \ell_{ni}(t)\widehat{U}'(t_{ni})\|_X^p \,\mathrm{d}t\\ &\leqslant \Big(\sum_{i=1}^{q} k\|\ell_{ni}\|_{L^\infty(J_n)}\|\widehat{U}'(t_{ni})\|_X\Big)^p\\ &= \Big(\sum_{i=1}^{q} k\|\ell_{i}\|_{L^\infty(0,1)}\|\widehat{U}'(t_{ni})\|_X\Big)^p\\ &\leqslant \Big(\sum_{i=1}^{q} \|\ell_{i}\|_{L^\infty(0,1)}^s\Big)^{p/s}\Big(\sum_{i=1}^{q} k\|\widehat{U}'(t_{ni})\|_X^p\Big),\end{aligned}$$

with s the dual exponent of p. This in combination with ($\star$) leads to the asserted estimate for $\widehat{U}'$ in ($\star\star$).

Taking (3.5) into account, AU can be estimated analogously. Finally, using the Lagrange polynomials for the points $c_0 = 0, c_1, \dots, c_q$, (3.5), and the fact that $\widehat{U}(t_{n0}) = \widehat{U}(t_n) = U_{n-1,q}$, we can also estimate $A\widehat{U}$ in the desired way. □

3.2.2 Consistency

We prove consistency of the Radau IIA methods for the initial value problem (1.1), assuming existence of a smooth solution. We recall that the stage order of the q-stage Radau IIA method is q, i.e.,

$$\text{(B}(q)\text{)} \qquad \sum_{i=1}^{q} b_i c_i^{\ell-1} = \frac{1}{\ell}, \qquad \ell = 1, \dots, q,$$

$$\text{(C}(q)\text{)} \qquad \sum_{j=1}^{q} a_{ij} c_j^{\ell-1} = \frac{c_i^\ell}{\ell}, \quad \ell = 1, \dots, q, \; i = 1, \dots, q.$$

The consistency errors E_{ni} and E_{n+1} of the method are determined by

$$\text{(3.6)} \qquad \begin{cases} u(t_{ni}) = u(t_n) - k\displaystyle\sum_{j=1}^{q} a_{ij}\big(Au(t_{nj}) - f(t_{nj})\big) + E_{ni}, \qquad i = 1, \dots, q,\\ u(t_{n+1}) = u(t_n) - k\displaystyle\sum_{i=1}^{q} b_i\big(Au(t_{ni}) - f(t_{ni})\big) + E_{n+1}.\end{cases}$$

Notice that $E_{n+1} = E_{nq}, n = 0, \dots, N-1$.

Lemma 3.2 (Consistency estimate) *If the solution u of (1.1) is sufficiently smooth, namely $u \in W^{q+1,p}((0,T);X)$, then the following consistency estimate holds*

$$\|E_{ni}\|_X \leqslant Ck^q \int_{t_n}^{t_{n+1}} \|u^{(q+1)}(\tau)\|_X \, \mathrm{d}\tau, \quad i=1,\dots,q, \quad n=0,\dots,N-1, \tag{3.7}$$

with a method-dependent constant C.

Proof In view of the differential equation in (1.1), (3.6) yields

$$u(t_{ni}) = u(t_n) + k\sum_{j=1}^{q} a_{ij}u'(t_{nj}) + E_{ni}, \quad i=1,\dots,q. \tag{3.8}$$

Notice that E_{ni} is the quadrature error over the interval $[t_n,t_{ni}]$ of the quadrature formula with weights $a_{ij}k$ and nodes $t_{nj}=t_n+c_jk, j=1,\dots,q$, for the function u'.

Taylor expansion about t_n yields

$$\begin{aligned} E_{ni} = &\sum_{\ell=1}^{q} \frac{k^\ell}{(\ell-1)!}\Big(\frac{c_i^\ell}{\ell} - \sum_{j=1}^{q} a_{ij}c_j^{\ell-1}\Big)u^{(\ell)}(t_n) + \frac{1}{q!}\int_{t_n}^{t_{ni}} (t_{ni}-\tau)^q u^{(q+1)}(\tau)\,\mathrm{d}\tau\\ &- \frac{k}{(q-1)!}\sum_{j=1}^{q} a_{ij}\int_{t_n}^{t_{nj}} (t_{nj}-\tau)^{q-1}u^{(q+1)}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

In view of the stage order conditions (C(q)), leading terms of order up to q vanish, and E_{ni} can be represented in the form

$$E_{ni} = k^q \int_{t_n}^{t_{n+1}} \kappa_i\big(\frac{\tau-t_n}{k}\big)u^{(q+1)}(\tau)\,\mathrm{d}\tau, \quad i=1,\dots,q, \tag{3.9}$$

with the bounded Peano kernels

$$\kappa_i(t) := \frac{1}{q!}\big((c_i-t)_+\big)^q - \frac{1}{(q-1)!}\sum_{j=1}^{q} a_{ij}\big((c_j-t)_+\big)^{q-1}, \quad 0\leqslant t\leqslant 1, \tag{3.10}$$

