MAXIMUM NORM ANALYSIS OF IMPLICIT–EXPLICIT BACKWARD DIFFERENCE FORMULAE FOR NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. We establish optimal order *a priori* error estimates for implicit–explicit BDF methods for abstract semilinear parabolic equations with time-dependent operators in a complex Banach space setting, under a sharp condition on the non-self-adjointness of the linear operator. Our approach relies on the discrete maximal parabolic regularity of implicit BDF schemes for autonomous linear parabolic equations, recently established in [19], and on ideas from [6]. We illustrate the applicability of our results to four initial and boundary value problems, namely two of second order, one of fractional order, and one of fourth order, that is the Cahn–Hilliard, parabolic equations.

1. INTRODUCTION

Let $V \hookrightarrow H = H' \hookrightarrow V'$ be a Gelfand triple of complex Hilbert spaces such that the restriction of the antiduality pairing $\langle \cdot, \cdot \rangle$ between V' and V to $H \times V$ coincides with the inner product $(\cdot, \cdot)$ on H. Let $T > 0, u_0 \in H$ and consider an abstract initial value problem for a possibly nonlinear parabolic equation,

$$(1.1) \qquad \begin{cases} u'(t) + A(t)u(t) = B(t, u(t)), & 0 < t < T, \\ u(0) = u_0; \end{cases}$$

here $A(t) : V \to V'$ are bounded linear operators, whereas $B(t, \cdot) : V \cap W \to V'$ are nonlinear operators defined in the intersection of V with another Banach space W. In this article, we study the stability of the implicit–explicit backward difference formula (BDF) methods for the time discretization of (1.1) and derive optimal order *a priori* error estimates.

1.1. Examples of initial and boundary value problems.
Examples of the abstract problem (1.1), to which our analysis applies, include (but are not restricted to) the following four types of nonlinear parabolic partial differential equations.

Example 1.1. Consider the following initial and boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^d$, with smooth boundary $\partial\Omega$,

(I)
$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \nabla \cdot \big((a(x,t) + \mathrm{i}\,b(x,t))\nabla u\big) \\
\qquad = f(u,x,t) + \nabla \cdot \boldsymbol{g}(u,x,t) & \text{in } \Omega \times (0,T), \\
u = 0 & \text{on } \partial\Omega \times (0,T), \\
u(\cdot,0) = u_0 & \text{in } \Omega,
\end{cases}
$$

where $a(x,t) > 0$ and $b(x,t)$ are smooth real-valued functions in $\overline{\Omega} \times [0,T]$, and the functions f and $\boldsymbol{g}$ are smooth with respect to u, not necessarily globally Lipschitz continuous. For example, $f(v,x,t) = -v^3$ and $\boldsymbol{g}(v,x,t) = (\mathrm{e}^v, 0, \ldots, 0)$ are allowed. In this case, we have $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $W = L^\infty(\Omega)$. Then, the operators $B(t,v) := f(v,x,t) + \nabla \cdot \boldsymbol{g}(v,x,t)$ are well defined as nonlinear maps from $V \cap W$ to V'.

Example 1.2. Consider the following initial and boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^d$, with smooth boundary $\partial\Omega$, this time with stronger nonlinearity,

(II)
$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \nabla \cdot \big((a(x,t) + \mathrm{i}\,b(x,t))\nabla u\big) \\
\qquad = f(u,\nabla u,x,t) + \nabla \cdot \boldsymbol{g}(u,\nabla u,x,t) & \text{in } \Omega \times (0,T), \\
u = 0 & \text{on } \partial\Omega \times (0,T), \\
u(\cdot,0) = u_0 & \text{in } \Omega,
\end{cases}
$$

where a and b are as in (I) while the functions f and $\boldsymbol{g}$ are smooth with respect to u and ∇u. For instance, $f(v,\nabla v,x,t) = -|\nabla v|^4 v$ and $\boldsymbol{g}(v,\nabla v,x,t) = |\nabla v|^4 \nabla v$ are allowed. In this case, we have $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $W = W^{1,\infty}(\Omega)$. Then, the operators $B(t,v) := f(v,\nabla v,x,t) + \nabla \cdot \boldsymbol{g}(v,\nabla v,x,t)$ are well defined as nonlinear maps from $V \cap W$ to V'.

Example 1.3. Consider the Cauchy problem for a fractional partial differential equation in $\mathbb{R}^d$,

(III)
$$
\begin{cases}
\dfrac{\partial u}{\partial t} + (-\Delta)^{1/2} u = f(u) & \text{in } \mathbb{R}^d \times (0,T), \\
u(\cdot,0) = u_0 & \text{in } \mathbb{R}^d,
\end{cases}
$$

with f a given smooth function of u such that $f(0) = 0$. For example, $f(v) = \mathrm{e}^v - 1$. In this case, we have $V = H^{\frac{1}{2}}(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^d)$ and $W = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, the operators $B(t,v) := f(v)$ are well defined as nonlinear maps from $V \cap W$ to V'.

Example 1.4. Consider the Cauchy problem for the Cahn–Hilliard equation in $\mathbb{R}^d$,

(IV)
$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \Delta^2 u = \Delta f(u) & \text{in } \mathbb{R}^d \times (0,T), \\
u(\cdot,0) = u_0 & \text{in } \mathbb{R}^d,
\end{cases}
$$

with f a given smooth function of u, such as $f(v) = v^3 - v$; see [12]. In this case, we have $V = H^2(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^d)$ and $W = H^2(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. Then, the

operators $B(t, v) := \Delta f(v) = f'(v)\Delta v + f''(v)|\nabla v|^2$ are well defined as nonlinear maps from $V \cap W$ to V'.

1.2. **Numerical methods.** For $k = 1, \ldots, 6$, consider the implicit k-step BDF method (δ, β) and the explicit k-step method (δ, γ) generated by the polynomials δ, β and γ,

(1.2)
$$\begin{cases} \delta(\zeta) = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-\zeta)^\ell = \sum_{i=0}^{k} \delta_i \zeta^i, \quad \beta(\zeta) = 1, \\ \gamma(\zeta) = \frac{1}{\zeta}\big[1 - (1-\zeta)^k\big] = \sum_{i=0}^{k-1} \gamma_i \zeta^i. \end{cases}$$

The BDF method (δ, β) is known to have order k and to be A(α_k)-stable with angles $\alpha_1 = \alpha_2 = 90°, \alpha_3 = 86.03°, \alpha_4 = 73.35°, \alpha_5 = 51.84°$ and $\alpha_6 = 17.84°$; see [17, Section V.2]. A(α)-stability is equivalent to $|\arg\delta(\zeta)| \le 180° - \alpha$ for $|\zeta| \le 1$. Note that the first- and second-order BDF methods are A-stable, that is $\operatorname{Re}\delta(\zeta) \ge 0$ for $|\zeta| \le 1$. For a given polynomial δ, the scheme (δ, γ) is the unique explicit k-step scheme of order k; the order of all other explicit k-step schemes $(\delta, \tilde{\gamma})$ is at most $k - 1$.

Let $N \in \mathbb{N}, N \ge k$, and consider a uniform partition $t_n := n\tau, n = 0, \ldots, N$, of the interval $[0, T]$, with time step $\tau := T/N$. Since the nonlinear operators $B(t, \cdot)$ on the right-hand side of (1.1) are only defined on $V \cap W$, we shall choose some Banach spaces $D \subset V \cap W$ and X such that

(1.3)
$$\begin{array}{ccccc} V & \subset & H & \subset & V' \\ \cup & & & & \cup \\ D & \subset & W & \subset & X \end{array} \quad \text{and} \quad \begin{array}{rl} A(t): & D \to X, \\ B(t, \cdot): & D \to X, \end{array}$$

and assume that we are given starting approximations $u_1, \ldots, u_{k-1} \in D$ to the nodal values $u_j^\star := u(t_j), j = 1, \ldots, k - 1$. We discretize (1.1) in time by the implicit–explicit k-step BDF method (δ, β, γ), i.e., we define approximations $u_m \in D$ to the nodal values $u_m^\star := u(t_m)$ of the exact solution as follows

(1.4)
$$\frac{1}{\tau}\sum_{i=0}^{k} \delta_i u_{n-i} + A(t_n)u_n = \sum_{i=0}^{k-1} \gamma_i B(t_{n-i-1}, u_{n-i-1}), \quad n = k, \ldots, N.$$

In other words, the linear part $A(t)u(t)$ of the equation in (1.1) is discretized by the implicit BDF scheme (δ, β), whereas the nonlinear part $B(t, u(t))$ is discretized by the explicit BDF scheme (δ, γ). As a result, the unknown u_n appears only on the left-hand side of the implicit–explicit BDF scheme (1.4); therefore, to advance in time, one only needs to solve one linear equation, which reduces to a linear system if one discretizes also in space, at each time level.

Motivated by Examples 1.1–1.4, we only require that the nonlinear operators $B(t, \cdot)$ are Lipschitz continuous in a tube $T_{u,r}^D$,

(1.5)
$$T_{u,r}^D := \{v \in D : \min_{0 \le t \le T}\|v - u(t)\|_W \le r\},$$

around the solution u, uniformly in t, where W may be a suitably chosen L^∞-based Sobolev space in practical applications, such as $L^\infty(\Omega)$ or $W^{1,\infty}(\Omega)$, depending on the type of the nonlinearity. The main difficulty in numerical analysis of such

problems is that one has to prove uniform boundedness of the numerical solutions $u_n, n = k, \ldots, N$, with respect to the norm of W. To overcome this difficulty, we study the stability of the implicit–explicit BDF methods for (1.1) in a Banach space setting, by using the mathematical tool of discrete maximal L^p-regularity. In contrast to the present approach, in [4, 2, 7, 5] the local Lipschitz condition was imposed in tubes $T_{u,r}^V$,

$$(1.6) \qquad T_{u,r}^V := \{v \in V : \min_{0 \le t \le T} \|v - u(t)\|_V \le r\},$$

defined in terms of the norm $\| \cdot \|_V$; as a consequence, the analysis of [4, 2, 7, 5] is not directly applicable to Examples 1.1–1.4, if we only consider the discretization in time, since it cannot ensure that the approximations are sufficiently close to the exact solution in the norm $\| \cdot \|_W$; it is, however, applicable, usually under mild mesh-conditions, in the fully discrete case, i.e., if we combine the time stepping schemes with discretization in space; cf., e.g., [4]. This analysis allows us to avoid growth conditions on the nonlinearities.

