

ERROR ESTIMATES FOR FINITE DIFFERENCE METHODS FOR A WIDE-ANGLE ‘PARABOLIC’ EQUATION

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Abstract. We consider a model initial- and boundary-value problem for a third-order p.d.e., a wide-angle ‘parabolic’ equation frequently used in underwater acoustics, with depth- and range-dependent coefficients in the presence of horizontal interfaces and dissipation. After commenting on the existence–uniqueness theory of solution of the equation, we discretize the problem by a second-order finite difference method of Crank–Nicolson type for which we prove stability and optimal-order error estimates in suitable discrete L^2 , H^1 and maximum norms. We also prove, under certain conditions, that the forward Euler scheme is also stable and convergent for the problem at hand.

Key words. wide-angle ‘parabolic’ equation, Underwater Acoustics, finite difference error estimates, interface problems

AMS subject classifications. 65M06, 65M15, 76Q05

1. Introduction. In this paper we shall study a finite difference method for approximating the solution of a model initial- and boundary-value problem for a third-order partial differential equation that represents a *wide-angle*, ‘*parabolic*’ approximation to the Helmholtz equation in cylindrical coordinates in the absence of azimuthal coupling.

‘Parabolic’ equations have been widely used to approximate far-field, outgoing, paraxial wave propagation phenomena in many different physical instances; cf., e.g., [29], [6] for references. We have in mind their application in underwater acoustics, where they were introduced by Tappert and Hardin, [29]. (For subsequent developments cf., e.g., the surveys [21], [30].) In fact, we shall consider a simple example of the class of wide-angle extensions of the standard parabolic equation (PE) of [29]. Such equations describe more accurately the long-range acoustic field in that they suffer from smaller phase discrepancies as approximations to the Helmholtz equation when compared with the standard PE. In addition, they are capable of approximating well modes of propagation that may interact strongly with the bottom layers. The most widely known such equation was originally introduced by Claerbout, cf. [11], in his study of migration processes in seismology.

The particular p.d.e. that we shall study here can be written in the form

$$(1.1) \quad (1 + qL)v_r = ik_0(p - q)Lv,$$

where $v = v(z, r)$ is a complex-valued function of depth (z) and range (distance from the source, r), describing the acoustic field generated by a harmonic point source placed in the water, and emitting sound at a frequency f . The constant k_0 is a reference wavenumber given by $k_0 = 2\pi f/c_0$, where c_0 is a reference sound speed; L

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is a second-order linear differential operator in z to be defined below, with coefficients depending in general on both z and r .

The p.d.e. (1.1) is obtained as an approximation to a pseudodifferential expression by a procedure whose formal steps are outlined e.g. in [15], [16], [21] and [30]. It is based on approximating $(1+x)^{\frac{1}{2}}$ near $x=0$ by a rational function of the form $(1+px)/(1+qx)$, $p \neq q$, real. Taking $p = \frac{3}{4}$, $q = \frac{1}{4}$ yields the (1,1)-Padé approximant of $(1+x)^{\frac{1}{2}}$ (Claerbout), while the choice $p = \frac{1}{2}$, $q = 0$ results in the linear Taylor polynomial approximation and the standard PE (Tappert). We refer the reader to the papers cited above and also to [31], [12], [7], among others, for discussions of the derivation of (1.1) and its validity as a wide-angle extension of the standard parabolic approximation. It should be remarked that in [16] Greene considers a slight generalization of (1.1) that is capable of handling propagation over wider angle intervals and corresponds to a rational approximation of $(1+x)^{\frac{1}{2}}$ of the form $(p_0 + p_1x)/(1 + q_1x)$. The coefficients are chosen to minimize the maximum error of this approximation over suitable intervals of the form $[\alpha, 1]$, $\alpha > 0$. The analysis of the difference scheme approximating Greene's equation proceeds along similar lines to the one presented here in the case of (1.1).

During the past few years there has developed considerable interest in studying extensions of the wide-angle equation (1.1) and their application to underwater acoustics problems. For example, *higher-order* analogs of (1.1), corresponding to rational approximations of $(1+x)^{\frac{1}{2}}$ with numerator and denominator of higher degree, are capable of approximating well propagation over very wide angles, cf., e.g., [27], [7], [13], [17]. In addition, *three-dimensional* extensions of (1.1) to problems with azimuthal dependence have been considered, [25], [26], [22], [9], [14]. In this paper however our study will be restricted to the simple equation (1.1).

We shall pose (1.1) as a model initial- and boundary-value problem in a medium with two horizontal layers of finite width. In physical variables, we are given constants $0 < z_B < z_{\max}$ defining a two-layered medium consisting of water of constant density ρ_1 occupying the strip $(0, z_B)$, $r \geq 0$, and of a bottom layer consisting of sediment of constant density ρ_2 in (z_B, z_{\max}) , $r \geq 0$. More layers can be incorporated into the analysis in a straightforward manner; we restrict ourselves to two for the sake of simplicity. (Usually the lowermost of these layers is an artificial one with suitable dissipation properties to model a medium of infinite extent in z ; an interesting alternative is to use at the bottom the nonlocal impedance boundary conditions due to Papadakis and originally presented in [15].) Given $R_{\max} > 0$, the maximum value of range of propagation contemplated, we seek the complex-valued field variable $v = v(z, r)$, defined for $0 \leq z \leq z_{\max}$, $0 \leq r \leq R_{\max}$, and satisfying the equation

$$\begin{aligned}
 (1.2) \quad & v_r + \frac{q}{k_0^2} v_{zrz} = \\
 & = ik_0(p-q) \left[\frac{1}{k_0^2} v_{zz} + (\beta(z, r) + i\gamma(z, r))v \right], \\
 & z \in (0, z_B) \cup (z_B, z_{\max}), \quad r \in (0, R_{\max}),
 \end{aligned}$$

which is of the form (1.1) with $Lv = k_0^{-2}v_{zz} + (\beta(z, r) + i\gamma(z, r))v$. Here the constants p, q, k_0 are as before, while $\beta(z, r)$ and $\gamma(z, r)$ are real-valued functions, smooth enough on $[0, z_B]$ and $[z_B, z_{\max}]$ for $r \in (0, R_{\max})$, but having a possible jump discontinuity across $\{z_B\} \times [0, R_{\max}]$. In the applications, $\beta = n^2 - 1$, where $n = n(z, r)$ is the index of refraction of the two layers, defined by $n(z, r) = c_0/c(z, r)$, where $c(z, r)$

is the sound speed in each medium. The function γ is nonnegative and incorporates empirically determined dissipative mechanisms such as attenuation (loss) coefficients in the various layers and volume absorption in the final, artificial layer, [10], [18]. Although the wide-angle equation (1.2) is strictly valid in horizontally stratified media wherein the speed of sound (and γ) depend only on z (cf., e.g., [15]), it is considered to be an acceptable model when β and γ are also range-dependent with mild variation in r , and it is in fact frequently solved in range-dependent domains with interfaces and bottom allowed to be slowly varying functions of r . In this paper we shall restrict ourselves to horizontal layers; the assumption of mild variation of β and γ with respect to r will not play any role in the analysis of the numerical schemes.

The equation (1.2) will be posed in $[0, z_{\max}] \times [0, R_{\max}]$ under the following set of auxiliary conditions. At the water surface $z = 0$ we shall assume ‘pressure-release’ boundary conditions, i.e., simply that

$$(1.3) \quad v(0, r) = 0, \quad r \in [0, R_{\max}].$$

At the boundary $z = z_B$ between the two layers the usual interface conditions of acoustics (continuity of pressure and of the normal component of velocity) are enforced. In terms of v these are expressed as the continuity requirements

$$(1.4) \quad v(z_B^-, r) = v(z_B^+, r), \quad r \in [0, R_{\max}],$$

and the transmission condition

$$(1.5) \quad v_z(z_B^-, r) = \rho v_z(z_B^+, r), \quad r \in [0, R_{\max}],$$

where $\rho = \rho_1/\rho_2$. At the bottom boundary $z = z_{\max}$ we shall assume that a homogeneous mixed boundary condition of the form

$$(1.6) \quad v_z(z_{\max}, r) + \tilde{\mu}v(z_{\max}, r) = 0, \quad r \in [0, R_{\max}],$$

holds, where $\tilde{\mu}$ is a nonnegative constant. (For reasons of convenience in the exposition we took the coefficient of v_z in (1.6) equal to one. The case of homogeneous Dirichlet boundary condition, i.e. $v(z_{\max}, r) = 0$, is thus not included in (1.6). However it is much easier to handle and all the results of the paper hold if (1.6) is replaced by it; we shall omit the details.)

At $r = 0$ we shall assume an initial condition in r , i.e. that

$$(1.7) \quad v(z, 0) = \tilde{v}^0(z), \quad 0 \leq z \leq z_{\max},$$

where $\tilde{v}^0(z)$ models the effect of the source placed at $r = 0$, cf., e.g., [29], [16].