$i=1,\dots,q$, where we used the standard notation $\tau_+=\tau$ for $\tau\geqslant 0$ and $\tau_+=0$ for $\tau<0$. We thus obtain the asserted consistency estimate (3.7). □

3.2.3 A priori error estimates in the discrete $\ell^p(X)$-norm

Let $e_n := u(t_n)-U_n, n=0,\dots,N$, and $e_{ni} := u(t_{ni})-U_{ni}, n=0,\dots,N-1$, denote the nodal and the intermediate errors, respectively, of the q-stage Radau IIA method (1.5). Of course, $e_{n+1}=e_{nq}, n=0,\dots,N-1$.

First, we rewrite (3.8) in a suitable for our purposes form, which will allow us to directly apply the discrete maximal parabolic regularity stability estimate of Lemma 3.1 to our error equations; see also [12, §4.1]. With $\widetilde{E}_{ni}, i=1,\dots,q$, defined by

$$k\sum_{j=1}^{q} a_{ij}\widetilde{E}_{nj} = E_{ni}, \quad i=1,\dots,q, \tag{3.11}$$

it is easily seen that (3.8) reads

$$u(t_{ni}) = u(t_n) - k\sum_{j=1}^{q} a_{ij}\big[Au(t_{nj}) - f(t_{nj}) - \widetilde{E}_{nj}\big], \quad i = 1,\dots,q. \tag{3.12}$$

Subtracting the first relation of (1.5) from (3.12), we obtain the error equations

$$e_{ni} = e_n - k\sum_{j=1}^{q} a_{ij}\big(Ae_{nj} - \widetilde{E}_{nj}\big), \quad i = 1,\dots,q, \tag{3.13}$$

$n = 0,\dots,N-1$. Now, the discrete maximal parabolic regularity stability estimates of Lemma 3.1 applied to (3.13) yield

$$\|(\partial e_n)_{n=1}^N\|_{\ell^p(X)} + \|(Ae_n)_{n=1}^N\|_{\ell^p(X)} \leqslant C_{p,X}\sum_{i=1}^{q}\|(\widetilde{E}_{ni})_{n=0}^{N-1}\|_{\ell^p(X)} \tag{3.14}$$

and

$$\sum_{i=1}^{q}\|(Ae_{ni})_{n=0}^{N-1}\|_{\ell^p(X)} \leqslant C_{p,X}\sum_{i=1}^{q}\|(\widetilde{E}_{ni})_{n=0}^{N-1}\|_{\ell^p(X)} \tag{3.15}$$

with a constant $C_{p,X}$ (depending also on the specific method) independent of N and the time step k; compare (3.13) to [11, (5.1)]. The notation ∂v_n and $\|(v_n)_{n=1}^M\|_{\ell^p(X)}$ was introduced immediately after (2.1) and in section 2.2, respectively; the notation $\|(v_n)_{n=0}^M\|_{\ell^p(X)}$ is completely analogous.

The estimate

$$\|\widetilde{E}_{ni}\|_X \leqslant Ck^{q-1}\int_{t_n}^{t_{n+1}}\|u^{(q+1)}(\tau)\|_X\,\mathrm{d}\tau, \quad i = 1,\dots,q, \tag{3.16}$$

$n = 0,\dots,N-1$, is an immediate consequence of (3.11) and the consistency estimate (3.7). Now, Hölder's inequality for integrals yields

$$\sum_{n=0}^{N-1}\Big(\int_{t_n}^{t_{n+1}}\|u^{(q+1)}(\tau)\|_X\,\mathrm{d}\tau\Big)^p \leqslant k^{p/s}\|u^{(q+1)}\|^p_{L^p((0,T);X)} \tag{3.17}$$

with s the dual exponent of p. Inserting (3.16) into (3.14) and (3.15), and using (3.17), we obtain the a priori error estimates

$$\|(\partial e_n)_{n=1}^N\|_{\ell^p(X)} + \|(Ae_n)_{n=1}^N\|_{\ell^p(X)} \leqslant \widetilde{C}_{p,X,q}k^q\|u^{(q+1)}\|_{L^p((0,T);X)} \tag{3.18}$$

and

$$\|(Ae_{ni})_{n=0}^{N-1}\|_{\ell^p(X)} \leqslant \widetilde{C}_{p,X,q}k^q\|u^{(q+1)}\|_{L^p((0,T);X)}, \quad i = 1,\dots,q, \tag{3.19}$$

respectively, with a constant $\widetilde{C}_{p,X,q}$, independent of T, N, the time step k, and the solution u.