1.3. **Continuous and discrete maximal parabolic regularity.** Our approach is based on the discrete maximal parabolic regularity property of the implicit BDF methods. Let us briefly recall the relevant definitions: An elliptic differential operator $-A$ on a Banach space $\left(X, \|\cdot\|_X\right)$ has maximal L^p-regularity, $1 < p < \infty$, if the solution u of the initial value problem

$$(1.7) \qquad u'(t) + Au(t) = f(t), \quad 0 < t < T, \quad u(0) = 0,$$

with forcing term $f \in L^p(0, T; X)$, satisfies the *a priori* estimate

$$(1.8) \qquad \int_0^T \|u'(t)\|_X^p \mathrm{d}t + \int_0^T \|Au(t)\|_X^p \mathrm{d}t \le C \int_0^T \|f(t)\|_X^p \mathrm{d}t$$

with some constant C. In other words, if both terms u' and Au on the left-hand side of the autonomous parabolic equation are well defined and have the same (i.e., maximal) regularity as the forcing term f. It is well known that if an operator has maximal L^p-regularity for some $1 < p < \infty$, then it has maximal L^p-regularity for all $1 < p < \infty$. Replacing A by $A(t)$ both in (1.7) and (1.8), the definition extends to nonautonomous parabolic equations with a family of elliptic differential operators $-A(t), t \in [0, T]$, on X with the same domain, $D = D(A(t)), t \in [0, T]$. Maximal regularity is an important tool in the theory of nonlinear parabolic equations. For an excellent account of the maximal regularity theory, in particular, for the important Weis' characterization on unconditional martingale difference (UMD) spaces (which include $L^q(\Omega), 1 < q < \infty$), and for relevant references, we refer to the lecture notes by Kunstmann and Weis [20]. Space-discrete analogues of the maximal parabolic regularity, uniform in the spatial mesh size, can be found in [14, 15, 21, 22].

For the time-discrete maximal parabolic regularity property of autonomous parabolic equations, uniformly in the time step, we refer to [19] and the references therein. The main result of [19] is that A-stable Runge–Kutta methods, satisfying minor additional conditions, such as Gauss–Legendre and Radau IIA methods, as well as one- and two-step BDF methods preserve maximal regularity; it is also shown in [19] that higher-order k-step BDF methods, $k = 3, \ldots, 6$, preserve

maximal regularity under some natural additional conditions on the operator accounting for the lack of A-stability of these methods.

Here we establish local stability of the implicit–explicit BDF methods (1.4) for (1.1) under smallness conditions on the *stability constants* λ, which is equal to 1 in the case of self-adjoint operators while $\lambda - 1$ may be viewed as a measure of the non-self-adjointness of the linear operators $A(t)$ (see (2.5) in the sequel), for $k = 3, \ldots, 6$, and $\tilde{\lambda}$, the constant in the local Lipschitz condition on the nonlinear operators $B(t, \cdot), t \in [0, T]$, (see (2.6) in the sequel). While we can quantify the condition on the stability constant λ, actually in a sharp way, unfortunately we cannot quantify the condition on $\tilde{\lambda}$, since we have no control on the constant C in the discrete maximal regularity of the implicit BDF schemes (see (2.3) in the sequel); therefore, we shall assume that $\tilde{\lambda}$ is sufficiently small, in the case $k = 3, \ldots, 6$ depending also on the value of λ. In the applications, in case the differential operators $B(t, \cdot), t \in [0, T]$, are of order lower than the order of the linear differential operators $A(t)$, the Lipschitz constant $\tilde{\lambda}$ in the local Lipschitz condition (2.6) can typically be chosen arbitrarily small; as we will see, this is, in particular, the case for Examples 1.1, 1.3, and 1.4.

More precisely, we shall assume that λ does not exceed $1/\cos \alpha_k$,

$$(1.9) \qquad \lambda < \frac{1}{\cos \alpha_k}.$$

This is actually a sharp condition on the non-self-adjointness of the linear operators $A(t)$, in the sense that if λ exceeds the right-hand side in (1.9), then the (implicit) k-step BDF method is in general unstable for the linear equation $u'(t) + A(t)u(t) = 0$. Indeed, for $k = 1$ and $k = 2$ condition (1.9) is void, and, for $k = 3, \ldots, 6$, letting $\tilde{A}$ be a positive definite self-adjoint operator and considering the "rotated" operator $A := e^{i\varphi}\tilde{A}$, we see that condition (2.5) is satisfied as an equality with $\lambda = 1/\cos \varphi$. According to the von Neumann stability criterion, the k-step BDF method is not unconditionally stable for the equation $u'(t) + Au(t) = 0$, if $\alpha_k < \varphi < 90°$. Indeed, the spectrum of A is unbounded and its eigenvalues lie on the ray $\ell_\varphi := \{\rho e^{i\varphi}, \rho > 0\}$, which is outside of the stability sector $\Sigma_{\alpha_k} := \{z = r e^{i\vartheta} : r \geq 0, |\vartheta| \leq \alpha_k\}$ of the k-step BDF scheme. By definition, for $\alpha_k < \varphi < 90°$ and φ sufficiently close to α_k, the ray ℓ_φ is not entirely contained in the stability region S of the method; if λ is an eigenvalue of $\tilde{A}$, then the method is unstable for $u'(t) + Au(t) = 0$ for all time steps τ such that $\tau e^{i\varphi}\lambda \notin S$; since there exists an unbounded sequence of positive eigenvalues of $\tilde{A}$, it is impossible to find a positive τ_0 such that the k-step BDF method be stable for this equation for all time steps $0 < \tau < \tau_0$.

Let us note that in the case of the linear operators of the differential equations in the initial and boundary value problems of Examples 1.1 and 1.2, condition (1.9) takes the form

$$(1.10) \qquad \lambda = \max_{\substack{x \in \bar{\Omega} \\ t \in [0,T]}} \frac{|a(x,t) + i\, b(x,t)|}{a(x,t)} < \frac{1}{\cos \alpha_k},$$

which can also be equivalently written as

$$(1.11) \qquad \max_{\substack{x \in \bar{\Omega} \\ t \in [0,T]}} \frac{|b(x,t)|}{a(x,t)} < \tan \alpha_k.$$

Implicit–explicit multistep methods, and in particular implicit–explicit BDF schemes, were introduced and analyzed for nonautonomous linear parabolic equations in [13]. In a Hilbert space setting, implicit–explicit BDF methods can be analyzed by various techniques, such as spectral and Fourier techniques (see, e.g., [4, 3]), and energy methods (see, e.g., [2, 7, 5]); both techniques have advantages and drawbacks. In a Banach space setting, the analysis of implicit BDF methods for autonomous linear parabolic equations in [19] is based on maximal regularity, whereas the analysis of implicit as well as of linearly implicit BDF methods for quasilinear parabolic equations with real symmetric coefficients in [6] combines maximal regularity and energy techniques. To our best knowledge, implicit–explicit BDF schemes for nonlinear parabolic equations, in particular with complex coefficients, have not been previously analyzed in a Banach space setting.

Error estimates under sharp stability conditions of the form (1.9) were established in the Hilbert space setting by spectral and Fourier techniques in [25] for implicit multistep methods, including BDF schemes, for linear parabolic equations, and in [3] for implicit–explicit multistep methods, including implicit–explicit BDF schemes, for a class of nonlinear parabolic equations with linear operators of a special form; more precisely, for linear operators of the form considered in (I) and (II), the assumption in [3] is that a is independent of t and b is of the form $b(x,t) = \tilde{b}(t)a(x)$.

For further stability analyses of implicit multistep methods for autonomous linear parabolic equations in a Banach space setting, we refer to [24] and references therein; in particular, in [24] stability under the optimal condition (1.9) is established.

An outline of the paper is as follows: In Section 2 we present our abstract framework and discuss its applicability to the cases of the initial and boundary value problems of Examples 1.1–1.4. In Section 3 we establish discrete maximal regularity of BDF methods for nonautonomous linear parabolic equations, thus extending recent results of [19] concerning autonomous parabolic equations. Section 4 is devoted to the angle of analyticity of nonautonomous linear operators under assumptions (A1) and (A3); a lower bound for this angle is given. Our main results are presented in Sections 5 and 6: We first establish local stability of the implicit–explicit BDF schemes (1.4) in Section 5, which is then combined with the consistency of the methods and leads to optimal order a priori error estimates in Section 6. In Section 7 we verify the applicability of our abstract framework to the initial and boundary value problems (I)–(IV) of Examples 1.1–1.4, respectively, in the concrete spaces given in Propositions 2.1–2.3.

2. Abstract framework and applications to Examples 1.1–1.4

In this section we present our abstract framework, which is, in particular, applicable to Examples 1.1–1.4.

For a sequence $(v_n)_{n=1}^N$ and a given stepsize τ, we shall use the notation

$$\left\|(v_n)_{n=1}^N\right\|_{L^p(X)} = \left(\tau \sum_{n=1}^N \|v_n\|_X^p\right)^{1/p},$$

which is the $L^p(0, N\tau; X)$ norm of the piecewise constant function taking the value v_n in the subinterval $(t_{n-1}, t_n], n = 1, \ldots, N$.

For any two Banach spaces X and Y which are imbedded into a common Hausdorff topological space, we denote by $X \cap Y$ the Banach space consisting of elements in both X and Y, equipped with the norm

$$\|v\|_{X \cap Y} := \|v\|_X + \|v\|_Y.$$

We will work with the Banach space setting under the following assumptions:

(A1) (*Generation of bounded analytic semigroups by the linear operators*)
   If $v_0 \in H$, then, for all $s \in [0, T]$, there exists a unique solution $v \in H^1(\mathbb{R}_+; V') \cap L^2(\mathbb{R}_+; V) \hookrightarrow C([0, \infty); H)$ of the initial value problem

(2.1)
$$\begin{cases} v'(t) + A(s)v(t) = 0, & t > 0, \\ v(0) = v_0. \end{cases}$$

   The solution map $E_s^H(t) : H \to H$, which maps v_0 to $v(t)$, extends to a bounded analytic semigroup $\{E_s^H(z)\}_{z \in \Sigma_{\theta_s}}$ on H, where $\theta_s \in (0, \pi/2]$ is the maximal angle of analyticity (i.e., the supremum of all such angles). Moreover, the domain $D(A_H(s)) = D_H \hookrightarrow V$ of the generator $-A_H(s)$ of the semigroup $\{E_s^H(z)\}_{z \in \Sigma_{\theta_s}}$ is supposed to be independent of $s \in [0, T]$ and compactly imbedded into H, with

$$\theta := \inf_{s \in [0,T]} \theta_s > 0.$$

(A2) (*Discrete maximal regularity of the implicit BDF schemes*)
   There exist Banach spaces D and X satisfying (1.3). Moreover, if $\theta > \pi/2 - \alpha_k$, then, for all $s \in [0, T]$, the k-step BDF solution determined by

(2.2)
$$\frac{1}{\tau} \sum_{j=0}^k \delta_j v_{n-j} + A(s)v_n = f_n, \quad n = k, \ldots, N,$$

   with given $f_n \in X$ and given starting values $v_0, \ldots, v_{k-1} \in D$, is bounded by

(2.3)
$$\frac{1}{\tau}\left\|(v_n - v_{n-1})_{n=k}^N\right\|_{L^p(X)} + \left\|(v_n)_{n=k}^N\right\|_{L^p(D)}$$
$$\leq C\left(\left\|(f_n)_{n=k}^N\right\|_{L^p(X)} + \frac{1}{\tau}\left\|(v_i)_{i=0}^{k-1}\right\|_{L^p(X)} + \left\|(v_i)_{i=0}^{k-1}\right\|_{L^p(D)}\right),$$

   where the constant C is independent of τ and $s \in [0, T]$.