Choosing z_{\max} and some arbitrary v_{ref} as scale coefficients in the length and field variables respectively, introducing nondimensional variables by $z' = \frac{z}{z_{\max}}$, $r' = \frac{r}{z_{\max}}$, $v' = \frac{v}{v_{\text{ref}}}$, writing (1.2)–(1.7) in terms of v' , z' and r' , and then relabeling the variables as v , β , γ , z , r again, we obtain the following set of equations

$$[1 + q(\beta + i\gamma)]v_r + \alpha qv_{zzr} = i\alpha\lambda v_{zz} + i\lambda(\beta + i\gamma)v, \\ z \in (0, z_*) \cup (z_*, 1), \quad r \in [0, R],$$

$$v(z, 0) = v^0(z), \quad z \in [0, 1],$$

(P') for $r \in [0, R]$:

$$v(0, r) = 0,$$

$$v(z_*^-, r) = v(z_*^+, r), \quad v_z(z_*^-, r) = \rho v_z(z_*^+, r),$$

$$v_z(1, r) + \mu v(1, r) = 0,$$

where $\alpha = (k_0 z_{\max})^{-2}$, $\lambda = (p - q)k_0 z_{\max}$, $z_* = z_B / z_{\max}$, $R = R_{\max} / z_{\max}$, $\mu = \tilde{\mu} z_{\max}$, $v^0(z) = \tilde{v}^0(z) / v_{\text{ref}}$, and where $\beta = \beta(z, r)$ and $\gamma = \gamma(z, r)$ have a possible jump discontinuity across $\{z_*\} \times [0, R]$.

Assuming $q \neq 0$, one may simplify (P') a bit further. If $q = 0$, i.e. in the standard PE case, it may be seen without difficulty that all the estimates of sections 3–6 hold; cf. also [1]. We shall assume therefore henceforth that $q \neq 0$. Then, it may be easily seen that the envelope-type transformation $u = v \exp(-i \frac{\lambda}{q} r)$ transforms the p.d.e. in (P') into one with the same operator in its left-hand side but with just a zeroth-order term with a constant coefficient in the right-hand side while leaving the auxiliary conditions unaltered. In the sequel, we shall study the transformed problem

$$\begin{aligned} [1 + q(\beta(z, r) + i\gamma(z, r))]u_r + \alpha q u_{z z r} &= -i \frac{\lambda}{q} u, \quad z \in (0, z_*) \cup (z_*, 1), \quad r \in [0, R], \\ u(z, 0) &= u^0(z) := v^0(z), \quad z \in [0, 1], \\ (P) \quad \text{for } r \in [0, R] : \\ u(0, r) &= 0, \\ u(z_*^-, r) &= u(z_*^+, r), \quad u_z(z_*^-, r) = \rho u_z(z_*^+, r), \\ u_z(1, r) + \mu u(1, r) &= 0. \end{aligned}$$

Postponing discussion of questions of existence and uniqueness of solutions of (P) until the next section, we proceed now to present the difference scheme that we shall use to approximate the solution of (P) and outline the contents of the paper. In addition to the hypothesis that $\mu \geq 0$, we shall require for the analysis of stability and convergence of the difference schemes that $\lambda\gamma(z, r) \geq 0$ in $[0, 1] \times [0, R]$. In some places we shall make the additional requirement that either $q > 0$ or $\gamma \equiv 0$. These conditions are fulfilled in practice since the most frequent choice are the Claerbout parameters $p = \frac{3}{4}$, $q = \frac{1}{4}$, and γ is nonnegative.

The solution of (P) will be approximated by a finite difference scheme of Crank–Nicolson type, of second-order accuracy in the depth and range variables. We define a version of this scheme by partitioning the intervals $[0, z_*]$ and $[z_*, 1]$ with uniform meshlength h_- and h_+ , respectively. To this effect we let J and m be positive integers, define h_- and h_+ by $z_* = mh_-$, $mh_- + (J - m)h_+ = 1$, and set $z_j = jh_-$, $j = 0, \dots, m$ (so that $z_m = z_*$) and $z_j = (j - m)h_+ + z_m$, $j = m + 1, \dots, J$. Let \mathbb{C}_0^{J+1} denote the set of complex $J+1$ -vectors $v = (v_0, \dots, v_J)^T$ with $v_0 = 0$. For $v \in \mathbb{C}_0^{J+1}$ we define $\Delta_h v \in \mathbb{C}_0^{J+1}$, $(\Delta_h v)_j = \Delta_h v_j$, by the relations $\Delta_h v_0 = 0$, $\Delta_h v_j = (v_{j-1} - 2v_j + v_{j+1})/h_-^2$, $j = 1, \dots, m-1$, $\Delta_h v_m = \frac{1}{h} \left(\rho \frac{v_{m+1} - v_m}{h_+} - \frac{v_m - v_{m-1}}{h_-} \right)$, $\Delta_h v_j = (v_{j-1} - 2v_j + v_{j+1})/h_+^2$, $j = m+1, \dots, J-1$, $\Delta_h v_J = 2(v_{J-1} - v_J)/h_+^2$, where $\hat{h} = (h_- + \rho h_+)/2$. Thus, $\Delta_h v_j$ is the usual centered difference quotient approximation to the second derivative at the interior points z_j , $j \neq m$, and is suitably defined at $j = m$ and $j = J$ in anticipation of the approximation of the interface conditions at z_* and the bottom mixed boundary condition.

To discretize (P) in r , let $R = kN$, N positive integer, $r^n = nk$, $n = 0, \dots, N$, and $r^{n+\frac{1}{2}} = r^n + \frac{k}{2}$. For $v^n \in \mathbb{C}_0^{J+1}$ define $\partial v^n = (v^{n+1} - v^n)/k$ and $v^{n+\frac{1}{2}} = (v^{n+1} + v^n)/2$. The finite difference approximations U_j^n to $u(z_j, r^n)$ are the components of the vectors $U^n = (U_0^n, \dots, U_J^n)^T \in \mathbb{C}_0^{J+1}$, $0 \leq n \leq N$, computed as follows. For $n = 0$ let

$$(1.8) \quad U_j^0 = u^0(z_j), \quad j = 0, \dots, J,$$

and for $0 \leq n \leq N-1$, $j = 1, \dots, J-1$, $j \neq m$

$$(1.9) \quad [1 + q(\beta(z_j, r^{n+\frac{1}{2}}) + i\gamma(z_j, r^{n+\frac{1}{2}}))] \partial U_j^n + \alpha q \partial(\Delta_h U_j^n) = -i \frac{\lambda}{q} U_j^{n+\frac{1}{2}}.$$

For a complex-valued function $f(z)$ on $[0, 1]$ whose right- and left-hand limits exist at $z_* = z_m$, denote $\hat{f}(z_m) = [h_- f(z_*^-) + \rho h_+ f(z_*^+)]/2\hat{h}$. With this notation in mind, discretizing the interface condition of (P) in the customary way (cf. [24], [23], [1]) we obtain for $0 \leq n \leq N-1$

$$(1.10) \quad [1 + q(\hat{\beta}(z_m, r^{n+\frac{1}{2}}) + i\hat{\gamma}(z_m, r^{n+\frac{1}{2}}))] \partial U_m^n + \alpha q \partial(\Delta_h U_m^n) = -i \frac{\lambda}{q} U_m^{n+\frac{1}{2}}.$$

Finally, discretizing the mixed boundary condition at $z = 1$ by centered differences in the customary way, we complete the definition of the difference approximations letting, for $n = 0, \dots, N-1$,

$$(1.11) \quad \begin{aligned} & \partial U_J^n \\ & + \alpha q \partial(\Delta_h U_J^n) - 2\alpha q \frac{\mu}{h_+} \partial U_J^n = -i \frac{\lambda}{q} U_J^{n+\frac{1}{2}}. \end{aligned}$$

To write the scheme (1.8)–(1.11) in a more compact form we extend our previous notation associating with a function $f : [0, 1] \rightarrow \mathbb{C}$ a vector $\hat{f} \in \mathbb{C}_0^{J+1}$ given by $\hat{f}_j = f(z_j)$, $1 \leq j \leq J$, $j \neq m$, and $\hat{f}_m = \hat{f}(z_m)$ defined as before. (The vectors $\hat{\beta}(r)$, $\hat{\gamma}(r)$ are defined analogously, i.e. as $\hat{\beta}_j(r) = \beta(z_j, r)$, $j \neq 0, m$ etc..) We also let $\delta = (0, \dots, 0, 1)^T \in \mathbb{C}_0^{J+1}$, and for $v, w \in \mathbb{C}_0^{J+1}$ we define $v \otimes w = (v_0 w_0, \dots, v_J w_J)^T$. With this notation in place, we may rephrase (1.8)–(1.11) as follows: For $0 \leq n \leq N$, we seek $U^n \in \mathbb{C}_0^{J+1}$ approximating $u^n \in \mathbb{C}_0^{J+1}$, $u_j^n := u(z_j, r^n)$, and satisfying

$$\begin{aligned} & U^0 = u^0 \\ & \text{for } n = 0, \dots, N-1 : \\ (\Delta) \quad & \partial U^n + q[\hat{\beta}(r^{n+\frac{1}{2}}) + i\hat{\gamma}(r^{n+\frac{1}{2}})] \otimes \partial U^n - 2\alpha q \frac{\mu}{h_+} \delta \otimes \partial U^n \\ & + \alpha q \partial(\Delta_h U^n) = -i \frac{\lambda}{q} U^{n+\frac{1}{2}}. \end{aligned}$$

The solution U^n of (Δ) approximates at $r = r^n$ the solution u^n of (P) . We may then compute approximations $V^n \in \mathbb{C}_0^{J+1}$ to the solution v^n of (P') by the formulas $V^n = U^n \exp(i \frac{\lambda}{q} r^n)$. This transformation will conserve all error estimates to be derived in the sequel for U^n , since $|U_j^n - u(z_j, r^n)| = |V_j^n - v(z_j, r^n)|$. Alternatively, we could have discretized problem (P') directly and obtained another difference scheme, similar to (Δ) , which may then be analyzed analogously; cf. [2].