Compare the a priori error estimates (3.18) and (3.19) with [12, (2.9a), (2.9b)], where a much more involved initial value problem is discretized.

3.2.4 Optimality of the a posteriori error estimate (1.11)

Assuming that the forcing term f and the solution u are sufficiently smooth, namely $f \in W^{q,p}((0,T);X)$ and $u \in W^{q+1,p}((0,T);X)$, and using the explicit representation (3.3) of the residual R, we shall show here that the a posteriori error estimator for the q-stage Radau IIA method on the right-hand side of (1.11) is also of order q, in analogy to the a priori error estimates (3.18) and (3.19). We mention that, as in the case of the implicit Euler method, this analysis leads to optimal order a priori error estimates for the collocation approximation $\widehat{U}$ in the continuous $L^p((0,T);X)$-norm, thus complementing the corresponding estimates (3.18) and (3.19) in the discrete $\ell^p(X)$-norm.

Let us first consider the second term,

$$R_f(t) := -\frac{1}{(q-1)!}\sum_{i=1}^{q}\ell_{ni}(t)\int_{t_{ni}}^{t}(t_{ni}-\tau)^{q-1}f^{(q)}(\tau)\,\mathrm{d}\tau, \quad t\in(t_n,t_{n+1}),$$

on the right-hand side of (3.3) of the residual $R(t)$. Obviously,

$$\|R_f(t)\|_X \leqslant \widetilde{C}_q k^{q-1}\int_{t_n}^{t_{n+1}}\|f^{(q)}(\tau)\|_X\,\mathrm{d}\tau, \quad t\in(t_n,t_{n+1}).$$

Therefore, using the analogue of (3.17) for $f^{(q)}$, we easily see that

$$\|R_f\|_{L^p((0,T);X)} \leqslant \widetilde{C}_q k^q\|f^{(q)}\|_{L^p((0,T);X)}. \tag{3.20}$$

It thus remains to estimate the first term on the right-hand side of (3.3), that is, to show that $\|A\widehat{U}^{(q)}\|_X$ is bounded, uniformly in the time step k.

Recall that $c_0=0, t_{n0}=t_n$, and let I_q be the interpolation operator by elements of $\mathcal{X}_k^{\mathrm{c}}(q)$ at the nodes $t_{ni}, i=0,1,\dots,q$. Furthermore, let $\widehat{\ell}_{ni}\in\mathbb{P}_q, i=0,1,\dots,q$, be the Lagrange polynomials for the nodes $t_{ni}, i=0,1,\dots,q$. Then, since the approximate solution $\widehat{U}$ is an element of $\mathcal{V}_k^{\mathrm{c}}(q)$, it can be written in the form

$$\widehat{U}(t)=\sum_{i=0}^{q}\widehat{\ell}_{ni}(t)\widehat{U}(t_{ni}), \quad t\in[t_n,t_{n+1}].$$

Therefore, with $e_{n0}=e_n$, we have

$$\widehat{U}(t)=-\sum_{i=0}^{q}\widehat{\ell}_{ni}(t)e_{ni}+\sum_{i=0}^{q}\widehat{\ell}_{ni}(t)u(t_{ni}), \quad t\in[t_n,t_{n+1}],$$

whence

$$\widehat{U}^{(q)}(t)=-\sum_{i=0}^{q}\widehat{\ell}_{ni}^{(q)}e_{ni}+\sum_{i=0}^{q}\widehat{\ell}_{ni}^{(q)}u(t_{ni}), \quad t\in(t_n,t_{n+1}). \tag{3.21}$$

We have

$$\widehat{\ell}_{ni}(t)=\prod_{\substack{j=0\\ j\neq i}}^{q}\frac{t-t_{nj}}{t_{ni}-t_{nj}}, \quad t\in[t_n,t_{n+1}],$$

and thus the (constant) derivative $\widehat{\ell}_{ni}^{\,(q)}$ of order q of $\widehat{\ell}_{ni}$ is

$$\widehat{\ell}_{ni}^{\,(q)}(t) = q! \prod_{\substack{j=0\\ j\neq i}}^{q} \frac{1}{t_{ni}-t_{nj}} = q!k^{-q} \prod_{\substack{j=0\\ j\neq i}}^{q} \frac{1}{c_i-c_j}.$$

Let

$$C_q := \max_{0\leqslant i\leqslant q} \prod_{\substack{j=0\\ j\neq i}}^{q} \frac{1}{|c_i-c_j|}.$$