(A3) (*Boundedness and bounded variation of $A(t) : D \to X$, and coercivity of $A(t) : V \to V'$*)
   There exist two positive constants M_1 and M_2 such that the operator norms

$\|A(t)\|_{\mathcal{L}(D,X)}$ are uniformly bounded by M_1, for all $t \in [0,T]$, and

(2.4) $$\sum_{i=1}^{m} \|A(\tau_i) - A(\tau_{i-1})\|_{\mathcal{L}(D,X)} \le M_2$$

for any partition $0 = \tau_0 < \tau_1 < \cdots < \tau_m = T$ of $[0,T]$. There exists a constant $\lambda \ge 1$ such that

(2.5) $$|\langle A(t)v, v\rangle| \le \lambda \operatorname{Re}\langle A(t)v, v\rangle \quad \forall v \in V.$$

(A4) (*Local Lipschitz continuity of the nonlinear operators* $B(t,\cdot)$)
There exists a constant $r_0 > 0$ such that the operators $B(t,\cdot) : D \to X$ satisfy a local Lipschitz condition in a tube T_{u,r_0}^D; see (1.5); more precisely, there exist nonnegative constants $\tilde{\lambda}$ and C_B, independent of $t \in [0,T]$, such that, for all $v, w \in T_{u,r_0}^D$,

(2.6) $$\|B(t,v) - B(t,w)\|_X \le \tilde{\lambda}\|v-w\|_D + C_B(\|v\|_D + \|w\|_D)\|v-w\|_W.$$

(A5) (*Control of the W-norm by the maximal L^p-regularity*)
For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

(2.7) $$\|v\|_W \le \varepsilon\|v\|_D + C_\varepsilon\|v\|_X \quad \forall v \in D.$$

For some $1 < p < \infty$, we have a time-space continuous imbedding $W^{1,p}(0,T; X) \cap L^p(0,T; D) \hookrightarrow L^\infty(0,T; W)$: there exists a positive constant C_W such that, for all $v \in W^{1,p}(0,T;X) \cap L^p(0,T;D)$ with $v(0) = 0$,

(2.8) $$\|v\|_{L^\infty(0,T;W)} \le C_W\big(\|v'\|_{L^p(0,T;X)} + \|v\|_{L^p(0,T;D)}\big).$$

Remark 2.1. Similar assumptions in the Banach space setting (1.3) were recently used in [6]. Assumptions (A1) and (A2) are now connected through θ, the angle of analyticity, since we study parabolic equations with complex coefficients. Assumption (A3) is now relaxed to operators of bounded variation in time, possibly discontinuous.

Remark 2.2. Assumption (A1) guarantees that $A_H(s)v = A(s)v$ for $v \in D_H \hookrightarrow V$. In other words, $A(s)$ is an extension of the operator $A_H(s)$.

(A1)–(A5) are natural assumptions for studying many PDE problems, as can be seen in the following three Propositions.

Proposition 2.1 (The abstract framework is applicable to Examples 1.1–1.2)**.** *Let $q \in (d,\infty) \cap [2,\infty)$ and $p \in (1,\infty)$ be such that $2/p + d/q < 1$. Then, assumptions (A1)–(A5) are satisfied for the initial and boundary value problem* (I) *in Example 1.1 with $D_H = H^2(\Omega) \cap H_0^1(\Omega), D = W_0^{1,q}(\Omega), W = L^\infty(\Omega)$ and $X = W^{-1,q}(\Omega)$, with λ as on the left-hand side of* (1.10) *and $\tilde{\lambda} = 0$.*
Let $q \in (d,\infty)$ and $p \in (2,\infty)$ be such that $2/p + d/q < 1$. Then, assumptions (A1)–(A5) are satisfied for the initial and boundary value problem (II) *in Example 1.2 with $D_H = H^2(\Omega) \cap H_0^1(\Omega), D = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), W = W^{1,\infty}(\Omega)$ and $X = L^q(\Omega)$, with λ as on the left-hand side of* (1.10) *and*

(2.9) $$\tilde{\lambda} = \sup_{t \in [0,T]} \sum_{i,j=1}^{d} \sup_{x \in \Omega} \sup_{\substack{|\xi - u(x,t)| \le r \\ |\vec{\eta} - \nabla u(x,t)| \le r}} \left| \frac{\partial g_i(\xi, \vec{\eta}, x, t)}{\partial \eta_j} \right|.$$

Proposition 2.2 (The abstract framework is applicable to Example 1.3). *Let $q \in (d, \infty) \cap [2, \infty)$ and $p \in (1, \infty)$ be such that $1/p + d/q < 1$. Then, assumptions (A1)–(A5) are satisfied for the Cauchy problem (III) in Example 1.3 with $D_H = H^1(\mathbb{R}^d), D = H^1(\mathbb{R}^d) \cap W^{1,q}(\mathbb{R}^d), X = L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $W = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \hookrightarrow X$, with the stability constants $\lambda = 1$ and $\tilde{\lambda} = 0$.*

Proposition 2.3 (The abstract framework is applicable to Example 1.4). *Let $q \in (d/2, \infty) \cap [2, \infty)$ and $p \in (2, \infty)$ be such that $4/p + d/q < 2$. Then, assumptions (A1)–(A5) are satisfied for the Cauchy problem (IV) in Example 1.4 with $D_H = H^4(\mathbb{R}^d), D = H^4(\mathbb{R}^d) \cap W^{4,q}(\mathbb{R}^d), X = L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $W = H^2(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d) \hookrightarrow X$, with the stability constants $\lambda = 1$ and $\tilde{\lambda} = 0$.*

The proofs of Propositions 2.1–2.3 will be given in Section 7; the proofs of Propositions 2.2 and 2.3 are based on [11, Corollary 2.7 and Proposition 2.9] and [18, Example 3.2 (A)], respectively.

Remark 2.3 (Condition (2.5) expressed in terms of time-dependent norms). Following [2, 7], we introduce time-dependent norms and rewrite (2.5) in an equivalent way. The time-dependent norms are based on the decomposition of the coercive operators $A(t)$ in their self-adjoint and anti-self-adjoint parts $A_s(t)$ and $A_a(t)$, respectively,

$$A_s(t) := \frac{1}{2}\big[A(t) + A(t)^\star\big], \quad A_a(t) := \frac{1}{2}\big[A(t) - A(t)^\star\big].$$

Now, we endow V with the time-dependent norms $\|\cdot\|_t$,

$$\|v\|_t := \langle A_s(t)v, v \rangle^{1/2} \quad \forall v \in V,$$

which are uniformly equivalent to the norm $\|\cdot\|_V$, and denote by $\|\cdot\|_{\star,t}$ the corresponding dual norm on V',

$$\forall v \in V' \quad \|v\|_{\star,t} := \sup_{w \in V \setminus \{0\}} \frac{|\langle v, w \rangle|}{\|w\|_t} = \sup_{\substack{w \in V \\ \|w\|_t = 1}} |\langle v, w \rangle|.$$

Then, condition (2.5) simply says that the operators $A(t) : V \to V'$ are uniformly bounded and their norms do not exceed λ,

$$(2.10) \qquad \|A(t)v\|_{\star,t} \le \lambda \|v\|_t \quad \forall v \in V.$$

In the case of self-adjoint operators $A(t)$, (2.5) and (2.10) are satisfied with $\lambda = 1$; otherwise $\lambda > 1$.

Remark 2.4 (Equivalent form of (2.3) in the case $v_0 = 0$). If $v_0 = 0$, then (2.3) in (A2) can be equivalently written in a more symmetric form as

$$(2.11) \quad \begin{aligned} &\frac{1}{\tau}\big\|(v_n - v_{n-1})_{n=k}^N\big\|_{L^p(X)} + \big\|(v_n)_{n=k}^N\big\|_{L^p(D)} \\ &\le C\Big(\big\|(f_n)_{n=k}^N\big\|_{L^p(X)} + \frac{1}{\tau}\big\|(v_i - v_{i-1})_{i=1}^{k-1}\big\|_{L^p(X)} + \big\|(v_i)_{i=1}^{k-1}\big\|_{L^p(D)}\Big). \end{aligned}$$

3. Discrete maximal L^p-regularity of BDF methods

We will work with the abstract assumptions (A1)–(A3) of the previous section and will show that the (implicit) BDF methods satisfy the discrete maximal parabolic regularity property, when applied to initial value problems of the form (1.1) for linear parabolic equations, i.e., with right-hand side $B(t, u(t)) = f(t)$; this property is of independent interest and will also play a crucial role in our stability analysis of the implicit–explicit BDF methods in section 5. We thus extend the corresponding discrete maximal parabolic regularity result (A2) for autonomous parabolic equations of [19, Theorems 4.1–4.2] to the case of nonautonomous equations, with operators of bounded variation with respect to time.

Proposition 3.1 (Discrete maximal parabolic regularity). *Under assumptions (A1)–(A3), there exist positive constants τ_0 and C (independent of τ, but possibly depending on T) such that, for every stepsize $\tau \le \tau_0$, the k-step BDF method,*

$$(3.1) \qquad \frac{1}{\tau} \sum_{j=0}^{k} \delta_j v_{n-j} + A(t_n) v_n = f_n, \quad n = k, \ldots, N,$$

with given starting values $v_0, \ldots, v_{k-1} \in D$, satisfies the following stability property

$$(3.2) \qquad \begin{aligned} &\frac{1}{\tau} \big\| (v_n - v_{n-1})_{n=k}^N \big\|_{L^p(X)} + \big\| (v_n)_{n=k}^N \big\|_{L^p(D)} \\ &\le C \Big(\big\| (f_n)_{n=k}^N \big\|_{L^p(X)} + \frac{1}{\tau} \big\| (v_i)_{i=0}^{k-1} \big\|_{L^p(X)} + \big\| (v_i)_{i=0}^{k-1} \big\|_{L^p(D)} \Big), \end{aligned}$$

i.e., discrete maximal parabolic regularity.