Many numerical studies of wide-angle equations using finite difference schemes have appeared in the computational underwater acoustics literature during the past decade, starting with the early contributions of Gilbert, Greene, and Thomson in [15]. We shall comment briefly on the more theoretical papers. Greene, [16], derived an implicit finite difference scheme for his rational approximation, using the Crank–Nicolson (trapezoidal) method for range-stepping, a fourth-order accurate Numerov–type differencing in the depth variable, and discretizing the interface condition at lower order. Using matrix spectral methods he argued the (unconditional) stability of the scheme in the case $\gamma = 0$, $\beta = \beta(z)$, $\mu = 0$, for uniform meshes.

A Crank–Nicolson type method (the IFD scheme), of second order local accuracy in z and r , was proposed and implemented by Botseas, Gilbert and Lee in [9]. The IFD scheme has the additional capability of handling more general interfaces. (The scheme to be analyzed in this paper is similar to the IFD. However, we have incorporated all dissipative mechanisms in the coefficient γ in the p.d.e. and discretized the variable coefficients and the bottom boundary condition in a different manner to render the scheme *conservative* in the absence of dissipation.) St.Mary and Lee, [27], analyzed the stability of IFD by matrix spectral techniques for uniform meshes in the absence of dissipation and interfaces. They proved that the scheme is unconditionally stable if β is a constant or $\beta = \beta(z)$ and $\mu = 0$. (They also stated a sufficient condition of mild change in $\beta(\cdot, r)$ that guarantees stability in the range-dependent case.) In [2], using energy techniques, two of present authors analyzed the stability and convergence of the present scheme (for uniform meshes, in the absence of dissipation and interfaces) proving unconditional stability and error estimates of second order of accuracy in the ℓ_2 norm.

For *finite element* discretizations in depth coupled with various range-stepping techniques we refer the reader to [12], [13], [19], [3], [4]; for a spectral-type method cf. [31].

The contents of the paper at hand are as follows. In section 2 we offer some commentary on questions of existence and uniqueness of the solutions of the wide-angle equations; such questions center of course around the fact that the operator $(1 + qL)$ in the left-hand side of (1.1) may not be invertible. It is proved in section 2 that if for each r $\gamma(z, r)$ is nonzero on a nonempty interval in $[0, 1]$, then this operator is invertible and (P) is well-posed. If $\gamma = 0$, a condition of the form $\alpha q \geq 1 + q \max_{z,r} \beta(z, r)$ (if $q > 0$, $\rho \leq 1$) is sufficient to ensure the well-posedness of (P) . Sections 3–5 are devoted to the analysis of the difference scheme (Δ) by energy techniques. For the purposes of deriving error estimates, the existence, uniqueness and sufficient regularity of the solution of (P) is assumed. In the preliminary section 3 we analyze an auxiliary two-point boundary-value problem in z , associated with (P) . The results of section 3 are used in the subsequent sections to define an ‘elliptic’ finite difference approximation to the solution of (P) , cf. [1], that enables us to express the *consistency* of the scheme in a way that permits proving optimal-order error estimates of order $O(k^2 + h^2)$, $h = \max(h_-, h_+)$, in an appropriate to the problem weighted ℓ_2 norm in section 4, and in a discrete H^1 sense (and also in the maximum norm) in section 5, under the additional hypothesis that either $\gamma = 0$ or $q > 0$. In section 6, we prove analogous error estimates in the case of an arbitrary (nonuniform) partition of the depth interval. The paper closes with section 7 in which we analyze the stability of the simple, first-order in range and second-order in depth, forward Euler difference scheme for (P) , assuming that either γ is bounded away from zero or a condition of the form $\alpha q > 1 + q \max_{z,r} \beta(z, r)$ if e.g. $\rho \leq 1$, $q > 0$, we prove the stability of the scheme in the weighted ℓ_2 norm, under no mesh conditions. This result is somewhat interesting, if one recalls that the Euler scheme is unstable for the standard PE. The reader is referred to section 7 for more commentary on this issue.

2. Mathematical preliminaries. In this section we shall briefly discuss some issues related to the existence and uniqueness of solutions of the initial- and boundary-value problem (P) . That the existence of solutions is called into question if the operator $(1 + qL)$ appearing in the left-hand side of (1.1) is not invertible, can be immediately seen e.g. by an explicit computation of the solution of constant-coefficient

versions of (P). For example, consider a problem of the form (P) *with no dissipation* (i.e., with $\gamma = 0$). To simplify matters, suppose that propagation takes place in a single-layered medium (i.e. no discontinuities at z_* and $\rho = 1$), with constant speed of sound equal to c_0 (so that $\beta = 0$) over a hard bottom (i.e. $\mu = 0$). Then (P) becomes

$$(2.1) \quad \begin{aligned} (1 + \alpha q \partial_z^2) u_r &= -i \frac{\lambda}{q} u, & 0 \leq z \leq 1, & \quad r \geq 0, \\ u(z, 0) &= u^0(z), & 0 \leq z \leq 1, \\ u(0, r) &= 0, \quad u_z(1, r) = 0, & \quad r \geq 0. \end{aligned}$$

The homogeneous, two-point boundary-value problem corresponding to the second-order operator appearing in the left-hand side of the p.d.e. above is

$$\begin{aligned} \omega''(z) + \frac{1}{\alpha q} \omega(z) &= 0, & 0 \leq z \leq 1, \\ \omega(0) = \omega'(1) &= 0. \end{aligned}$$

This will have only the trivial solution if

$$(2.2) \quad \frac{1}{\alpha q} \neq \mu_j, \quad j = 1, 2, \dots,$$

where $\mu_j = (2j - 1)^2(\pi/2)^2$, $j = 1, 2, \dots$, are the eigenvalues of the operator $-\partial_z^2$ with the boundary conditions under consideration. Letting for $j = 1, 2, \dots$, $\varphi_j = A_j \sin \sqrt{\mu_j} z$ be the associated orthonormal eigenfunctions, and separating variables in (2.1) we immediately see that if (2.2) holds, then the solution of (2.1) may be represented as

$$u(z, r) = \sum_{j=1}^{\infty} c_j \exp(-i\sigma_j \frac{\lambda}{q} r) \varphi_j(z),$$

where $\sigma_j = (1 - \mu_j \alpha q)^{-1}$ and the c_j are the Fourier coefficients of u^0 with respect to $\{\varphi_j\}$. If $\frac{1}{\alpha q} = \mu_k$ for some k , the series is missing the term corresponding to $j = k$ and the initial value u^0 must have a vanishing component in the direction of φ_k if a solution is to exist. Note that if $q > 0$, a sufficient condition for (2.2) to hold is of course that $\frac{1}{\alpha q} < \mu_1$. Since here $\alpha = (k_0 z_B)^{-2}$, we see that if $\frac{f z_B}{c_0} < \frac{\sqrt{q}}{4}$, i.e. in the case of shallow-water propagation of sufficiently low frequency, eigenvalues will not be hit and the solution will exist for an arbitrary u^0 .

This state of affairs persists in the variable coefficient case under consideration. We may now write the p.d.e. in (P) in the form

$$(2.3) \quad (1 - \alpha q M(r)) u_r = -i \frac{\lambda}{q} u,$$

where $M(r)$ is the second-order linear differential operator in z with complex-valued variable coefficients given by

$$(2.4) \quad M(r)v = -v_{zz} - \frac{1}{\alpha}(\beta(z, r) + i\gamma(z, r))v.$$

The existence, uniqueness and regularity of solutions of initial- and boundary-value problems for such complex *Sobolev type* p.d.e.'s (indeed in multidimensional

domains, with more general second-order operators in both sides of (2.3) and more general boundary conditions, but in the presence of smooth coefficients) have been studied by Lagnese, [20]. From the results of [20] one may infer that if for each $r \in [0, R]$ $1/\alpha q$ is not an eigenvalue of the operator $M(r)$ posed with the boundary conditions of (P), then the existence, uniqueness and regularity of the solution of (P) follows, under standard hypotheses such as appropriate smoothness of u^0 etc.. In the case that M has range-independent coefficients, it is shown in [20] that if $1/\alpha q$ is an eigenvalue of M , then, existence of solutions is guaranteed only for special u^0 satisfying compatibility conditions involving the spectrum of M . Although the analysis of [20] properly holds for smooth coefficients only, it is reasonable to expect that an analogous theory is valid for problems like (P), i.e. in the case of operators with discontinuous coefficients provided the solution satisfies the appropriate transmission conditions at the interfaces.