Then, obviously,

$$|\widehat{\ell}_{ni}^{\,(q)}(t)| \leqslant C_q q! k^{-q}, \quad i=0,1,\dots,q, \quad t\in(t_n,t_{n+1}). \tag{3.22}$$

We now use (3.21) and split the first term, $R_{\widehat{U}}$, say, on the right-hand side of (3.3) of the residual R in the form $R_{\widehat{U}}(t) = R_1(t) + R_2(t)$ with

$$\begin{cases} R_1(t) := k^q \Phi_q\big(\dfrac{t-t_n}{k}\big) \displaystyle\sum_{i=0}^{q} \widehat{\ell}_{ni}^{\,(q)} Au(t_{ni}), \\ R_2(t) := -k^q \Phi_q\big(\dfrac{t-t_n}{k}\big) \displaystyle\sum_{i=0}^{q} \widehat{\ell}_{ni}^{\,(q)} Ae_{ni}. \end{cases} \tag{3.23}$$

Obviously, Φ_q is bounded, $|\Phi_q(s)| \leqslant c/q!$, uniformly in the time step k.

We shall first estimate R_1. With $r\in\mathbb{P}(q-1)$ the Taylor polynomial of u about $t_n = t_{n0}$, we have $I_q r = r$ and $r^{(q)} = 0$, whence

$$\begin{aligned} \sum_{i=0}^{q} \widehat{\ell}_{ni}^{\,(q)} u(t_{ni}) &= \sum_{i=0}^{q} \widehat{\ell}_{ni}^{\,(q)} \big[u(t_{ni}) - r(t_{ni})\big] \\ &= \frac{1}{(q-1)!} \sum_{i=0}^{q} \widehat{\ell}_{ni}^{\,(q)} \int_{t_n}^{t_{ni}} (t_{ni}-\tau)^{q-1} u^{(q)}(\tau)\,\mathrm{d}\tau. \end{aligned}$$

This relation, in combination with (3.22) and the analogue of (3.17) for $Au^{(q)}$, leads to the desired estimate

$$\|R_1\|_{L^p((0,T);X)} \leqslant \hat{c}_q k^q \|Au^{(q)}\|_{L^p((0,T);X)}$$

for R_1, whence to

$$\|R_1\|_{L^p((0,T);X)} \leqslant \hat{c}_q k^q \big(\|u^{(q+1)}\|_{L^p((0,T);X)} + \|f^{(q)}\|_{L^p((0,T);X)}\big). \tag{3.24}$$

Next, we estimate $R_2(t)$. Utilizing (3.22), we obtain from (3.23)

$$\int_{t_n}^{t_{n+1}} \|R_2(t)\|_X^p \,\mathrm{d}t \leqslant (cC_q)^p k \Big(\sum_{i=0}^{q} \|Ae_{ni}\|_X\Big)^p,$$

whence

$$\|R_1\|_{L^p((0,T);X)} \leqslant cC_q \Big(k \sum_{n=1}^{N} \Big(\sum_{i=0}^{q} \|Ae_{ni}\|_X\Big)^p\Big)^{1/p}.$$

Using here the triangle inequality for the ℓ^p norm on $\mathbb{R}^N$ for $q+1$ vectors, we infer that

$$\|R_2\|_{L^p((0,T);X)} \leqslant cC_q \sum_{i=0}^{q} \Big(k \sum_{n=1}^{N} \|Ae_{ni}\|_X^p\Big)^{1/p} = cC_q \sum_{i=0}^{q} \|(Ae_{ni})_{n=1}^N\|_{\ell^p(X)}.$$

In view of the a priori error estimates (3.18) and (3.19), this yields the desired optimal order estimate for R_2,

$$\|R_2\|_{L^p((0,T);X)} \leqslant \widetilde{C}_{p,X,q} k^q \|u^{(q+1)}\|_{L^p((0,T);X)}. \tag{3.25}$$

The optimality of the a posteriori error estimate (1.11) is now an obvious consequence of (3.3), (3.20), (3.24) and (3.25). Summarizing, we have proved the following:

Theorem 3.1 (A posteriori error estimate) *Consider the Radau IIA approximations defined in (1.5). Let $\widehat{U} \in \mathcal{V}_k^c(q)$ be the continuous piecewise polynomial function such that $\widehat{U}(t_{ni}) = U_{ni}, i = 1, \dots, q, n = 0, \dots, N-1$, where, in particular, $\widehat{U}(t_{n+1}) = \widehat{U}(t_{nq}) = U_{n+1} = \widehat{U}(t_{n+1}^+)$. With u being the solution of (1.1), the following maximal regularity a posteriori error estimate holds*

$$\|(u - \widehat{U})'\|_{L^p((0,t);X)} + \|A(u - \widehat{U})\|_{L^p((0,t);X)} \leqslant c_{p,X} \|R\|_{L^p((0,t);X)}, \tag{3.26}$$

for $0 < t \leqslant T$, where the a posteriori estimator is given by

$$R(t) = \widehat{U}'(t) + A\widehat{U}(t) - f(t) = (I - I_{q-1})(A\widehat{U}(t) - f(t)), \quad t \in (t_n, t_{n+1}]. \tag{3.27}$$