Proof. For $k \le n \le m \le N$, we rewrite the numerical method (3.1) in the form

$$(3.3) \qquad \frac{1}{\tau} \sum_{j=0}^{k} \delta_j v_{n-j} + A_m v_n = f_n + (A_m - A_n) v_n,$$

with $A_j := A(t_j)$, and shall use a discrete perturbation argument. First, applying the discrete maximal regularity of the implicit k-step BDF method for autonomous equations, namely (2.3), to (3.3), we obtain the estimate

$$(3.4) \qquad \begin{aligned} &\frac{1}{\tau} \big\| (v_n - v_{n-1})_{n=k}^m \big\|_{L^p(X)} + \big\| (v_n)_{n=k}^m \big\|_{L^p(D)} \le C \big\| (f_n)_{n=k}^m \big\|_{L^p(X)} \\ &+ C \big\| ((A_m - A_n) v_n)_{n=k}^m \big\|_{L^p(X)} + C \Big(\frac{1}{\tau} \big\| (v_i)_{i=0}^{k-1} \big\|_{L^p(X)} + \big\| (v_i)_{i=0}^{k-1} \big\|_{L^p(D)} \Big). \end{aligned}$$

We now let $E_{k-1} := 0$ and

$$(3.5) \qquad E_\ell := \big\| (v_n)_{n=k}^\ell \big\|_{L^p(D)}^p = \tau \sum_{n=k}^{\ell} \| v_n \|_D^p, \quad \ell = k, \ldots, N,$$

and focus on the second term on the right-hand side of (3.4). Denoting for notational simplicity the operator norm $\| \cdot \|_{\mathcal{L}(D,X)}$ by $\| \cdot \|$, we first note that

$$\big\| ((A_m - A_n) v_n)_{n=k}^m \big\|_{L^p(X)}^p = \tau \sum_{n=k}^{m} \| (A_m - A_n) v_n \|_X^p$$

$$\leq \tau \sum_{n=k}^{m} \|A_m - A_n\|^p \|v_n\|_D^p = \sum_{n=k}^{m} \|A_m - A_n\|^p (E_n - E_{n-1}),$$

whence

$$(3.6) \qquad \big\|\big((A_m - A_n)v_n\big)_{n=k}^{m}\big\|_{L^p(X)}^p \leq \sum_{n=k}^{m-1} \big(\|A_m - A_n\|^p - \|A_m - A_{n+1}\|^p\big) E_n.$$

Now, since $\|A_j\| \leq M_1$, it is easily seen that

$$\big|\|A_m - A_n\|^p - \|A_m - A_{n+1}\|^p\big| \leq c_\star \big|\|A_m - A_n\| - \|A_m - A_{n+1}\|\big|$$

with $c_\star := p(2M_1)^{p-1}$; therefore,

$$(3.7) \qquad \big|\|A_m - A_n\|^p - \|A_m - A_{n+1}\|^p\big| \leq c_\star \|A_{n+1} - A_n\|,$$

and (3.6) yields

$$(3.8) \qquad \big\|\big((A_m - A_n)v_n\big)_{n=k}^{m}\big\|_{L^p(X)}^p \leq c_\star \sum_{n=k}^{m-1} \|A_{n+1} - A_n\| E_n.$$

Now, letting

$$F_m := \big\|(f_n)_{n=k}^{m}\big\|_{L^p(X)}^p + \Big(\frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(X)} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(D)}\Big)^p,$$

considering the pth power of the second term on the left-hand side of (3.4), and using (3.8), we have

$$E_m \leq Cc_\star \sum_{n=k}^{m-1} \|A_{n+1} - A_n\| E_n + CF_m,$$

i.e.,

$$(3.9) \qquad E_m \leq C \sum_{n=k}^{m-1} a_n E_n + CF_m, \quad m = k, \dots, N,$$

with $a_n := c_\star \|A_{n+1} - A_n\|$. In view of the bounded variation condition (2.4), the sum $\sum_{n=k}^{N} a_n$ is uniformly bounded by a constant independent of the time step τ; therefore, a discrete Gronwall-type argument applied to (3.9) yields

$$(3.10) \qquad E_m \leq CF_m, \quad m = k, \dots, N.$$

Now, in view of (3.10), estimate (3.8) yields

$$\big\|\big((A_m - A_n)v_n\big)_{n=k}^{m}\big\|_{L^p(X)}^p \leq c_\star \Big(\sum_{n=k}^{m-1} \|A_{n+1} - A_n\|\Big) CF_m,$$

whence

$$\big\|\big((A_m - A_n)v_n\big)_{n=k}^{m}\big\|_{L^p(X)}^p \leq cF_m.$$

Using here the definition of F_m, we obtain the desired estimate of the second term on the right-hand side of (3.4), namely

$$(3.11) \quad \begin{aligned} \big\|\big((A_m - A_n)v_n\big)_{n=k}^{m}\big\|_{L^p(X)} &\leq C\big\|(f_n)_{n=k}^{m}\big\|_{L^p(X)} \\ &+ C\Big(\frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(X)} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(D)}\Big). \end{aligned}$$

Combining (3.11) with (3.4), and letting $m = N$, we get the stability estimate (3.2). □

4. Angle of analyticity

Assuming (A1) and (A3), we show here that the angle θ of analyticity of the semigroups generated by $-A_H(s), s \in [0, T]$, exceeds $\arcsin(1/\lambda)$.

Lemma 4.1. *Under assumptions* (A1) *and* (A3), *we have* $\theta \geq \arcsin(1/\lambda)$.

Proof. It is known that $-A_H(s)$ generates an analytic semigroup in the sector Σ_ϑ, if and only if the following two conditions hold (see, e.g., [8, Theorem 3.7.11]):

  (i) $z + A_H(s)$ is invertible for $z \in \Sigma_{\varphi + \pi/2}$, for $\varphi \in (0, \vartheta)$;
  (ii) $z(z + A_H(s))^{-1}$ is uniformly bounded on H for $z \in \Sigma_{\varphi + \pi/2}$, for $\varphi \in (0, \vartheta)$.

The analyticity of the semigroup implies that $z + A_H(s)$ is invertible for $\mathrm{Re}(z) \geq 0$, and the compact imbedding $D_H \hookrightarrow\hookrightarrow H$ implies that $z + A_H(s)$ is a Fredholm operator of index zero. Hence, to verify (i), we need to prove only the injectivity of the operator $z + A_H(s)$. In fact, if $z \in \Sigma_{\varphi + \pi/2}$ with $\varphi = \arcsin(1/\lambda)$, then $w \in D_H \hookrightarrow V$ and $(z + A_H(s))w = 0$ imply

$$z\|w\|_H^2 + (A(s)w, w) = (zw, w) + (A_H(s)w, w) = 0;$$

thus, taking real parts, we have

$$\mathrm{Re}(z)\|w\|_H^2 + \mathrm{Re}(A(s)w, w) = 0.$$

By using (2.5) of assumption (A3), from the last two relations we see that

$$|z|\|w\|_H^2 = |(A(s)w, w)| \leq \lambda \mathrm{Re}(A(s)w, w) = -\lambda \mathrm{Re}(z)\|w\|_H^2,$$

whence

$$(\lambda \mathrm{Re}(z) + |z|)\|w\|_H^2 \leq 0.$$

Hence, since $\lambda \mathrm{Re}(z) + |z| > 0$ for $z \in \Sigma_{\varphi + \pi/2}$, it follows that $w = 0$. This shows the injectivity of the map $z + A_H(s) : D_H \to H$, which implies invertibility of this Fredholm operator. This proves (i) for $\vartheta = \arcsin(1/\lambda)$.

To verify (ii), we assume that $z \in \Sigma_{\varphi + \pi/2}$ with $\varphi < \arcsin(1/\lambda)$, and $z(z + A_H(s))^{-1}v = w$. Then $(z + A_H(s))w = zv$. Taking in this relation the inner product with w, we get

$$z\|w\|_H^2 + (A(s)w, w) = (zw, w) + (A_H(s)w, w) = (zv, w),$$

whence, taking real parts,

$$\mathrm{Re}(z)\|w\|_H^2 + \mathrm{Re}(A(s)w, w) = \mathrm{Re}(zv, w).$$

In view of the last two relations, we have

$$\begin{aligned}
|z|\|w\|_H^2 &\leq |(zv, w)| + |(A(s)w, w)| \\
&\leq |z|\|v\|_H\|w\|_H + \lambda \mathrm{Re}(A(s)w, w) \qquad \text{(in view of (A3))} \\
&= |z|\|v\|_H\|w\|_H + \lambda \mathrm{Re}(zv, w) - \lambda \mathrm{Re}(z)\|w\|_H^2 \\
&\leq (1 + \lambda)|z|\|v\|_H\|w\|_H - \lambda \mathrm{Re}(z)\|w\|_H^2,
\end{aligned}$$

which yields

$$(\mathrm{Re}(z)/|z| + 1/\lambda)\|w\|_H \leq (1/\lambda + 1)\|v\|_H.$$

Since $\operatorname{Re}(z)/|z| + 1/\lambda \geq -\sin\varphi + 1/\lambda > 0$ for $z \in \Sigma_{\varphi+\pi/2}$ with $\varphi < \arcsin(1/\lambda)$, it follows that

$$\|z(z + A_H(s))^{-1}v\|_H = \|w\|_H \leq \frac{1/\lambda + 1}{1/\lambda - \sin\varphi}\|v\|_H \quad \forall z \in \Sigma_\varphi.$$

Since this estimate is valid for arbitrary $\varphi < \arcsin(1/\lambda)$, it follows that (ii) is valid for $\vartheta = \arcsin(1/\lambda)$. The proof is complete. $\qquad\square$

5. STABILITY

In this section, we prove local stability of the implicit–explicit BDF schemes (1.4). We shall combine this stability result with the easily established consistency of the schemes to derive optimal order error estimates in Section 6.

Besides the approximations $u_n \in D, n = 0, \ldots, N$, satisfying (1.4), we consider the nodal values $u_m^\star := u(t_m)$ of the solution u of the initial value problem (1.1), which satisfy the perturbed equation

$$(5.1) \qquad \frac{1}{\tau}\sum_{i=0}^{k}\delta_i u_{n-i}^\star + A(t_n)u_n^\star = \sum_{i=0}^{k-1}\gamma_i B(t_{n-i-1}, u_{n-i-1}^\star) + d_n, \quad n = k, \ldots, N.$$

We assume for the time being, and shall verify in the next section, that the consistency error (d_n) is bounded by

$$(5.2) \qquad \big\|(d_n)_{n=k}^N\big\|_{L^p(X)} \leq \delta$$

and also that the errors of the starting approximations are bounded by

$$(5.3) \qquad \frac{1}{\tau}\big\|(u_i - u_i^\star)_{i=0}^{k-1}\big\|_{L^p(X)} + \big\|(u_i - u_i^\star)_{i=0}^{k-1}\big\|_{L^p(D)} \leq \delta,$$

with δ a sufficiently small constant. We then have the following stability results for the BDF solutions.