In order to investigate whether $1/\alpha q$ is an eigenvalue of $M(r)$ in the case of the specific problem (P) at hand, fix $r > 0$, let $\varphi : [0, 1] \rightarrow \mathbb{C}$ be the solution of the interface problem

$$(2.5) \quad \begin{aligned} (1 - \alpha q M)\varphi &= 0, & z \in (0, z_*) \cup (z_*, 1), \\ \varphi(0) &= 0, & \varphi'(1) + \mu\varphi(1) = 0, \\ \varphi(z_*^-) &= \varphi(z_*^+), & \varphi'(z_*^-) = \rho\varphi'(z_*^+), \end{aligned}$$

where M is given by (2.4), and ask whether (2.5) has only the trivial solution. (The r -dependence has been suppressed in the notation.) Denoting complex conjugation by overbars, we introduce the weighted $L^2(0, 1)$ inner product

$$(u, v) = \int_0^{z_*} u(z)\bar{v}(z)dz + \rho \int_{z_*}^1 u(z)\bar{v}(z)dz,$$

which is natural to the problem under consideration and will be used in the sequel instead of the customary $L^2(0, 1)$ inner product. (We also let $\|\cdot\| = (\cdot, \cdot)^{\frac{1}{2}}$ be the associated weighted L^2 norm.) Taking this inner product of both sides of the o.d.e. in (2.5) with φ , using the boundary and interface conditions indicated, integrating by parts, and separating real and imaginary parts we obtain the equations

$$(2.6) \quad \|\varphi\|^2 + q(\beta\varphi, \varphi) - \alpha q(\|\varphi'\|^2 + \rho\mu|\varphi(1)|^2) = 0,$$

and

$$(2.7) \quad (\gamma\varphi, \varphi) = 0.$$

If for each $r > 0$ $\gamma(z, r)$ is nonzero at least on a nonempty subinterval of $[0, 1]$, we may argue, in view of (2.7), that $\varphi = 0$ on $[0, 1]$. For instance, suppose that γ is of constant sign on an interval (ζ, η) . Then (2.7) implies that $\varphi \equiv 0$ on (ζ, η) , whence $\varphi(\zeta) = \varphi'(\zeta^+) = \varphi(\eta) = \varphi'(\eta^-) = 0$. Since φ satisfies homogeneous, second-order linear o.d.e.'s with smooth coefficients in each interval $(0, z_*)$ and $(z_*, 1)$, we may argue that φ is identically zero outside (ζ, η) too, in view of the uniqueness of solutions of the initial-value problem for such o.d.e.'s, the initial conditions at ζ and η and the transmission equations that φ satisfies at z_* . We conclude that adding for each $r > 0$ a small amount of ‘‘dissipation’’ or ‘‘absorption’’ (actually an imaginary part to the index of refraction that should be nonzero just in a small portion of some layer), ensures the well-posedness of the problem (P) for the wide-angle equation. This

should not be surprising as it is related, of course, to the effect of adding a small imaginary absorption coefficient to the Helmholtz equation, cf., e.g., [32].

In the absence of a dissipative term, i.e. if $\gamma = 0$, we may derive from (2.6) a sufficient condition on the coefficients of the equation that will ensure that $\varphi = 0$. To this end consider the Poincaré inequality

$$(2.8) \quad \|\varphi\|^2 \leq C_\rho \|\varphi'\|^2,$$

which is valid for all φ in V , the subspace of the Sobolev space $H^1(0, 1)$ whose elements vanish at $z = 0$. (By elementary means we may find the upper bound $\max(1, \rho/2)$ for the constant C_ρ ; thus it may be taken equal to one in (2.8) in the physically relevant case $\rho \leq 1$.) Using (2.8) in (2.6) we may easily see that, if $q > 0$, a sufficient condition that φ vanish on $[0, 1]$ is that

$$(2.9) \quad \frac{\alpha q}{C_\rho} \geq 1 + q \max_{z,r} \beta(z, r).$$

In the applications that we have in mind the condition (2.9) leads, as in the constant coefficient case, to an upper bound that the nondimensional number fz_{\max}/c_0 should not exceed in order that well-posedness of (P) is ensured. This upper bound is usually too small to cover many realistic cases of sound propagation in the sea that are properly modelled by the wide-angle equation, but then, in practice, there is always some absorption present that renders (P) well-posed. (We remark that if $q < 0$, a similar to (2.9) sufficient condition for invertibility of $(1 - \alpha q M)$ may be derived. The latter condition is usually satisfied in practical situations but, of course, the usual choice for q is $1/4$.)

In the sequel we shall assume that (P) has a unique solution which is sufficiently regular in $[0, z_*] \times [0, R]$ and in $[z_*, 1] \times [0, R]$ so that the finite difference approximations converge at the rates to be derived. To motivate analogous finite difference energy estimates we shall establish at appropriate places below additional properties of the solution of (P) . Thus, we prove in Section 2 that if $\lambda\gamma(z, r) \geq 0$ on $[0, 1] \times [0, R]$, then the *a priori* L^2 estimate

$$\|u(\cdot, r)\| \leq \|u^0\|$$

is valid for $0 \leq r \leq R$ (in fact as an equality if $\gamma = 0$). In section 5 we show that if $\gamma = 0$ or $q > 0$, then

$$\|u_z(\cdot, r)\| \leq C \|(u^0)'\|$$

for some constant $C = O(R^{\frac{1}{2}})$.

As a final note of interest it is worthwhile to point out that when (P) is discretized in range by an implicit finite difference method, such as the Crank–Nicolson scheme, an *artificial* absorption term is introduced due to the range-discretization, so that, even if $\gamma = 0$, the discretized problem still has a unique solution. (This was actually first observed by Greene in [16].) To see this, consider for simplicity a one-layer problem with $\gamma = 0$, $\rho = 1$ and no discontinuities across z_* . Discretizing (P) only in the range variable by the Crank–Nicolson scheme we obtain a sequence of functions $u_n(z)$ defined for $0 \leq z \leq 1$ and approximating $u(z, r^n)$ for $n = 0, 1, 2, \dots$. The $u_n(z)$ satisfy the boundary conditions of (P) at $z = 0$ and 1 , and are given for $n = 0$ by the

initial condition $u_0(z) = u^0(z)$ and for $n \geq 0$ by the scheme

$$(2.10) \quad \begin{aligned} & \frac{1}{k}(1 + q\beta(z, r^{n+\frac{1}{2}}))(u_{n+1}(z) - u_n(z)) \\ & + \frac{\alpha q}{k}(u''_{n+1}(z) - u''_n(z)) = -i\frac{\lambda}{2q}(u_{n+1}(z) + u_n(z)). \end{aligned}$$

Considering the associated homogenous o.d.e. (i.e. setting $u_n = 0$ in the above), and taking inner products of both sides with u_{n+1} we readily obtain the equation

$$([1 + q\beta(\cdot, r^{n+\frac{1}{2}})]u_{n+1}, u_{n+1}) - \alpha q(\|u'_{n+1}\|^2 + \mu|u_{n+1}(1)|^2) + i\frac{\lambda k}{2q}\|u_{n+1}\|^2 = 0,$$

from which, separating real and imaginary parts, we see that $u_{n+1} = 0$, with the nonzero artificial ‘‘absorption’’ term $\frac{\lambda k}{2q}$ appearing now as a consequence of the range discretization. As a result, (2.10) has a unique solution.

3. A two-point boundary-value problem. In this section we shall introduce notation and derive some auxiliary results that will be used in the error estimates of subsequent sections. They concern the approximation of the solution $w : [0, 1] \rightarrow \mathbb{C}$ of a two-point boundary-value problem associated with (P) , namely of

$$(3.1) \quad \begin{aligned} w''(z) &= f(z) \quad \text{in } (0, z_*) \cup (z_*, 1), \\ w'(z_*^-) &= \rho w'(z_*^+), \\ w(z_*^-) &= w(z_*^+), \\ w(0) &= 0, \\ w'(1) + \mu w(1) &= 0, \end{aligned}$$

where $f : [0, 1] \rightarrow \mathbb{C}$ is given, and z_* , ρ and μ are as in section 1. In the space V , introduced in section 2, we may reformulate (3.1) variationally as follows: seek $w \in V$ such that

$$(w', \varphi') + \rho\mu w(1)\bar{\varphi}(1) = -(f, \varphi)$$

is satisfied for all $\varphi \in V$. Since $\mu \geq 0$ this variational problem has a unique solution as is easily seen by an application of the Lax–Milgram theorem. Accordingly, we shall henceforth assume that f is such that (3.1) possesses a solution w which is continuous on $[0, 1]$ and smooth enough for our purposes on $[0, z_*]$ and on $[z_*, 1]$. We shall approximate the vector $\mathbf{w} \in \mathbb{C}_0^{J+1}$, where $\mathbf{w}_j := w(z_j)$, $0 \leq j \leq J$, by the solution $W \in \mathbb{C}_0^{J+1}$ of the difference scheme

$$(3.2) \quad \Delta_h W - \frac{2\mu}{h_+} \delta \otimes W = \hat{f},$$

wherein we have employed the difference operator notation introduced in section 1.