Furthermore, the estimator is of optimal asymptotic order of accuracy in the sense that, if the forcing term f and the solution u are sufficiently smooth, namely $f \in W^{q,p}((0,T);X)$ and $u \in W^{q+1,p}((0,T);X)$, there exists a constant $\widetilde{C}_{p,X,q}$ such that

$$\|R\|_{L^p((0,T);X)} \leqslant \widetilde{C}_{p,X,q} k^q \big(\|u^{(q+1)}\|_{L^p((0,T);X)} + \|f^{(q)}\|_{L^p((0,T);X)}\big). \tag{3.28}$$

Remark 3.1 (Order of convergence) In the case of a triple of Hilbert spaces $V \subset H \subset V^\star$, the standard order of convergence for the nodal errors $e_n = u(t_n) - U_n$ as well as for the errors $e_{ni} = u(t_{ni}) - U_{ni}$ at the intermediate nodes of the q-stage Radau IIA method, $q \geqslant 2$, is $q+1$, i.e., the minimum of the stage order plus 1 and of the order $p = 2q-1$ of the method; the errors are measured in the discrete maximum norm in time and in the Hilbert space norm in space, i.e., in the discrete $L^\infty(H)$ norm. The a posteriori bounds in the $L^\infty(H)$- and $L^2(V)$-norms, established via the energy technique, are also of order $q+1$, provided the exact solution u is sufficiently smooth; cf. [3], [4], [19].

However, the order q of the a posteriori estimator in (3.26), as well as of the a priori error estimates (3.18) and (3.19), is optimal. This is due to the fact that our estimates are in stronger norms in time. More precisely, the error is measured in the $W^{1,p}(X)$ (semi)norm in the first term on the left-hand side of (3.26), rather than in the standard $L^\infty(H)$-norm in the case of Hilbert spaces. Since the collocation approximation $\widehat{U}$ is a piecewise polynomial of degree at most q in time, its derivative $\widehat{U}'$ is a piecewise polynomial of degree at most $q-1$ in time; therefore, the highest possible attainable order of convergence of the derivative $e' = u' - \widehat{U}'$ of the error is q, even if $\widehat{U}'$ were the best approximation to u' from the space of piecewise polynomials of degree at most $q-1$.

Remark 3.2 (Regularity requirements) To establish optimal order $O(k^q)$ a priori error estimates in the discrete $\ell^p(X)$-norm, we assumed that $u \in W^{q+1,p}((0,T);X)$. To show that the a posteriori error estimator is of optimal order $O(k^q)$, and thus to obtain optimal order a priori error estimates in the continuous $L^p((0,T);X)$-norm, we furthermore assumed that $f \in W^{q,p}((0,T);X)$ or equivalently $Au \in W^{q,p}((0,T);X)$. The additional regularity requirement $f \in W^{q,p}((0,T);X)$ for the optimality of the a posteriori error estimator is due to the explicit appearance of the interpolation error $f - I_{q-1}f$ of the forcing term in the residual R; see (3.27). In contrast, the consistency errors E_{ni} can be expressed in terms of the solution u only; see (3.9).

Remark 3.3 (A posteriori estimates for fully discrete methods) In actual computations for parabolic equations, time stepping methods, such as the Radau IIA methods, are combined with space discretization, for instance by the finite element method. The finite element solutions u_h are, in general, *not* in the domain $\mathscr{D}(A)$ of the continuous operator. Among other technical challenges, this fact is quite important since our approach is based on the maximal regularity properties of A. Hence, the derivation of a posteriori error estimates for fully discrete methods is not straightforward. One possibility to bypass this issue and to extend the present analysis is to use the *elliptic reconstruction* $U \in \mathscr{D}(A)$ of the finite element solutions u_h. By construction, u_h is then the finite element approximation of the corresponding elliptic problem with solution U; see [18]. Roughly speaking, the errors $u - U$ and $U - u_h$ are estimated separately. The spatial error $U - u_h$ is controlled by elliptic estimators, while $u - U$ satisfies an error equation of a form similar to (1.10), where maximal regularity a posteriori estimates are applicable. The derivation of a posteriori error estimates for fully discrete methods will be addressed in a forthcoming work.