Proposition 5.1 (Stability of the implicit–explicit BDF schemes (1.4))**.** *Consider time discretization of the initial value problem* (1.1) *by the implicit–explicit k-step BDF method* (1.4)–(1.2)*, with $1 \leq k \leq 6$ and starting approximations $u_0, \ldots, u_{k-1} \in D$, and assume that the stability condition* (1.9) *is satisfied. Under the assumptions* (A1)–(A5) *and* (5.2)–(5.3)*, there exist positive constants $\tilde\lambda_0$ and δ_0 such that, for $\tilde\lambda \leq \tilde\lambda_0$ and $\delta \leq \delta_0$, the errors $e_n = u_n - u_n^\star$ between the solutions of* (1.4) *and* (5.1) *are bounded by*

$$(5.4) \qquad \frac{1}{\tau}\big\|(e_n - e_{n-1})_{n=k}^N\big\|_{L^p(X)} + \big\|(e_n)_{n=k}^N\big\|_{L^p(D)} \leq C\delta,$$

$$(5.5) \qquad \big\|(e_n)_{n=k}^N\big\|_{L^\infty(W)} \leq C\delta,$$

with a constant C depending on $\|(u_n^\star)_{n=0}^N\|_{L^\infty(W)}$, $\|(u_n^\star)_{n=0}^N\|_{L^p(D)}$, and T, but independent of δ and τ.

Proof. Subtracting (5.1) from (1.4), we obtain the following error equation, for the errors $e_n := u_n - u_n^\star$,

$$(5.6) \qquad \frac{1}{\tau}\sum_{i=0}^{k}\delta_i e_{n-i} + A(t_n)e_n = \sum_{i=0}^{k-1}\gamma_i b_{n-i-1} - d_n, \quad n = k, \ldots, N,$$

with the abbreviation $b_\ell := B(t_\ell, u_\ell) - B(t_\ell, u_\ell^\star), \ell = 0, \ldots, N-1$.

We let $r \in (0, r_0]$ be a small number to be determined later, and let $M \leq N$ be maximal such that

$$(5.7) \qquad \|(e_n)_{n=0}^{M-1}\|_{L^\infty(W)} \leq r.$$

Then $u_n \in T_{u,r}^D, n = 0, \ldots, M-1$, and assumption (A4) implies

$$\|b_\ell\|_X \leq \tilde{\lambda}\|e_\ell\|_D + C_B(\|u_\ell\|_D + \|u_\ell^\star\|_D)\|e_\ell\|_W$$
$$\leq \tilde{\lambda}\|e_\ell\|_D + C_B(\|e_\ell\|_D + 2\|u_\ell^\star\|_D)\|e_\ell\|_W$$
$$= (\tilde{\lambda} + C_B\|e_\ell\|_W)\|e_\ell\|_D + 2C_B\|u_\ell^\star\|_D\|e_\ell\|_W$$

and thus

$$(5.8) \qquad \|b_\ell\|_X \leq (\tilde{\lambda} + C_B r)\|e_\ell\|_D + 2C_B\|u_\ell^\star\|_D\|e_\ell\|_W, \quad \ell = 0, \ldots, M-1.$$

Let us denote by J_m the quantity

$$(5.9) \qquad J_m := \frac{1}{\tau}\|(e_n - e_{n-1})_{n=k}^m\|_{L^p(X)} + \|(e_n)_{n=k}^m\|_{L^p(D)},$$

which we want to estimate; cf. the left-hand side of (5.4). First, applying Proposition 3.1 to (5.6), we obtain, for all $m \leq M$,

$$J_m \leq C \left\|\left(\sum_{i=0}^{k-1} \gamma_i b_{n-i-1} - d_n\right)_{n=k}^m\right\|_{L^p(X)} + \frac{C}{\tau}\|(e_i)_{i=0}^{k-1}\|_{L^p(X)} + C\|(e_i)_{i=0}^{k-1}\|_{L^p(D)}$$

$$\leq C \left\|(b_n)_{n=0}^{m-1}\right\|_{L^p(X)} + C\|(d_n)_{n=k}^m\|_{L^p(X)} + \frac{C}{\tau}\|(e_i)_{i=0}^{k-1}\|_{L^p(X)} + C\|(e_i)_{i=0}^{k-1}\|_{L^p(D)}$$

$$\leq C \left\|(b_n)_{n=0}^{m-1}\right\|_{L^p(X)} + C\delta,$$

where we used (5.2) and (5.3); therefore, in view of (5.8),

$$J_m \leq C(\tilde{\lambda} + r)\|(e_n)_{n=0}^{m-1}\|_{L^p(D)} + C\|(e_n)_{n=0}^{m-1}\|_{L^p(W)} + C\delta.$$

Since (A5) implies

$$\|(e_n)_{n=0}^{m-1}\|_{L^p(W)} \leq \varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^p(D)} + C_\varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^p(X)},$$

the last two estimates yield

$$(5.10) \quad J_m \leq C(\tilde{\lambda}+r)\|(e_n)_{n=0}^{m-1}\|_{L^p(D)} + \varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^p(D)} + C_\varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^p(X)} + C\delta.$$

Now, with a sufficiently small positive constant $\tilde{\lambda}_0$, for $\tilde{\lambda} \leq \tilde{\lambda}_0$ and suitably small r and ε, we have $C(\tilde{\lambda} + r) + \varepsilon \leq 1/2$ (r is a fixed constant from now on). As a consequence, (5.10) yields

$$(5.11) \qquad J_m \leq C\|(e_n)_{n=0}^{m-1}\|_{L^p(X)} + C\delta.$$

Combining this estimate with the upper bound (5.3) of the starting errors, and setting $e_{-1} = 0$, we obtain

$$(5.12) \qquad \frac{1}{\tau}\|(e_n - e_{n-1})_{n=0}^m\|_{L^p(X)} + \|(e_n)_{n=0}^m\|_{L^p(D)} \leq C\|(e_n)_{n=0}^{m-1}\|_{L^p(X)} + C\delta.$$

Now, with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\|(e_n)_{n=0}^m\|_{L^\infty(X)} \leq \|e_0\|_X + \sum_{n=1}^m \|e_n - e_{n-1}\|_X$$

$$= \|e_0\|_X + \frac{1}{\tau}\|(e_n - e_{n-1})_{n=1}^m\|_{L^1(X)}$$

$$\leq \tau^{1/p'}\delta + \frac{T^{\frac{1}{p'}}}{\tau}\|(e_n - e_{n-1})_{n=1}^m\|_{L^p(X)}$$

$$\leq \tau^{1/p'}\delta + C\|(e_n)_{n=0}^{m-1}\|_{L^p(X)} + C\delta$$

$$\leq \varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^\infty(X)} + C_\varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^1(X)} + C\delta;$$

in the last step of this estimate we used the following inequality of [1, Theorem 2.11]:

$$\|(e_n)_{n=0}^{m-1}\|_{L^p(X)} \leq \|(e_n)_{n=0}^{m-1}\|_{L^\infty(X)}^{1-\frac{1}{p}}\|(e_n)_{n=0}^{m-1}\|_{L^1(X)}^{\frac{1}{p}}$$

$$\leq \varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^\infty(X)} + C_\varepsilon\|(e_n)_{n=0}^{m-1}\|_{L^1(X)}.$$

Therefore,

$$(5.13) \qquad \|(e_n)_{n=0}^m\|_{L^\infty(X)} \leq C\|(e_n)_{n=0}^{m-1}\|_{L^1(X)} + C\delta,$$

which holds for all $0 \leq m \leq M$. Then Gronwall's inequality implies $\|(e_n)_{n=0}^M\|_{L^\infty(X)} \leq C\delta$. Substituting this estimate into (5.12), we obtain

$$(5.14) \qquad \frac{1}{\tau}\|(e_n - e_{n-1})_{n=0}^M\|_{L^p(X)} + \|(e_n)_{n=0}^M\|_{L^p(D)} \leq C\delta.$$

Let $\tilde{e} \in W^{1,p}(-\tau, M\tau; X) \cap L^p(-\tau, M\tau; D)$ denote the piecewise linear interpolant of e_n, $n = -1, 0, 1, \ldots, M$, at the nodes t_n. Then we have

$$\|\tilde{e}'\|_{L^p(-\tau, M\tau; X)} + \|\tilde{e}\|_{L^p(-\tau, M\tau; D)} \leq C\left(\frac{1}{\tau}\|(e_n - e_{n-1})_{n=0}^M\|_{L^p(X)} + \|(e_n)_{n=0}^M\|_{L^p(D)}\right),$$

whence, in view of (5.14),

$$(5.15) \qquad \|\tilde{e}'\|_{L^p(-\tau, M\tau; X)} + \|\tilde{e}\|_{L^p(-\tau, M\tau; D)} \leq C\delta.$$

Now, according to assumption (A5), we have

$$\|(e_n)_{n=0}^M\|_{L^\infty(W)} = \|\tilde{e}\|_{L^\infty(-\tau, M\tau; W)} \leq C(\|\tilde{e}'\|_{L^p(-\tau, M\tau; X)} + \|\tilde{e}\|_{L^p(-\tau, M\tau; D)}),$$

and, using (5.15), we obtain

$$(5.16) \qquad \|(e_n)_{n=0}^M\|_{L^\infty(W)} \leq C\delta.$$

Estimates (5.14) and (5.16) imply the existence of a positive constant δ_0 such that $\|(e_n)_{n=0}^M\|_{L^\infty(W)} \leq r$ for $\delta \leq \delta_0$, contradicting the maximality of M unless $M = N$. Hence, (5.7), (5.14) and (5.16) are valid for $M = N$. The proof is complete. $\qquad \square$

Remark 5.1 (On the stability condition (1.9)). The stability condition (1.9) is void for $k = 1, 2$, and takes for the implicit–explicit k-step BDF method (1.4), $k = 3, \ldots, 6$, the form $\lambda < \lambda_k$ with

$$(5.17) \qquad \lambda_3 = 14.45087, \quad \lambda_4 = 3.49040, \quad \lambda_5 = 1.62892979, \quad \lambda_6 = 1.050513.$$

6. Error estimates

In this section we present our main result, maximum norm optimal order error estimates for the implicit–explicit BDF schemes (1.4).