For the purpose of studying the stability and convergence of the various difference schemes we introduce in \mathbb{C}_0^{J+1} the weighted ℓ_2 inner product defined for $u, v \in \mathbb{C}_0^{J+1}$ by

$$(u, v)_h := h_- \sum_{j=0}^m u_j \bar{v}_j + \rho h_+ \sum_{j=m}^J u_j \bar{v}_j,$$

where $\sum_{j=s}^r \alpha_j := \frac{1}{2}(\alpha_s + \alpha_r) + \sum_{j=s+1}^{r-1} \alpha_j$. The norm $\|\cdot\|_h$ induced on \mathbb{C}_0^{J+1} by the inner product $(\cdot, \cdot)_h$ is then given for $v \in \mathbb{C}_0^{J+1}$ by

$$\|v\|_h = \left\{ h_- \sum_{j=1}^{m-1} |v_j|^2 + \hat{h} |v_m|^2 + \rho h_+ \sum_{j=m+1}^{J-1} |v_j|^2 + \frac{\rho h_+}{2} |v_J|^2 \right\}^{\frac{1}{2}}.$$

In addition, we shall also have occasion to use a discrete H^1 -type norm $|\cdot|_{1,h}$ defined for $v \in \mathbb{C}_0^{J+1}$ as

$$|v|_{1,h} = \left\{ h_- \sum_{j=0}^{m-1} \left| \frac{v_{j+1} - v_j}{h_-} \right|^2 + \rho h_+ \sum_{j=m}^{J-1} \left| \frac{v_{j+1} - v_j}{h_+} \right|^2 \right\}^{\frac{1}{2}}.$$

The difference operator $-\Delta_h$ is Hermitian and positive definite on \mathbb{C}_0^{J+1} with respect to the inner product $(\cdot, \cdot)_h$; straightforward computations using summation by parts yield

$$(3.3) \quad (\Delta_h u, v)_h = (u, \Delta_h v)_h \quad \forall u, v \in \mathbb{C}_0^{J+1},$$

and

$$(3.4) \quad (\Delta_h v, v)_h = -|v|_{1,h}^2 \quad \forall v \in \mathbb{C}_0^{J+1}.$$

Defining now the sesquilinear form $a_h(\cdot, \cdot)$ on \mathbb{C}_0^{J+1} by the formula

$$a_h(u, v) = -(\Delta_h u, v)_h + \rho \mu u_J \bar{v}_J \quad \text{for } u, v \in \mathbb{C}_0^{J+1},$$

we have, in view of (3.4),

$$(3.5) \quad a_h(v, v) = |v|_{1,h}^2 + \rho \mu |v_J|^2 \quad \text{for } v \in \mathbb{C}_0^{J+1}.$$

In particular $[a_h(\cdot, \cdot)]^{\frac{1}{2}}$ is a norm on \mathbb{C}_0^{J+1} since $\mu \geq 0$. Let us also note that the inequality

$$(3.6) \quad \max_{1 \leq j \leq J} |v_j|^2 \leq \max\left(1, \frac{1}{\rho}\right) |v|_{1,h}^2 \quad \forall v \in \mathbb{C}_0^{J+1}$$

(a discrete analog of Sobolev's inequality on V), may be established in a straightforward manner.

The existence–uniqueness of the solution $W \in \mathbb{C}_0^{J+1}$ of the tridiagonal system represented by the difference scheme (3.2) may now be easily established by taking the $(\cdot, \cdot)_h$ inner product of both sides of the homogeneous analog of (3.2) with W ; this yields precisely that $a_h(W, W) = 0$ enabling us to infer that the homogeneous system has only the solution $W = 0$, q.e.d..

The main result of this section is contained in the following Lemma. Here, and in the next two sections, we let $h := \max(h_-, h_+)$.

LEMMA 3.1. *Let the solution w of (3.1) be four times continuously differentiable on $[0, z_*]$ and on $[z_*, 1]$. Then, there exists a constant C , independent of h_- and h_+ , such that*

$$(3.7) \quad |w - W|_{1,h} \leq Ch^2$$

and

$$(3.8) \quad \max_{1 \leq j \leq J} |\mathbf{w}_j - W_j| \leq Ch^2.$$

Proof. Let $e = \mathbf{w} - W$. Then, by Taylor's theorem,

$$(3.9) \quad \Delta_h e - \frac{2\mu}{h_+} \delta \otimes e = \omega,$$

where $\omega \in \mathbb{C}_0^{J+1}$ with $|\omega_j| \leq Ch^2$ for $j \neq m, J$ and $|\omega_j| \leq Ch$ for $j = m, J$. Taking in (3.9) the $(\cdot, \cdot)_h$ inner product of both sides with e and using (3.5) we obtain

$$a_h(e, e) = -(\omega, e)_h \leq \max_j |e_j| (h_- \sum_{j=0}^{m'} |\omega_j| + \rho h_+ \sum_{j=m}^J |\omega_j|),$$

from which $a_h(e, e) \leq Ch^2 \max_j |e_j|$. Hence, (3.7) follows (in view of (3.5) and (3.6)); (3.8) is a consequence of (3.6) and (3.7). \square

4. Stability and convergence in the discrete L^2 -norm. In this section we shall study the stability and convergence in the $\|\cdot\|_h$ norm of the difference scheme (Δ) . To this end we shall assume throughout that $\lambda\gamma(z, r) \geq 0$ in $[0, 1] \times [0, R]$.

4.1. Stability. We consider first the continuous problem. Multiplying both sides in the p.d.e. in (P) by \bar{u}_r , integrating by parts on $[0, z_*]$ and taking imaginary parts we obtain

$$(4.1i) \quad q \int_0^{z_*} \gamma |u_r|^2 dz + \rho \alpha q \operatorname{Im}[u_{zr}(z_*^+, r) \bar{u}_r(z_*, r)] = -\frac{\lambda}{q} \operatorname{Re} \int_0^{z_*} u \bar{u}_r dz.$$

Analogously,

$$(4.1ii) \quad q \int_{z_*}^1 \gamma |u_r|^2 dz - \alpha q \operatorname{Im}[u_{zr}(z_*^+, r) \bar{u}_r(z_*, r)] = -\frac{\lambda}{q} \operatorname{Re} \int_{z_*}^1 u \bar{u}_r dz.$$

Multiplying (4.1ii) by ρ and adding to (4.1i) we get $q(\gamma u_r, u_r) = -\frac{\lambda}{q} \operatorname{Re}(u, u_r)$. Recalling that $\|\cdot\|$ denotes the weighted norm induced by the inner product (\cdot, \cdot) on $L^2(0, 1)$, we thus have

$$(4.2) \quad \frac{1}{2} \frac{d}{dr} \|u(r)\|^2 = -\frac{q^2}{\lambda} (\gamma u_r, u_r).$$

Since $\lambda\gamma \geq 0$, this yields

$$(4.3) \quad \|u(r)\| \leq \|u(s)\| \quad \text{for } 0 \leq s \leq r \leq R,$$

with *equality* if $\gamma = 0$.

The *stability* of the finite difference scheme (Δ) in the discrete L^2 -norm $\|\cdot\|_h$ may be derived by following the discrete analogs of the steps that led to the derivation of the *a priori* L^2 estimate (4.3). Specifically, taking the $(\cdot, \cdot)_h$ inner product of both sides of the difference equation in (Δ) with ∂U^n , using the commutativity of ∂ and Δ_h and (3.4), and finally taking imaginary parts yields

$$(4.4) \quad \frac{1}{2} (\|U^{n+1}\|_h^2 - \|U^n\|_h^2) = -\frac{q^2}{\lambda} k(\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial U^n, \partial U^n)_h,$$

from which the discrete analog of (4.3)

$$(4.5) \quad \|U^{n+1}\|_h \leq \|U^n\|_h, \quad 0 \leq n \leq N-1,$$

follows. Again, (4.5) holds as an equality in the conservative case $\gamma = 0$. Needless to say, the existence and uniqueness of solutions of the difference scheme (Δ) follow immediately from (4.5).

4.2. Convergence. The theorem that follows contains the main result of this section, namely that the finite difference method (Δ) converges to the solution of (P) with optimal rates in the $\|\cdot\|_h$ norm. For the proof we use the device of comparing the solution of the scheme (Δ) with an *elliptic approximation* in \mathbb{C}_0^{J+1} of the solution of (P) , cf. [1]. This enables us to obtain the optimal-order error estimate in view of the results of Lemma 3.1; a straightforward estimation would not yield the optimal rates since the z -component of the truncation error of the difference scheme applied to the exact solution u is of second order in h for $j \neq m, J$ but only of first order if $j = m$ or J .

THEOREM 4.1. *Let $\lambda\gamma \geq 0$ and suppose that the solution u of (P) is sufficiently smooth in $[0, z_*] \times [0, R]$ and in $[z_*, 1] \times [0, R]$. Let $\{U^n\}_{n=0}^N$ be the solution of the finite difference scheme (Δ) . Then, there exists a constant $C = C(u, R)$, independent of h_-, h_+ and k , such that*

$$(4.6) \quad \max_{0 \leq n \leq N} \|u^n - U^n\|_h \leq C(k^2 + h^2).$$

Proof. Given a continuous function $v : [0, 1] \rightarrow \mathbb{C}$ that is smooth on $[0, z_*]$ and on $[z_*, 1]$ and satisfies $v'(z_*^-) = \rho v'(z_*^+)$, $v(0) = 0$, and $v'(1) + \mu v(1) = 0$, let $Pv \in \mathbb{C}_0^{J+1}$ be its *elliptic approximation*, defined by the equations

$$(4.7) \quad \begin{aligned} \Delta_h(Pv)_j - \frac{2\mu}{h_+} \delta_j(Pv)_j &= v''(z_j), \quad 1 \leq j \leq J, \quad j \neq m, \\ \Delta_h(Pv)_m &= \frac{1}{2h} [\rho h_+ v''(z_*^+) + h_- v''(z_*^-)], \end{aligned}$$

wherein the notation introduced in section 1 has been employed again. In vector notation (4.7) is written as $\Delta_h(Pv) - \frac{2\mu}{h_+} \delta \otimes (Pv) = \widehat{v''}$, i.e. as an equation of the form (3.2); it follows that Pv is well-defined for all v with the aforementioned properties.