4 Extension to nonautonomous equations

In this section, we extend the maximal parabolic regularity stability estimates for Radau IIA methods to nonautonomous parabolic equations by a perturbation argument; for similar ideas and results, we refer to [12, §3.6]. Furthermore, we establish optimal order a posteriori error estimates.

We consider an initial value problem for a nonautonomous linear parabolic equation,

$$\begin{cases} u'(t) + A(t)u(t) = f(t), \quad 0 < t < T, \\ u(0) = 0, \end{cases} \tag{4.1}$$

in a Banach space X.

Our structural assumptions on $A(t)$ are that all operators $A(t), t \in [0,T]$, share the same domain $\mathscr{D}(A)$, $A(t)$ is the generator of an analytic semigroup on X having maximal L^p-regularity, for every $t \in [0,T]$, $A(t)$ induce equivalent norms on $\mathscr{D}(A)$,

$$\|A(t)v\|_X \leqslant c\|A(\tilde t)v\|_X \quad \forall t, \tilde t \in [0,T] \ \forall v \in \mathscr{D}(A), \tag{4.2}$$

and $A(t) : \mathscr{D}(A) \to X$ satisfies a Lipschitz condition with respect to t, i.e.,

$$\|\big(A(t) - A(\tilde t)\big)v\|_X \leqslant L|t - \tilde t|\, \|A(\tau)v\|_X \quad \forall t, \tilde t \in [0,T] \ \forall v \in \mathscr{D}(A), \tag{4.3}$$

for all $\tau \in [0,T]$.

4.1 Maximal parabolic regularity

With our notation, and starting value $U_0 = 0$, the q-stage Radau IIA method for the initial value problem (4.1) yields approximations $U_\ell \in \mathscr{D}(A)$ to the nodal values $u(t_\ell)$, as well as internal approximations $U_{\ell i} \in \mathscr{D}(A)$ to the intermediate values $u(t_{\ell i})$, by

$$\begin{cases} U_{ni} = U_n - k \sum_{j=1}^{q} a_{ij} \big(A(t_{nj}) U_{nj} - f(t_{nj}) \big), \quad i = 1, \dots, q, \\ U_{n+1} = U_n - k \sum_{i=1}^{q} b_i \big(A(t_{ni}) U_{ni} - f(t_{ni}) \big), \end{cases} \tag{4.4}$$

$n = 0, \dots, N-1$. Notice that $U_{n+1} = U_{nq}$.

Proposition 4.1 (Maximal parabolic regularity for nonautonomous equations) *Assume that the operator $A(t)$ is the generator of an analytic semigroup on X having* maximal L^p-regularity, *for every $t \in [0,T]$, and satisfies the structural conditions (4.2) and (4.3). Then, the Radau II approximations U_n, U_{ni} for the initial value problem (4.1), given in (4.4), satisfy the maximal parabolic regularity stability estimates*

$$\sum_{i=1}^{q} \|(A(t_m)U_{ni})_{n=0}^{m-1}\|_{\ell^p(X)} \leqslant C_{p,X,T} \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)} \tag{4.5}$$

and

$$\|(\partial U_n)_{n=1}^{m}\|_{\ell^p(X)} + \|(A(t_m)U_n)_{n=1}^{m}\|_{\ell^p(X)} \leqslant C_{p,X,T} \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}, \tag{4.6}$$

$m = 1, \dots, N$, *with a constant $C_{p,X,T}$ independent of m and k.*

Proof We fix an m, and, for $n = 0, \dots, m-1$, rewrite (4.4) in the form

$$\begin{cases} U_{ni} = U_n - k \sum_{j=1}^{q} a_{ij} \big(A(t_m) U_{nj} - g_{nj} - f(t_{nj}) \big), \quad i = 1, \dots, q, \\ U_{n+1} = U_n - k \sum_{i=1}^{q} b_i \big(A(t_m) U_{ni} - g_{ni} - f(t_{ni}) \big), \end{cases} \tag{4.7}$$

with

$$g_{ni} := \big(A(t_m) - A(t_{ni}) \big) U_{ni}, \quad i = 1, \dots, q. \tag{4.8}$$

Since the time t is frozen at t_m in the operator $A(t_m)$ in (4.7), we can apply the known maximal parabolic regularity estimates (3.4) and (3.5) for Radau IIA methods for autonomous equations, and obtain

$$\begin{aligned} \|(\partial U_n)_{n=1}^{m}\|_{\ell^p(X)} + \|(A(t_m)U_n)_{n=1}^{m}\|_{\ell^p(X)} \leqslant{}& C_{p,X} \sum_{i=1}^{q} \|(g_{ni})_{n=0}^{m-1}\|_{\ell^p(X)} \\ &+ C_{p,X} \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)} \end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
\sum_{i=1}^{q} \|(A(t_m)U_{ni})_{n=0}^{m-1}\|_{\ell^p(X)} \leqslant C_{p,X} &\sum_{i=1}^{q} \|(g_{ni})_{n=0}^{m-1}\|_{\ell^p(X)} \\
&+ C_{p,X} \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}
\end{aligned} \tag{4.10}
$$

with a constant $C_{p,X}$ independent of m and T; notice that to choose $C_{p,X}$ independently of m we resort to the equivalence of the norms in (4.2).