Proposition 6.1 (Optimal order error estimates). *Assume* (A1)–(A5), *with stability constant satisfying* (1.9), *and that the starting approximations are such that*

$$(6.1) \qquad \frac{1}{\tau}\left\|(u_i - u_i^\star)_{i=0}^{k-1}\right\|_{L^p(X)} + \left\|(u_i - u_i^\star)_{i=0}^{k-1}\right\|_{L^p(D)} \leq C\tau^k.$$

If the solution u of (1.1) *is sufficiently regular, $u \in C^{k+1}([0,T];X)$ and $\widetilde{B}(t) := B(t,u(t)) \in C^k([0,T];X)$, then there exist positive constants $\tilde{\lambda}_0$ and τ_0 such that, for $\tilde{\lambda} \leq \tilde{\lambda}_0$ and $\tau \leq \tau_0$, the errors $e_n = u_n - u_n^\star$ between the approximate solutions u_n of* (1.4) *and the nodal values $u_n^\star$ of the solution u of* (1.1) *are bounded by*

$$(6.2) \qquad \frac{1}{\tau}\left\|(e_n - e_{n-1})_{n=k}^{N}\right\|_{L^p(X)} + \left\|(e_n)_{n=k}^{N}\right\|_{L^p(D)} \leq C\tau^k,$$

$$(6.3) \qquad \left\|(e_n)_{n=k}^{N}\right\|_{L^\infty(W)} \leq C\tau^k,$$

with a constant C independent of τ.

Proof. With Proposition 5.1 on the stability of the BDF solutions, we need to establish estimates for the consistency errors d_n only.

The order of the k-step methods (δ, β) and (δ, γ) is k, i.e.,

$$(6.4) \qquad \sum_{i=0}^{k}(k-i)^\ell \delta_i = \ell k^{\ell-1} = \ell \sum_{i=0}^{k-1}(k-i-1)^{\ell-1}\gamma_i, \quad \ell = 0, 1, \ldots, k.$$

The consistency error d_n of the scheme (1.4) for the solution u of (1.1), i.e., the amount by which the exact solution misses satisfying the implicit–explicit BDF scheme (1.4), is given by

$$(6.5) \qquad \tau d_n = \sum_{i=0}^{k}\delta_i u(t_{n-i}) + \tau A(t_n)u(t_n) - \tau\sum_{i=0}^{k-1}\gamma_i B\big(t_{n-i-1}, u(t_{n-i-1})\big),$$

$n = k, \ldots, N$; cf. (5.1). Letting

$$\begin{cases} d_{n,1} := \displaystyle\sum_{i=0}^{k}\delta_i u(t_{n-i}) - \tau u'(t_n), \\[2mm] d_{n,2} := \tau B\big(t_n, u(t_n)\big) - \tau\displaystyle\sum_{i=0}^{k-1}\gamma_i B\big(t_{n-i-1}, u(t_{n-i-1})\big), \end{cases}$$

and using the differential equation in (1.1), we infer that $\tau d_n = d_{n,1} + d_{n,2}$. Now, by Taylor expanding about t_{n-k} and using the order conditions of the implicit (δ, β)-scheme, i.e., the first equality in (6.4), and the second equality in (6.4), respectively, we obtain

$$d_{n,1} = \frac{1}{k!}\left[\sum_{i=0}^{k}\delta_i \int_{t_{n-k}}^{t_{n-i}}(t_{n-i}-s)^k u^{(k+1)}(s)\mathrm{d}s - k\tau\int_{t_{n-k}}^{t_n}(t_n-s)^{k-1}u^{(k+1)}(s)\mathrm{d}s\right],$$

$$d_{n,2} = \frac{\tau}{(k-1)!}\left[\int_{t_{n-k}}^{t_n}(t_n-s)^{k-1}\widetilde{B}^{(k)}(s)\mathrm{d}s - \sum_{i=0}^{k}\gamma_i\int_{t_{n-k}}^{t_{n-i-1}}(t_{n-i-1}-s)^{k-1}\widetilde{B}^{(k)}(s)\mathrm{d}s\right],$$

with $\widetilde{B}(t) := B(t, u(t)), t \in [0, T]$. Thus, under the regularity condition

$$u \in C^{k+1}([0, T]; X) \quad \text{and} \quad \widetilde{B} \in C^k([0, T]; X),$$

we obtain the desired optimal order consistency estimate

$$(6.6) \qquad \max_{k \le n \le N} \|d_n\|_X \le C\tau^k.$$

Now, in view of the consistency estimate (6.6) and our assumption (6.1) on the starting approximations, conditions (5.2) and (5.3) are valid with $\delta = C\tau^k$. Therefore, for sufficiently small time step τ, the desired error estimates (6.2) and (6.3) follow immediately from the corresponding estimates (5.4) and (5.5), respectively. $\qquad \square$

Remark 6.1 (On the accuracy requirement for the starting approximations). The accuracy requirement (6.1) for the starting approximations $u_0, \dots, u_{k-1}$ can be equivalently written in the form

$$(6.7) \qquad \max_{0 \le i \le k-1} \|u_i - u_i^\star\|_X \le C\tau^{k+(1-\frac{1}{p})}, \qquad \max_{0 \le i \le k-1} \|u_i - u_i^\star\|_D \le C\tau^{k-\frac{1}{p}}.$$

The larger p is, the stronger is the accuracy requirement (6.7) on the starting approximations. Suitable choices of p depend on the concrete application; for instance, for the problems in Examples 1.1–1.2, 1.3, and 1.4, respectively, the conditions $2/p + d/q < 1, 1/p + d/q < 1$, and $4/p + d/q < 2$, respectively, are required; see Propositions 2.1–2.3 and Section 7.

Proposition 6.1 applies directly to Examples 1.1–1.4; for instance, in the cases of Examples 1.1 and 1.2, we have:

Corollary 6.2 (Application to Examples 1.1 and 1.2). *In the case of Example 1.1, if the solution u of* (I) *is sufficiently regular,*

$$u \in C^{k+1}\big([0, T]; W^{-1,q}(\Omega)\big) \cap C^k\big([0, T]; W_0^{1,q}(\Omega)\big)$$

(and thus $\widetilde{B} = f(u, x, t) + \nabla \cdot \boldsymbol{g}(u, x, t) \in C^k([0, T]; W^{-1,q}(\Omega))$), the starting approximations are such that

$$(6.8) \qquad \frac{1}{\tau} \big\|(u_i - u_i^\star)_{i=0}^{k-1}\big\|_{L^p(W^{-1,q}(\Omega))} + \big\|(u_i - u_i^\star)_{i=0}^{k-1}\big\|_{L^p(W^{1,q}(\Omega))} \le C\tau^k,$$

with $q \in (d, \infty) \cap [2, \infty)$ and p such that $2/p + d/q < 1$, and the coefficients a and b satisfy (1.11), *then for sufficiently small time step τ, the errors $e_n = u_n - u_n^\star$ between the approximate solutions u_n of* (1.4) *and the nodal values $u_n^\star$ of the solution u of* (I) *are bounded by*

$$(6.9) \qquad \frac{1}{\tau}\big\|(e_n - e_{n-1})_{n=k}^N\big\|_{L^p(W^{-1,q}(\Omega))} + \big\|(e_n)_{n=k}^N\big\|_{L^p(W^{1,q}(\Omega))} \le C\tau^k,$$

$$(6.10) \qquad \max_{k \le n \le N} \|e_n\|_{L^\infty(\Omega)} \le C\tau^k,$$

with a constant C independent of τ.

Analogously, in the case of Example 1.2, if the solution u of (II) *is sufficiently regular,*

$$u \in C^{k+1}\big([0, T]; L^q(\Omega)\big) \cap C^k\big([0, T]; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)\big)$$

(and thus $\widetilde{B} = f(u, \nabla u, x, t) + \nabla \cdot \boldsymbol{g}(u, \nabla u, x, t) \in C^k([0, T]; L^q(\Omega)))$, the starting approximations are such that

$$(6.11) \qquad \frac{1}{\tau} \left\| (u_i - u_i^\star)_{i=0}^{k-1} \right\|_{L^p(L^q(\Omega))} + \left\| (u_i - u_i^\star)_{i=0}^{k-1} \right\|_{L^p(W^{2,q}(\Omega))} \leq C\tau^k,$$

with $q > d$ and p such that $2/p + d/q < 1$, and the coefficients a and b satisfy (1.11), then for a sufficiently small constant $\tilde{\lambda}$ in (2.9) and for sufficiently small time step τ, the errors $e_n = u_n - u_n^\star$ between the approximate solutions u_n of (1.4) and the nodal values $u_n^\star$ of the solution u of (II) are bounded by

$$(6.12) \qquad \frac{1}{\tau} \left\| (e_n - e_{n-1})_{n=k}^N \right\|_{L^p(L^q(\Omega))} + \left\| (e_n)_{n=k}^N \right\|_{L^p(W^{2,q}(\Omega))} \leq C\tau^k,$$

$$(6.13) \qquad \max_{k \leq n \leq N} \| e_n \|_{W^{1,\infty}(\Omega)} \leq C\tau^k,$$

with a constant C independent of τ.

7. Proofs of Propositions 2.1, 2.2, and 2.3

In this section we show that our abstract framework is applicable in the cases of Examples 1.1–1.4.

7.1. **Proof of Proposition 2.1.** In this subsection we show that our abstract framework is applicable in the cases of Examples 1.1 and 1.2; more precisely, we verify our abstract conditions (A1)–(A5) for the initial and boundary value problems (I) and (II) in the spaces given in Proposition 2.1.

It follows easily from the smoothness of the diffusion coefficients and the positivity of the coefficient $a(x, t)$ that assumption (A3) is satisfied; the smallest possible value of the stability constant λ in (2.5) is as on the left-hand side of (1.10). Assumption (A4) is satisfied, if the functions f and $\boldsymbol{g}$ are locally Lipschitz continuous with respect to the arguments u and ∇u. The proof of (A5), under the condition $2/p + d/q < 1$, is given in [6, Section 9]. Thus, it remains to verify assumptions (A1) and (A2).