Let now $W(r) = Pu(r) := Pu(\cdot, r)$. It follows by (4.7) that W is a smooth function of r , since P commutes with differentiation with respect to r . For $0 \leq n \leq N$ define $W^n := W(r^n)$, and $\zeta^n := u^n - W^n$, $\vartheta^n := W^n - U^n$, so that $u^n - U^n = \zeta^n + \vartheta^n$. By (3.8) we infer that $\max_n \max_j |u_j^n - W_j^n| \leq Ch^2$; hence, by the definition of the $\|\cdot\|_h$ norm we see that

$$(4.8) \quad \max_{0 \leq n \leq N} \|\zeta^n\|_h \leq Ch^2.$$

There remains to estimate $\|\vartheta^n\|_h$. With this aim in mind, we study the truncation error of the finite difference scheme (Δ) when applied to $W(r)$. Using (4.7), it is not hard to see that for $0 \leq n \leq N-1$

$$(4.9) \quad \begin{aligned} \partial W^n + q[\hat{\beta}(r^{n+\frac{1}{2}}) + i\hat{\gamma}(r^{n+\frac{1}{2}})] \otimes \partial W^n \\ - 2\alpha q \frac{\mu}{h_+} \delta \otimes \partial W^n + \alpha q \partial \Delta_h W^n = -i \frac{\lambda}{q} W^{n+\frac{1}{2}} + \omega^n. \end{aligned}$$

Here $\omega^n \in \mathbb{C}_0^{J+1}$, $\omega^n = \omega_1^n + \omega_2^n + \omega_3^n + \omega_4^n + \omega_5^n$, where the ω_{ij}^n are defined, for $1 \leq i \leq 5$ and $1 \leq j \leq J$, as follows:

$$\begin{aligned}\omega_{1j}^n &:= [1 + q\hat{\beta}_j(r^{n+\frac{1}{2}}) + iq\hat{\gamma}_j(r^{n+\frac{1}{2}})](\partial W_j^n - \partial u_j^n), \\ \omega_{2j}^n &:= [1 + q\hat{\beta}_j(r^{n+\frac{1}{2}}) + iq\hat{\gamma}_j(r^{n+\frac{1}{2}})][\partial u_j^n - u_r(z_j, r^{n+\frac{1}{2}})], \\ \omega_{3j}^n &:= \alpha q[(\partial \widehat{u}_{zz}(r^n))_j - (\widehat{u}_{zzr}(r^{n+\frac{1}{2}}))_j], \\ \omega_{4j}^n &:= i\frac{\lambda}{q}(W_j^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}), \\ &\text{and} \\ \omega_{5j}^n &:= i\frac{\lambda}{q}[u_j^{n+\frac{1}{2}} - u(z_j, r^{n+\frac{1}{2}})].\end{aligned}$$

Noting that the elliptic approximation operator P commutes with differentiation with respect to r and taking into account Lemma 3.1 we obtain e.g. that

$$|\partial W_j^n - \partial u_j^n| = \frac{1}{k} \left| \int_{r^n}^{r^{n+1}} \left(\frac{\partial W_j}{\partial r}(r) - u_r(z_j, r) \right) dr \right| \leq Ch^2,$$

from which $\max_{j,n} |\omega_{1j}^n| \leq Ch^2$. Obviously, $\max_{j,n} |\omega_{4j}^n| \leq Ch^2$. The remaining ω 's can be estimated by an $O(k^2)$ bound in a straightforward manner, using Taylor expansions in r ; we finally conclude that

$$(4.10) \quad \max_n \|\omega^n\|_h \leq C(k^2 + h^2).$$

For the estimation of ϑ^n below we shall also need an estimate of the form

$$(4.11) \quad \max_n \|\partial \omega^n\|_h \leq C(k^2 + h^2).$$

This also follows in a straightforward manner from Lemma 3.1 and Taylor's theorem. For example, easy calculations yield that

$$\partial \omega_{1j}^{n-1} = \frac{1}{k^2} [1 + q\hat{\beta}_j(r^n) + iq\hat{\gamma}_j(r^n)] \int_{r^n}^{r^{n+1}} \int_{r-k}^r \left(\frac{\partial^2 W_j}{\partial r^2}(s) - \frac{\partial^2 u}{\partial r^2}(z_j, s) \right) ds dr + O(h^2),$$

from which $\max_{j,n} |\partial \omega_{1j}^n| \leq Ch^2$. Similarly, $\max_{j,n} |\partial \omega_{4j}^n| \leq Ch^2$, whereas Taylor expansions in r yield $\max_{i=2,3,5} \max_{j,n} |\partial \omega_{ij}^n| \leq Ck^2$, and (4.11) follows.

With the truncation error under control we proceed now to estimate $\vartheta^n = W^n - U^n$. Subtracting the difference equation in (Δ) from (4.9) yields, for $0 \leq n \leq N-1$

$$(4.12) \quad \begin{aligned} &\partial \vartheta^n + q[\hat{\beta}(r^{n+\frac{1}{2}}) + i\hat{\gamma}(r^{n+\frac{1}{2}})] \otimes \partial \vartheta^n \\ &\quad - 2\alpha q \frac{\mu}{h_+} \delta \otimes \partial \vartheta^n + \alpha q \partial \Delta_h \vartheta^n = -i\frac{\lambda}{q} \vartheta^{n+\frac{1}{2}} + \omega^n. \end{aligned}$$

Taking now the $(\cdot, \cdot)_h$ inner product of both sides of this equation with $\partial \vartheta^n$, using the commutativity of ∂ and Δ_h and (3.4), and taking imaginary parts we obtain

$$(4.13) \quad \|\vartheta^{n+1}\|_h^2 - \|\vartheta^n\|_h^2 = -\frac{2q^2}{\lambda} k(\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial \vartheta^n, \partial \vartheta^n)_h + \frac{2q}{\lambda} k \operatorname{Im}(\omega^n, \partial \vartheta^n)_h,$$

wherefrom, since $\lambda\gamma \geq 0$, it follows that

$$(4.14) \quad \|\vartheta^{n+1}\|_h^2 - \|\vartheta^n\|_h^2 \leq \frac{2q}{\lambda} k \operatorname{Im}(\omega^n, \partial \vartheta^n)_h.$$

Sum now both sides of this inequality from $n = 0$ to $n = M - 1$, where $1 \leq M \leq N$. Summation by parts in the right-hand side yields

$$(4.15) \quad \|\vartheta^M\|_h^2 - \|\vartheta^0\|_h^2 \leq \frac{2q}{\lambda} \operatorname{Im}\{(\omega^{M-1}, \vartheta^M)_h - k \sum_{n=1}^{M-1} (\partial\omega^{n-1}, \vartheta^n)_h - (\omega^0, \vartheta^0)_h\}.$$

Since $\|\vartheta^0\|_h = \|\zeta^0\|_h \leq Ch^2$, using the Cauchy–Schwarz inequality in the above we obtain, in view of (4.10) and (4.11), that

$$\|\vartheta^M\|_h^2 \leq C(k^2 + h^2)^2 + Ck \sum_{n=0}^{M-1} \|\vartheta^n\|_h^2, \quad 1 \leq M \leq N.$$

We conclude, by Gronwall’s discrete inequality that

$$(4.16) \quad \max_{0 \leq n \leq N} \|\vartheta^n\|_h \leq C(k^2 + h^2),$$

and (4.6) follows, in view of (4.8). \square

5. Stability and convergence in the discrete H^1 – and maximum norms.

In addition to our hypothesis $\lambda\gamma(z, r) \geq 0$ in $[0, 1] \times [0, R]$, we assume in this section that $\gamma = 0$ or $q > 0$.

5.1. Stability. Consider first the continuous problem. Multiplying both sides of the p.d.e. in (P) by \bar{u} and integrating by parts yields

$$(5.1i) \quad \int_0^{z_*} (1 + q\beta + iq\gamma)u_r \bar{u} dz + \rho\alpha q u_{zr}(z_*^+, r)\bar{u}(z_*, r) - \alpha q \int_0^{z_*} u_{zr} \bar{u}_z dz = -i \frac{\lambda}{q} \int_0^{z_*} |u|^2 dz.$$

Similarly,

$$(5.1ii) \quad \int_{z_*}^1 (1 + q\beta + iq\gamma)u_r \bar{u} dz - \alpha q \mu u_r(1, r)\bar{u}(1, r) - \alpha q u_{zr}(z_*^+, r)\bar{u}(z_*, r) - \alpha q \int_{z_*}^1 u_{zr} \bar{u}_z dz = -i \frac{\lambda}{q} \int_{z_*}^1 |u|^2 dz.$$

Multiplying (5.1ii) by ρ , adding to (5.1i) and taking real parts yields now

$$(5.2) \quad \alpha q \frac{d}{dr} \{ \|u_z(r)\|^2 + \rho\mu |u(1, r)|^2 \} = \frac{d}{dr} \|u(r)\|^2 + q \frac{d}{dr} (\beta u, u) - q(\beta_r u, u) - 2q \operatorname{Im}(\gamma u_r, u).$$

If $\gamma = 0$, integrating (5.2) from 0 to r , and using the L^2 *a priori* bound (4.3), and the Poincaré–Friedrichs and Sobolev inequalities, we conclude that

$$(5.3) \quad \|u_z(r)\| \leq C \|(u^0)'\|, \quad 0 \leq r \leq R,$$

for some constant $C = O(\sqrt{R})$. In case $\gamma \neq 0$, suppose $q > 0$ and note that for $\varepsilon > 0$ $|q(\gamma u_r, u)| = |\lambda q| \lambda^{-1} (\gamma u_r, u) \leq \varepsilon \lambda^{-1} (\gamma u_r, u_r) + C_\varepsilon \|u\|^2$. Choose $\varepsilon \leq q^2$; then, by (4.2)

$$\frac{d}{dr} \|u(r)\|^2 + \frac{2\varepsilon}{\lambda} (\gamma u_r, u_r) \leq 0.$$

These inequalities, inserted into the right-hand side of (5.2), give

$$(5.4) \quad \alpha q \frac{d}{dr} \{ \|u_z(r)\|^2 + \rho \mu |u(1, r)|^2 \} \leq C \|u(r)\|^2 + q \frac{d}{dr} (\beta u, u),$$

which in turn implies (5.3).