To obtain the asserted result, it remains to estimate the first term on the right-hand sides of (4.9) and (4.10) in a suitable way. Let

$$
Z_m := \sum_{i=1}^{q} \|(g_{ni})_{n=0}^{m-1}\|_{\ell^p(X)}^p = k \sum_{\ell=0}^{m-1} \sum_{i=1}^{q} \|\big(A(t_m) - A(t_{\ell i})\big)U_{\ell i}\|_X^p
$$

and

$$
E_\ell := k \sum_{j=0}^{\ell-1} \sum_{i=1}^{q} \|A(t_m)U_{ji}\|_X^p, \quad \ell = 1, \dots, m, \quad E_0 := 0.
$$

Now, according to estimate (4.10), we have

$$
E_m \leqslant C \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}^p + C Z_m. \tag{4.11}
$$

Furthermore, in view of the Lipschitz condition (4.3),

$$
Z_m \leqslant kL \sum_{\ell=0}^{m-1} (t_m - t_\ell)^p \sum_{i=1}^{q} \|A(t_m)U_{\ell i}\|_X^p = L \sum_{\ell=1}^{m} (t_m - t_\ell)^p (E_\ell - E_{\ell-1}),
$$

whence, by summation by parts, we have

$$
Z_m \leqslant L \sum_{\ell=1}^{m} a_\ell E_\ell, \tag{4.12}
$$

with $a_\ell := (t_m - t_{\ell-1})^p - (t_m - t_\ell)^p$, and (4.11) yields

$$
E_m \leqslant C \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}^p + C \sum_{\ell=1}^{m} a_\ell E_\ell. \tag{4.13}
$$

Since the sum $\sum_{\ell=1}^{m} a_\ell$ is uniformly bounded,

$$
\sum_{\ell=1}^{m} a_\ell = \big(t_m - t_0\big)^p \leqslant T^p,
$$

a discrete Gronwall-type argument applied to (4.13) leads to

$$
E_m \leqslant C \sum_{i=1}^{q} \|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}^p. \tag{4.14}
$$

Combining (4.10) with (4.12) and (4.14), we obtain the asserted maximal parabolic regularity stability estimate (4.5) with a constant $C_{p,X,T}$ independent of m and k. Analogously, from (4.9), we obtain (4.6). $\square$

Remark 4.1 (Bounded variation condition) The Lipschitz condition (4.3) in Proposition 4.1 can be relaxed to a bounded variation condition, namely,

$$\|\big(A(t)-A(\tilde t)\big)v\|_X \leqslant [\sigma(t)-\sigma(\tilde t)]\|A(\tau)v\|_X, \quad 0\leqslant \tilde t\leqslant t\leqslant T,\ \forall v\in \mathscr{D}(A), \tag{4.15}$$

for all $\tau\in[0,T]$, with an increasing function $\sigma:[0,T]\to\mathbb{R}$; cf., e.g., [9].

Indeed, in this case, we have

$$Z_m \leqslant k\sum_{\ell=0}^{m-1}[\sigma(t_m)-\sigma(t_\ell)]^p\sum_{i=1}^{q}\|A(t_m)U_{\ell i}\|_X^p=\sum_{\ell=1}^{m}[\sigma(t_m)-\sigma(t_{\ell-1})]^p(E_\ell-E_{\ell-1}),$$

whence, by summation by parts,

$$Z_m\leqslant \sum_{\ell=1}^{m}a_\ell E_\ell, \tag{4.16}$$

with $a_\ell:=[\sigma(t_m)-\sigma(t_{\ell-1})]^p-[\sigma(t_m)-\sigma(t_\ell)]^p\geqslant 0$, and (4.11) yields

$$E_m\leqslant C\sum_{i=1}^{q}\|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}^p+C\sum_{\ell=1}^{m}a_\ell E_\ell. \tag{4.17}$$

Since the sum $\sum_{\ell=1}^m a_\ell$ is uniformly bounded by a constant independent of m and the time step k,

$$\sum_{\ell=1}^{m}a_\ell=\big[\sigma(t_m)-\sigma(t_0)\big]^p\leqslant\big[\sigma(T)-\sigma(0)\big]^p,$$

a discrete Gronwall-type argument applied to (4.17) leads to

$$E_m\leqslant C\sum_{i=1}^{q}\|(f(t_{ni}))_{n=0}^{m-1}\|_{\ell^p(X)}^p \tag{4.18}$$

and the proof can be completed as in the case of the Lipschitz condition.