Existence and uniqueness of a solution $v \in H^1(\mathbb{R}_+; V') \cap L^2(\mathbb{R}_+; V) \hookrightarrow C([0, \infty); H)$ for (2.1) are simple consequences of the positivity of $a(x, t)$ and the boundedness of $a(x, t)$ and $b(x, t)$. It is easy to see that the operator $-A_H(s) : D_H \to H$,

$$-A_H(s)v = -\nabla \cdot \big((a(x, s) + \mathrm{i}\, b(x, s))\nabla v\big),$$

is densely defined, closed and invertible. Following Section 4, one can prove that the operator $-A_H(s)$ generates a bounded analytic semigroup in the sector Σ_{θ_s}, with

$$\theta_s \geq \inf_{x \in \Omega} \arctan \frac{a(x, s)}{|b(x, s)|} \geq \inf_{(x,t) \in \Omega \times (0,T)} \arctan \frac{a(x, t)}{|b(x, t)|} = \arcsin \frac{1}{\lambda}.$$

In fact, we have the sharper result

$$(7.1) \qquad \theta_s = \inf_{x \in \Omega} \arctan \frac{a(x, s)}{|b(x, s)|},$$

which immediately implies $\theta := \inf_{s \in [0,T]} \theta_s = \arcsin \frac{1}{\lambda}$. To prove (7.1), simply note that otherwise we could choose φ satisfying

$$(7.2) \qquad \theta_s > \varphi > \arctan \frac{a(x_0, s)}{|b(x_0, s)|} \quad \text{for some } x_0 \in \Omega \text{ such that } b(x_0, s) \neq 0.$$

Without loss of generality, we can assume that $b(x_0, s) > 0$. Then the operator

$$-\mathrm{e}^{\mathrm{i}\varphi} A_H(s) = \nabla \cdot \left(\left(a(x,s)\cos\varphi - b(x,s)\sin\varphi + \mathrm{i}\left[a(x,s)\sin\varphi + b(x,s)\cos\varphi\right]\right)\nabla \right)$$

would generate a bounded analytic semigroup. But this is impossible, because (7.2) implies that $a(x,s)\cos\varphi - b(x,s)\sin\varphi < 0$ at some point $x \in \Omega$, which means that the operator $-\mathrm{e}^{\mathrm{i}\varphi} A_H(s)$ would have eigenvalues with positive real parts, thus the resolvent operator $z + \mathrm{e}^{\mathrm{i}\varphi} A_H(s)$ would not be invertible for some z on the right half-plane, which contradicts the bounded analyticity of the semigroup generated by $-\mathrm{e}^{\mathrm{i}\varphi} A_H(s)$ (see, e.g., [8, Theorem 3.7.11]). This proves (A1) for both Problems (I) and (II), with (7.1).

Next, we verify (A2) for Problem (II). In this case, $X = L^q(\Omega)$. With the positivity of $a(x,t)$ and the Hölder continuity of $a(x,t)$ and $b(x,t)$, Auscher, McIntosh and Tchamitchian [9, Theorem 4.19] have proved that the kernel of the semigroup $E_s^H(t)$ has a Gaussian upper bound:

$$(7.3) \qquad |G_s(t,x,y)| \leq \frac{C_0}{t^{d/2}} \mathrm{e}^{-\frac{|x-y|^2}{C_0 t}}, \quad \forall\, t > 0, \ \forall\, x, y \in \Omega,$$

where the constant C_0 depends only on the lower bound of $a(x,t)$, on the upper bounds of $|a(x,t) + \mathrm{i}\, b(x,t)|$, and on the Hölder norms of $a(\cdot, t)$ and $b(\cdot, t)$, but is independent of $s \in [0, T]$. Similarly, when $\varphi \in (0, \theta)$ the semigroup generated by $-\mathrm{e}^{\mathrm{i}\varphi} A_H(s)$ has also a Gaussian upper bound. Hence, [20, Theorem 8.5] implies that the semigroup generated by $-\mathrm{e}^{\mathrm{i}\varphi} A_H(s)$ satisfies the conditions of [20, Theorem 8.6], for all $\varphi \in (0, \theta)$, which further implies that $\{E_s^H(t)\}_{t>0}$ extends to an R-bounded analytic semigroup $\{E_s^X(z)\}_{z \in \Sigma_\varphi}$ on X (in view of [20, Remark 8.23] and [26, Theorem 4.2]). If we denote the generator of $\{E_s^X(t)\}_{t>0}$ by $-A_X(s)$, then [26, Theorem 4.2] implies that the family of operators $\{z(z + A_X(s))^{-1} : z \in \Sigma_\varphi\}$ is R-bounded. As a consequence, [19, Theorems 4.1–4.2 and Remark 4.3] implies that, when $\theta > \pi/2 - \alpha_k$, the solution of (2.2) satisfies (with $v_n = 0$ for $n < 0$)

$$(7.4) \quad \begin{aligned} &\frac{1}{\tau}\left\| \left(\sum_{j=0}^{k} \delta_j v_{n-j} \right)_{n=0}^{N} \right\|_{L^p(L^q(\Omega))} + \left\| (v_n)_{n=k}^{N} \right\|_{L^p(W^{2,q}(\Omega))} \\ &\leq C\left(\left\| (f_n)_{n=k}^{N} \right\|_{L^p(L^q(\Omega))} + \frac{1}{\tau}\left\| (v_i)_{i=0}^{k-1} \right\|_{L^p(L^q(\Omega))} + \left\| (v_i)_{i=0}^{k-1} \right\|_{L^p(W^{2,q}(\Omega))} \right). \end{aligned}$$

Then (A2) follows from (7.4) and the inequality

$$\frac{1}{\tau}\left\| (v_n - v_{n-1})_{n=0}^{N} \right\|_{L^p(L^q(\Omega))} \leq \frac{C}{\tau}\left\| \left(\sum_{j=0}^{k} \delta_j v_{n-j} \right)_{n=0}^{N} \right\|_{L^p(L^q(\Omega))}.$$

To prove the last inequality, let $\dot{v}_n := \frac{1}{\tau}\sum_{j=0}^{k}\delta_j v_{n-j}$. Recall that $\delta(\zeta) = (1 - \zeta)\mu(\zeta)$, where the polynomial $\mu(\zeta)$ of degree $k-1$ has no zeros in the closed unit disc; therefore

$$\frac{1}{\mu(\zeta)} = \sum_{n=0}^{\infty} \chi_n \zeta^n, \qquad \text{with } |\chi_n| \leq C\rho^n \text{ for some } \rho < 1.$$

Then, with $\eta_\ell = 1$ for $\ell \geq 0$ and $\eta_\ell = 0$ for $\ell < 0$, we have

$$\frac{v_n - v_{n-1}}{\tau} = \sum_{m=0}^{n} \dot{v}_{n-m}\,\chi_m = \sum_{m=0}^{N} \eta_{n-m}\dot{v}_{n-m}\,\chi_m,$$

because both sides have the same generating function. Therefore,

$$\frac{1}{\tau}\Big\|(v_n - v_{n-1})_{n=0}^{N}\Big\|_{L^p(X)} \leq \sum_{m=0}^{N} |\chi_m| \, \|(\eta_{n-m}\dot{v}_{n-m})_{n=0}^{N}\|_{L^p(X)}$$

$$\leq \sum_{m=0}^{N} |\chi_m| \, \|(\dot{v}_n)_{n=0}^{N}\|_{L^p(X)}$$

$$\leq C\big\|(\dot{v}_n)_{n=0}^{N}\big\|_{L^p(X)}.$$

Finally, we verify (A2) for Problem (I). In this case, we want to derive from (7.4) the following estimate:

$$(7.5) \quad \frac{1}{\tau}\Big\|\Big(\sum_{j=0}^{k} \delta_j v_{n-j}\Big)_{n=k}^{N}\Big\|_{L^p(W^{-1,q}(\Omega))} + \big\|(v_n)_{n=k}^{N}\big\|_{L^p(W^{1,q}(\Omega))} \leq$$

$$C\Big(\big\|(f_n)_{n=k}^{N}\big\|_{L^p(W^{-1,q}(\Omega))} + \frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(W^{-1,q}(\Omega))} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(W^{1,q}(\Omega))}\Big).$$

In fact, [10, Theorem 1] implies that the Riesz transform $\nabla A_X(s)^{-1/2}$ is bounded on $L^q(\Omega)$ for all $1 < q < \infty$. Once we have (7.4) and the boundedness of the Riesz transform, the discrete maximal regularity (2.3) can be proved in the same way as in [6, Proposition 8.7]. $\qquad\square$

7.2. **Proof of Proposition 2.2.** Since the operator $-(-\Delta)^{1/2}$ is self-adjoint and nonpositive definite, it generates a bounded analytic semigroup of angle $\pi/2$ on the Hilbert space $H = L^2(\mathbb{R}^d)$, with domain $D_H = H^1(\mathbb{R}^d)$. This implies (A1).

Assumption (A3) is due to the time independence and self-adjointness of the operator $(-\Delta)^{1/2}$. The self-adjointness also implies $\lambda = 1$.

If w, v are bounded in $W = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, with $\|w\|_W + \|v\|_W \leq K$, then the local Lipschitz continuity of f implies, for $q \geq 2$,

$$\|f(w) - f(v)\|_X = \|f(w) - f(v)\|_{L^2(\mathbb{R}^d)} + \|f(w) - f(v)\|_{L^q(\mathbb{R}^d)}$$

$$\leq C_K\big(\|w - v\|_{L^2(\mathbb{R}^d)} + \|w - v\|_{L^q(\mathbb{R}^d)}\big)$$

$$\leq C_K\Big(\|w - v\|_{L^2(\mathbb{R}^d)} + \|w - v\|_{L^2(\mathbb{R}^d)}^{\frac{2}{q}}\|w - v\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{2}{q}}\Big)$$

$$\leq C_K\big(\|w - v\|_{L^2(\mathbb{R}^d)} + \|w - v\|_{L^\infty(\mathbb{R}^d)}\big) = C_K\|w - v\|_W.$$

This implies assumption (A4) in the case $q \geq 2$, with $\tilde{\lambda} = 0$.

If $q > d$, then [1, Theorems 5.2 and 5.9] implies that, for any positive ε, there exists positive C_ε such that

$$\|v\|_{L^q(\mathbb{R}^d)} + \|v\|_{L^\infty(\mathbb{R}^d)} \leq \varepsilon\|v\|_{W^{1,q}(\mathbb{R}^d)} + C_\varepsilon\|v\|_{L^q(\mathbb{R}^d)} \quad \forall v \in W^{1,q}(\mathbb{R}^d).$$

For $D = H^1(\mathbb{R}^d) \cap W^{1,q}(\mathbb{R}^d)$ and $X = L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, we have

$$W^{1,p}(0,T;X) \cap L^p(0,T;D) \hookrightarrow L^\infty(0,T;X) \hookrightarrow L^\infty(0,T;L^2(\mathbb{R}^d)), \quad \text{when } p > 1,$$

and

$$W^{1,p}(0,T;X) \cap L^p(0,T;D) \hookrightarrow W^{1,p}(0,T;L^q(\mathbb{R}^d)) \cap L^p(0,T;W^{1,q}(\mathbb{R}^d))$$

$$\hookrightarrow L^\infty(0,T;(L^q(\mathbb{R}^d),W^{1,q}(\mathbb{R}^d))_{1-1/p,p}) \qquad \text{see [23, Proposition 1.2.10]}$$

$$= L^\infty(0,T;B^{1-1/p;q,p}(\mathbb{R}^d)) \qquad \text{by the definition of Besov spaces [1, §7.32]}$$

$$\hookrightarrow L^\infty(0,T;L^\infty(\mathbb{R}^d)), \qquad \text{when } 1/p + d/q < 1, \text{ see [1, §7.34].}$$

The last two imbedding results imply

$$W^{1,p}(0,T;X) \cap L^p(0,T;D) \hookrightarrow L^\infty(0,T;L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) = L^\infty(0,T;W).$$

This proves assumption (A5) in the case $q > d$.