Motivated by these estimates we now proceed to prove a discrete analog of (5.3). Taking the $(\cdot, \cdot)_h$ inner product of the difference equation in (Δ) with $U^{n+\frac{1}{2}}$, using (3.3)–(3.5), and taking real parts yields the following analog of (5.2):

$$(5.5) \quad \begin{aligned} \alpha q \{ a_h(U^{n+1}, U^{n+1}) - a_h(U^n, U^n) \} &= \|U^{n+1}\|_h^2 - \|U^n\|_h^2 \\ &+ q \{ (\hat{\beta}(r^{n+\frac{1}{2}}) \otimes U^{n+1}, U^{n+1})_h - (\hat{\beta}(r^{n+\frac{1}{2}}) \otimes U^n, U^n)_h \} \\ &- 2qk \operatorname{Im}(\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial U^n, U^{n+\frac{1}{2}})_h. \end{aligned}$$

If $\gamma = 0$ this implies

$$\begin{aligned} a_h(U^{n+1}, U^{n+1}) - \frac{1}{\alpha q} \|U^{n+1}\|_h^2 - \frac{1}{\alpha} (\hat{\beta}(r^{n+1}) \otimes U^{n+1}, U^{n+1})_h \\ \leq a_h(U^n, U^n) - \frac{1}{\alpha q} \|U^n\|_h^2 - \frac{1}{\alpha} (\hat{\beta}(r^n) \otimes U^n, U^n)_h + Ck (\|U^{n+1}\|_h^2 + \|U^n\|_h^2). \end{aligned}$$

Hence, summing from $n = 0$ to $n = M - 1 \leq N - 1$, using (4.5), (3.5) and (3.6) we conclude that for some constant $C = O(\sqrt{R})$

$$(5.6) \quad |U^n|_{1,h} \leq C |U^0|_{1,h}, \quad 1 \leq n \leq N.$$

Let now $\gamma \neq 0$ and $q > 0$. As in the continuous estimates we have

$$|q(\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial U^n, U^{n+\frac{1}{2}})_h| \leq \varepsilon \lambda^{-1} (\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial U^n, \partial U^n)_h + C_\varepsilon \|U^{n+\frac{1}{2}}\|_h^2,$$

and (choosing $\varepsilon \leq q^2$) by (4.4)

$$\|U^{n+1}\|_h^2 - \|U^n\|_h^2 + \frac{2\varepsilon}{\lambda} k (\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial U^n, \partial U^n)_h \leq 0.$$

Therefore (5.5) now yields

$$\begin{aligned} \alpha q \{ a_h(U^{n+1}, U^{n+1}) - a_h(U^n, U^n) \} &\leq q \{ (\hat{\beta}(r^{n+1}) \otimes U^{n+1}, U^{n+1})_h \\ &- (\hat{\beta}(r^n) \otimes U^n, U^n)_h \} + Ck (\|U^{n+1}\|_h^2 + \|U^n\|_h^2), \end{aligned}$$

from which an estimate of the type (5.6) follows easily.

5.2. Convergence. **THEOREM 5.1.** *Let $\lambda\gamma \geq 0$, $\gamma = 0$ or $q > 0$ and suppose that the solution u of (P) is sufficiently smooth in $[0, z_*] \times [0, R]$ and in $[z_*, 1] \times [0, R]$. Let $\{U^n\}_{n=0}^N$ be the solution of the finite difference scheme (Δ) . Then, for some constant $C = C(u, R)$ we have*

$$(5.7) \quad \max_{0 \leq n \leq N} |u^n - U^n|_{1,h} \leq C(k^2 + h^2)$$

and

$$(5.8) \quad \max_{0 \leq n \leq N} \max_{0 \leq j \leq J} |u_j^n - U_j^n| \leq C(k^2 + h^2).$$

Proof. The estimate (5.8) follows from (5.7) and (3.6). In order to show (5.7) we first note —using notation introduced in the course of the proof of Theorem 4.1— that, in view of Lemma 3.1, we have

$$(5.9) \quad \max_{0 \leq n \leq N} |\zeta^n|_{1,h} \leq Ch^2.$$

There remains to estimate $|\vartheta^n|_{1,h}$. To this end, taking the $(\cdot, \cdot)_h$ inner product of both sides of (4.12) with $\vartheta^{n+\frac{1}{2}}$, using (3.3)–(3.5), and taking real parts yields, for $0 \leq n \leq N-1$,

$$(5.10) \quad \begin{aligned} \alpha q \{a_h(\vartheta^{n+1}, \vartheta^{n+1}) - a_h(\vartheta^n, \vartheta^n)\} &= \|\vartheta^{n+1}\|_h^2 - \|\vartheta^n\|_h^2 \\ &+ q \{(\hat{\beta}(r^{n+\frac{1}{2}}) \otimes \vartheta^{n+1}, \vartheta^{n+1})_h - (\hat{\beta}(r^{n+\frac{1}{2}}) \otimes \vartheta^n, \vartheta^n)_h\} \\ &- 2qk \operatorname{Im}(\hat{\gamma}(r^{n+\frac{1}{2}}) \otimes \partial \vartheta^n, \vartheta^{n+\frac{1}{2}})_h - 2k \operatorname{Re}(\omega^n, \vartheta^{n+\frac{1}{2}})_h. \end{aligned}$$

If $\gamma = 0$, using (4.10) and (4.16), and then summing both sides from $n = 0$ to $n = M-1 \leq N-1$ yields

$$a_h(\vartheta^M, \vartheta^M) - \frac{1}{\alpha q} \|\vartheta^M\|_h^2 - \frac{1}{\alpha} (\hat{\beta}(r^M) \otimes \vartheta^M, \vartheta^M)_h \leq a_h(\vartheta^0, \vartheta^0) + C(k^2 + h^2)^2.$$

Since $\vartheta^0 = -\zeta^0$, (3.5)–(3.7) imply

$$(5.11) \quad \max_{0 \leq n \leq N} |\vartheta^n|_{1,h} \leq C(k^2 + h^2).$$

In case $\gamma \neq 0$ and $q > 0$, using in (5.10) the estimation idea of the stability proof in the discrete H^1 norm, and also (4.13) and (4.16), yields

$$\begin{aligned} \alpha q \{a_h(\vartheta^{n+1}, \vartheta^{n+1}) - a_h(\vartheta^n, \vartheta^n)\} &\leq q \{(\hat{\beta}(r^{n+1}) \otimes \vartheta^{n+1}, \vartheta^{n+1})_h \\ &- (\hat{\beta}(r^n) \otimes \vartheta^n, \vartheta^n)_h\} + \frac{2q}{\lambda} k \operatorname{Im}(\omega^n, \partial \vartheta^n)_h + Ck(k^2 + h^2). \end{aligned}$$

Summing both sides of this equation from $n = 0$ to $n = M-1 \leq N-1$ and using (4.16), and summation by parts gives

$$\begin{aligned} \alpha q \{a_h(\vartheta^M, \vartheta^M) - a_h(\vartheta^0, \vartheta^0)\} &\leq C(k^2 + h^2)^2 \\ &+ \frac{2q}{\lambda} \operatorname{Im}\{(\omega^{M-1}, \vartheta^M)_h - k \sum_{n=1}^{M-1} (\partial \omega^{n-1}, \vartheta^n)_h - (\omega^0, \vartheta^0)\}. \end{aligned}$$

From this estimate and with the aid of (5.9), (4.10), (4.11) and (4.16) we conclude that (5.11) is valid in this case as well, and (5.7) follows. \square

6. Arbitrary partitions in the depth variable. The technique of error estimation of the two previous sections requires uniform partitions in the range variable; otherwise, no estimates of the form (4.11) would be available. It may be readily generalized, however, to cover the case of an arbitrary partition in the *depth* variable z , provided, of course, that the endpoints and the interface point z_* are nodes of this partition. This is what will be studied in this section.