4.2 A posteriori error estimates

Let R be the *residual* of the collocation approximate solution $\widehat U$,

$$R(t):=\widehat U'(t)+A(t)\widehat U(t)-f(t),\quad t\in(t_n,t_{n+1}],\quad n=0,\dots,N-1, \tag{4.19}$$

cf. (1.9), i.e., the amount by which $\widehat U$ misses being an exact solution of the differential equation in (4.1). Then, the error $e:=u-\widehat U$ satisfies the *error equation*

$$e'(t)+A(t)e(t)=-R(t),\quad t\in(t_n,t_{n+1}],\quad n=0,\dots,N-1. \tag{4.20}$$

Let us now fix a $\tau\in(0,T)$. To apply the maximal L^p-regularity of the operator $A(\tau)$, for a frozen τ, we rewrite (4.20) in the form

$$e'(t)+A(\tau)e(t)=[A(\tau)-A(t)]e(t)-R(t),\quad t\in(0,\tau]. \tag{4.21}$$

Then, the maximal L^p-regularity of $A(\tau)$, applied to (4.21) yields the preliminary estimate

$$\begin{aligned}\|e'\|_{L^p((0,\tau);X)} + \|A(\tau)e\|_{L^p((0,\tau);X)} \leqslant c_{p,X}\|[A(\tau)-A(\cdot)]e\|_{L^p((0,\tau);X)}\\ + c_{p,X}\|R\|_{L^p((0,\tau);X)}\end{aligned} \tag{4.22}$$

for all $0 < \tau \leqslant T$, for any $p \in (1,\infty)$, with a constant $c_{p,X}$ independent of τ, depending only on p and X.

With

$$\eta(t) := \|A(\tau)e\|^p_{L^p((0,t);X)} = \int_0^t \|A(\tau)e(s)\|^p_X \,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau,$$

estimate (4.22) yields

$$\eta(\tau) \leqslant C\|[A(\tau)-A(\cdot)]e\|^p_{L^p((0,\tau);X)} + C\|R\|^p_{L^p((0,\tau);X)}, \quad 0 \leqslant \tau \leqslant T. \tag{4.23}$$

Now, in view of the Lipschitz condition (4.3),

$$\begin{aligned}\|[A(\tau)-A(\cdot)]e\|^p_{L^p((0,\tau);X)} &= \int_0^\tau \|[A(\tau)-A(t)]e(t)\|^p_X \,\mathrm{d}t\\ &\leqslant L^p \int_0^\tau (\tau-t)^p \|A(\tau)e(t)\|^p_X \,\mathrm{d}t,\end{aligned}$$

i.e.,

$$\|[A(\tau)-A(\cdot)]e\|^p_{L^p((0,\tau);X)} \leqslant L^p \int_0^\tau (\tau-t)^p \eta'(t)\,\mathrm{d}t,$$

and integration by parts yields

$$\|[A(\tau)-A(\cdot)]e\|^p_{L^p((0,\tau);X)} \leqslant L^p p \int_0^\tau (\tau-t)^{p-1} \eta(t)\,\mathrm{d}t. \tag{4.24}$$

From (4.23) and (4.24), we obtain

$$\eta(\tau) \leqslant C \int_0^\tau (\tau-t)^{p-1}\eta(t)\,\mathrm{d}t + C\|R\|^p_{L^p((0,\tau);X)}, \quad 0 \leqslant \tau \leqslant T,$$

whence, via a Gronwall inequality,

$$\eta(\tau) \leqslant C'\|R\|^p_{L^p((0,\tau);X)}, \quad 0 \leqslant \tau \leqslant T, \tag{4.25}$$

with a constant C' depending also on L and T.

Now, (4.24) and (4.25) yield

$$\|[A(\tau)-A(\cdot)]e\|_{L^p((0,\tau);X)} \leqslant c\|R\|_{L^p((0,\tau);X)}$$

and, in combination with (4.22), the desired a posteriori error estimate

$$\|e'\|_{L^p((0,\tau);X)} + \|A(\tau)e\|_{L^p((0,\tau);X)} \leqslant c\|R\|_{L^p((0,\tau);X)}, \quad 0 < \tau \leqslant T, \tag{4.26}$$

for any $p \in (1,\infty)$, with a constant c depending on p, X, L, and T. Notice that, in view of the equivalence of the norms $\|A(t)\cdot\|_X$, the constant c can be chosen independently of τ.

As in the case of autonomous equations, we can see that the a posteriori error estimator on the right-hand side of (4.26) is of optimal order.

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