It is well known that the operator $-(-\Delta)^{1/2}$ generates a bounded analytic semigroup $\{E^H(z)\}_{z \in \Sigma_{\pi/2}}$ with the kernel

$$(7.6) \qquad E(z,x,y) = \frac{c_d}{z^d}\left(1 + \frac{|x-y|^2}{z^2}\right)^{-\frac{d+1}{2}},$$

where c_d is a positive constant; see [16, §1.1.3]. It is easy to check that the kernel $E(z,x,y)$ satisfies the condition of [11, Proposition 2.9-(b)] with $g(s) = C(1+s^2)^{-\frac{d+1}{2}}$. Hence, [11, Proposition 2.9-(a), with $p = 1$ and $q = \infty$ therein] is satisfied. Substituting $(s,p,q,p_o) = (2,1,\infty,q)$ and $\Omega = \mathbb{R}^d$ into [11, Corollary 2.7], we see that the analytic semigroup generated by the operator $-(-\Delta)^{1/2}$ is R-bounded on $L^q(\mathbb{R}^d)$ in the sector Σ_φ, for any $\varphi \in (0,\pi/2)$. Equivalently, in view of [26, Theorem 4.2], the family of operators $\{z(z+(-\Delta)^{\frac{1}{2}})^{-1} : z \in \Sigma_{\varphi+\pi/2}\}$ is R-bounded on $L^q(\mathbb{R}^d)$, for any $1 < q < \infty$ and $\varphi \in (0,\pi/2)$. Then [19, Theorems 4.1–4.2 and Remark 4.3] implies that, for BDF methods up to order 6,

$$(7.7) \quad \begin{aligned} &\frac{1}{\tau}\big\|(v_n - v_{n-1})_{n=k}^N\big\|_{L^p(L^2(\mathbb{R}^d))} + \big\|(v_n)_{n=k}^N\big\|_{L^p(H^1(\mathbb{R}^d))} \\ &\le C\Big(\big\|(f_n)_{n=k}^N\big\|_{L^p(L^2(\mathbb{R}^d))} + \frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(L^2(\mathbb{R}^d))} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(H^1(\mathbb{R}^d))}\Big) \end{aligned}$$

and

$$(7.8) \quad \begin{aligned} &\frac{1}{\tau}\big\|(v_n - v_{n-1})_{n=k}^N\big\|_{L^p(L^q(\mathbb{R}^d))} + \big\|(v_n)_{n=k}^N\big\|_{L^p(W^{1,q}(\mathbb{R}^d))} \\ &\le C\Big(\big\|(f_n)_{n=k}^N\big\|_{L^p(L^q(\mathbb{R}^d))} + \frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(L^q(\mathbb{R}^d))} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(W^{1,q}(\mathbb{R}^d))}\Big). \end{aligned}$$

Estimates (7.7) and (7.8) imply (A2).

Overall, assumptions (A1)–(A5) are satisfied for $q \in (d,\infty) \cap [2,\infty)$.

7.3. **Proof of Proposition 2.3.** Similarly, the operator $-\Delta^2$ is self-adjoint and nonpositive definite. Hence, it generates a bounded analytic semigroup of angle $\pi/2$ on the Hilbert space $H = L^2(\mathbb{R}^d)$, with domain $D_H = H^4(\mathbb{R}^d)$. This implies (A1).

Assumption (A3) is due to the time independence and self-adjointness of the operator Δ^2. The self-adjointness also implies $\lambda = 1$.

Since $W = H^2(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d) \hookrightarrow W^{2,q}(\mathbb{R}^d)$ for $q \ge 2$, if w,v are bounded in W with $\|w\|_W + \|v\|_W \le K$, then the local Lipschitz continuity of f implies

$$\|\Delta f(w) - \Delta f(v)\|_X = \|\Delta f(w) - \Delta f(v)\|_{L^2(\mathbb{R}^d)} + \|\Delta f(w) - \Delta f(v)\|_{L^q(\mathbb{R}^d)}$$

$$\leq C_K(\|w - v\|_{H^2(\mathbb{R}^d)} + \|w - v\|_{W^{2,q}(\mathbb{R}^d)})$$
$$\leq C_K(\|w - v\|_{H^2(\mathbb{R}^d)} + \|w - v\|_{W^{2,\infty}(\mathbb{R}^d)}) = C_K\|w - v\|_W.$$

This implies assumption (A4) in the case $q \geq 2$, with $\tilde{\lambda} = 0$.

If $1 < q < \infty$ and $2q > d$, then [1, Theorems 5.2 and 5.9] imply that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying

$$\|v\|_{H^2(\mathbb{R}^d)} \leq \varepsilon\|v\|_{H^4(\mathbb{R}^d)} + C_\varepsilon\|v\|_{L^2(\mathbb{R}^d)} \qquad \forall v \in H^4(\mathbb{R}^d),$$
$$\|v\|_{W^{2,\infty}(\mathbb{R}^d)} \leq \varepsilon\|v\|_{W^{4,q}(\mathbb{R}^d)} + C_\varepsilon\|v\|_{L^q(\mathbb{R}^d)} \quad \forall v \in W^{4,q}(\mathbb{R}^d).$$

Since $D = H^4(\mathbb{R}^d) \cap W^{4,q}(\mathbb{R}^d)$ and $X = L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, the last two inequalities imply (2.7) in assumption (A5). Moreover, we have

$$W^{1,p}(0,T;X) \cap L^p(0,T;D) \hookrightarrow W^{1,p}(0,T;L^2(\mathbb{R}^d)) \cap L^p(0,T;H^4(\mathbb{R}^d))$$
$$\hookrightarrow W^{1-\theta,p}(0,T;H^{4\theta}(\mathbb{R}^d)) \qquad \text{by using complex interpolation}$$
$$\hookrightarrow L^\infty(0,T;H^2(\mathbb{R}^d)), \qquad \text{when } (1-\theta)p > 1 \text{ and } 4\theta > 2,$$

and

$$W^{1,p}(0,T;X) \cap L^p(0,T;D) \hookrightarrow W^{1,p}(0,T;L^q(\mathbb{R}^d)) \cap L^p(0,T;W^{4,q}(\mathbb{R}^d))$$
$$\hookrightarrow W^{1-\theta,p}(0,T;W^{4\theta,q}(\mathbb{R}^d)) \quad \text{by using complex interpolation}$$
$$\hookrightarrow L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d)), \qquad \text{when } (1-\theta)p > 1 \text{ and } (4\theta - 2)q > d.$$

It remains to prove the existence of a θ satisfying the conditions above. In fact, if $1 < p, q < \infty$ and $d/q + 4/p < 2$, then $\frac{1}{4}\left(2 + \frac{d}{q}\right) < 1 - \frac{1}{p}$ and $\frac{1}{2} < 1 - \frac{1}{p}$. Hence, there exists $\theta \in (0,1)$ satisfying

$$\frac{1}{4}\left(2 + \frac{d}{q}\right) < \theta < 1 - \frac{1}{p} \quad \text{and} \quad \frac{1}{2} < \theta < 1 - \frac{1}{p}.$$

Then θ satisfies $(1-\theta)p > 1$, $4\theta > 2$ and $(4\theta - 2)q > d$. This proves (A5) in the case $q > \max(d/2, 1)$.

Now, according to [18, Example 3.2 (A)], the semigroup generated by $-e^{i\varphi}\Delta^2$ satisfies a Gaussian estimate

$$|E(te^{i\varphi}, x, y)| \leq \frac{C_\varphi}{t^{d/4}} \exp\left(-\frac{|x - y|^{4/3}}{C_\varphi\, t^{1/3}}\right)$$

for any $\varphi \in (0, \pi/2)$. Substituting $(s, p, q, p_o) = (2, 1, \infty, q)$ and $\Omega = \mathbb{R}^d$ into [11, Corollary 2.7], we see that the analytic semigroup generated by the operator $-\Delta^2$ is R-bounded on $L^q(\mathbb{R}^d)$, $1 < q < \infty$, in the sector Σ_φ, for any $\varphi \in (0, \pi/2)$. Equivalently, in view of [26, Theorem 4.2], the family of operators $\{z(z + \Delta^2)^{-1} : z \in \Sigma_{\varphi+\pi/2}\}$ is R-bounded on $L^q(\mathbb{R}^d)$, for any $1 < q < \infty$ and $\varphi \in (0, \pi/2)$. Then [19, Theorems 4.1–4.2 and Remark 4.3] implies that, for BDF methods of order up to 6,

$$(7.9) \quad \begin{aligned} &\frac{1}{\tau}\big\|(v_n - v_{n-1})_{n=k}^N\big\|_{L^p(L^2(\mathbb{R}^d))} + \big\|(v_n)_{n=k}^N\big\|_{L^p(H^4(\mathbb{R}^d))} \\ &\leq C\Big(\big\|(f_n)_{n=k}^N\big\|_{L^p(L^2(\mathbb{R}^d))} + \frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(L^2(\mathbb{R}^d))} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(H^4(\mathbb{R}^d))}\Big) \end{aligned}$$

and

$$(7.10) \quad \frac{1}{\tau}\big\|(v_n - v_{n-1})_{n=k}^N\big\|_{L^p(L^q(\mathbb{R}^d))} + \big\|(v_n)_{n=k}^N\big\|_{L^p(W^{4,q}(\mathbb{R}^d))}$$

$$\leq C\Big(\big\|(f_n)_{n=k}^N\big\|_{L^p(L^q(\mathbb{R}^d))} + \frac{1}{\tau}\big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(L^q(\mathbb{R}^d))} + \big\|(v_i)_{i=0}^{k-1}\big\|_{L^p(W^{4,q}(\mathbb{R}^d))}\Big).$$

Estimates (7.9) and (7.10) imply (A2).

Overall, assumptions (A1)–(A5) are satisfied for $q \in (d/2, \infty) \cap [2, \infty)$.

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