In what follows, let $0 = z_0 < z_1 < \dots < z_J = 1$ be an arbitrary partition of $[0,1]$ such that $z_m = z_*$. Let $h_j := z_j - z_{j-1}$, $1 \leq j \leq J$, $h_{J+1} := 0$, $\hat{h}_j := (h_j + h_{j+1})/2$,

$j \neq m$, and $\hat{h}_m := (h_m + \rho h_{m+1})/2$. For the purposes of this section, if $v \in \mathbb{C}_0^{J+1}$ we define $\Delta_h v \in \mathbb{C}_0^{J+1}$ as

$$\begin{aligned}\Delta_h v_0 &:= 0, \\ \Delta_h v_j &:= \frac{1}{\hat{h}_j} \left(\frac{v_{j+1} - v_j}{h_{j+1}} - \frac{v_j - v_{j-1}}{h_j} \right), \quad 1 \leq j \leq J-1, \quad j \neq m, \\ \Delta_h v_m &:= \frac{1}{\hat{h}_m} \left(\rho \frac{v_{m+1} - v_m}{h_{m+1}} - \frac{v_m - v_{m-1}}{h_m} \right), \\ \Delta_h v_J &:= \frac{2}{h_J^2} (v_{J-1} - v_J).\end{aligned}$$

The difference scheme to be analyzed is then the following generalization of (Δ) : for $0 \leq n \leq N$ approximate $u^n \in \mathbb{C}_0^{J+1}$, where $u_j^n := u(z_j, r^n)$, by vectors $U^n \in \mathbb{C}_0^{J+1}$ satisfying

$$(6.1) \quad \begin{aligned}U^0 &= u^0 \\ \partial U^n + q[\hat{\beta}(r^{n+\frac{1}{2}}) + i\hat{\gamma}(r^{n+\frac{1}{2}})] \otimes \partial U^n \\ &\quad - 2\alpha q \frac{\mu}{h_J} \delta \otimes \partial U^n + \alpha q \partial \Delta_h U^n = -i \frac{\lambda}{q} U^{n+\frac{1}{2}}, \quad 0 \leq n \leq N-1.\end{aligned}$$

Here and in the remainder of this section, we associate with a function $f : [0, 1] \rightarrow \mathbb{C}$ a vector $\hat{f} \in \mathbb{C}_0^{J+1}$ given by $\hat{f}_j = f(z_j)$, $1 \leq j \leq J$, $j \neq m$, $\hat{f}_m = \frac{1}{2h_m} [h_m f(z_*^-) + \rho h_{m+1} f(z_*^+)]$.

Using the notation $\tilde{h}_j := \hat{h}_j$, $1 \leq j \leq m$, $\tilde{h}_j := \rho \hat{h}_j$, $m+1 \leq j \leq J$, we define now a discrete weighted L^2 inner product on \mathbb{C}_0^{J+1} by

$$(v, w)_H := \sum_{j=1}^J \tilde{h}_j v_j \bar{w}_j,$$

and denote by $\|\cdot\|_H$ the associated norm. We shall also use discrete weighted H^1 and H^{-1} type norms, given, in case $\mu \neq 0$, for $v \in \mathbb{C}_0^{J+1}$ by

$$(6.2i) \quad \|v\|_{1,H} = \left\{ \sum_{j=1}^m h_j \left| \frac{v_j - v_{j-1}}{h_j} \right|^2 + \rho \sum_{j=m+1}^J h_j \left| \frac{v_j - v_{j-1}}{h_j} \right|^2 + \rho \mu |v_J|^2 \right\}^{\frac{1}{2}},$$

$$(6.2ii) \quad \|v\|_{-1,H} = \left\{ \sum_{j=1}^m h_j \left| \sum_{\ell=1}^{j-1} \tilde{h}_\ell v_\ell \right|^2 + \frac{1}{\rho} \sum_{j=m+1}^J h_j \left| \sum_{\ell=1}^{j-1} \tilde{h}_\ell v_\ell \right|^2 + \frac{1}{\rho \mu} \left| \sum_{\ell=1}^J \tilde{h}_\ell v_\ell \right|^2 \right\}^{\frac{1}{2}}.$$

(Note that the quantities $(v, w)_H$ and $\|v\|_{1,H}$ are extensions to the arbitrary mesh case of $(v, w)_h$ and $[a_h(v, v)]^{\frac{1}{2}}$, respectively.) For $\mu = 0$ we slightly modify the definitions of $\|\cdot\|_{1,H}$ and $\|\cdot\|_{-1,H}$ by setting $\mu = 0$ in (6.2i), and replacing μ by one in (6.2ii). With this notation in place we may prove that

$$(6.3) \quad |(v, w)_H| \leq C \|v\|_{-1,H} \|w\|_{1,H} \quad \forall v, w \in \mathbb{C}_0^{J+1},$$

with $C = 1$ for $\mu \neq 0$. Indeed, for such v and w and letting $\tilde{h}_0 = 0$ we have

$$\begin{aligned}
(v, w)_H &= \sum_{j=1}^J \tilde{h}_j v_j \bar{w}_j = \sum_{j=1}^J \sum_{\ell=1}^j (\tilde{h}_\ell v_\ell - \tilde{h}_{\ell-1} v_{\ell-1}) \bar{w}_j \\
&= \sum_{j=1}^J \left(\sum_{\ell=1}^j \tilde{h}_\ell v_\ell \right) \bar{w}_j - \sum_{j=0}^{J-1} \left(\sum_{\ell=0}^j \tilde{h}_\ell v_\ell \right) \bar{w}_{j+1} \\
&= - \sum_{j=2}^J h_j \left(\sum_{\ell=1}^{j-1} \tilde{h}_\ell v_\ell \right) \frac{\bar{w}_j - \bar{w}_{j-1}}{h_j} + \sum_{\ell=1}^J \tilde{h}_\ell v_\ell \bar{w}_J,
\end{aligned}$$

from which (6.3) follows for $\mu \neq 0$ directly by the Cauchy–Schwarz inequality. In case $\mu = 0$ we have to use also the easily established fact that $\|\cdot\|_{1,H}$ dominates (modulo a multiplicative constant depending on ρ) the discrete maximum norm.

The *consistency* of the scheme (6.1) may be studied again in terms of an elliptic approximation $W \in \mathbb{C}_0^{J+1}$ of the solution w of (3.1) given by

$$(6.4) \quad \Delta_h W - \frac{2\mu}{h_J} \delta \otimes W = \hat{f}.$$

Assuming that the solution w of (3.1) is sufficiently smooth in $[0, z_*]$ and in $[z_*, 1]$, defining again $\mathbf{w} \in \mathbb{C}_0^{J+1}$, where $\mathbf{w}_j := w(z_j)$, $0 \leq j \leq J$, and letting $\omega \in \mathbb{C}_0^{J+1}$ be the local error of (6.4) defined by

$$(6.5) \quad \omega := \Delta_h \mathbf{w} - \frac{2\mu}{h_J} \delta \otimes \mathbf{w} - \hat{f},$$

we have by straightforward Taylor expansions

$$(6.6) \quad \omega_j = \begin{cases} \frac{1}{3}(h_{j+1} - h_j)w'''(z_j) + \tilde{\omega}_j & \text{if } 1 \leq j \leq J-1, \quad j \neq m, \\ \frac{1}{6\hat{h}_m}[\rho h_{m+1}^2 w'''(z_*^+) - h_m^2 w'''(z_*^-)] + \tilde{\omega}_m & \text{if } j = m, \\ -\frac{h_J}{3}w'''(z_J) + \tilde{\omega}_J & \text{if } j = J, \end{cases}$$

where

$$(6.7) \quad \max_{1 \leq j \leq J} |\tilde{\omega}_j| \leq Ch^2,$$

with $h = \max_j h_j$. It is straightforward to check that the analogs of the relations (3.3) and (3.5) hold in the arbitrary mesh case as well. Specifically, for $u, v \in \mathbb{C}_0^{J+1}$, we have that $(\Delta_h u, v)_H = (u, \Delta_h v)_H$ and $-(\Delta_h v, v)_H + \frac{2\mu}{h_J}(\delta \otimes v, v)_H = \|v\|_{1,H}^2$. We conclude, taking the $(\cdot, \cdot)_H$ inner product of both sides of (6.4), that if $\hat{f} = 0$, then $W = 0$, i.e. that the discrete problem (6.4) has a unique solution. To prove its convergence to the solution of (3.1), let $e = \mathbf{w} - W$ and subtract (6.4) from (6.5) to obtain $\Delta_h e - \frac{2\mu}{h_J} \delta \otimes e = \omega$. Taking inner products yields $\|e\|_{1,H}^2 = -(\omega, e)_H$, from which, using (6.3) we conclude that

$$(6.8) \quad \|e\|_{1,H} \leq C\|\omega\|_{-1,H}.$$

It is now straightforward to prove, using the definition of the discrete H^{-1} norm and the relations (6.6) and (6.7) that

$$(6.9) \quad \|\omega\|_{-1,H} \leq Ch^2.$$

Hence, it follows by (6.8) that

$$(6.10) \quad \|e\|_{1,H} \leq Ch^2.$$

Since, in addition, the $\|\cdot\|_{1,H}$ norm dominates (modulo a multiplicative constant) the discrete maximum norm, (6.10) implies

$$(6.11) \quad \max_{1 \leq j \leq J} |e_j| \leq Ch^2.$$

The estimates (6.10) and (6.11) of the error e of the elliptic approximation W to the solution w of (3.1) are precisely what is needed to generalize the results of sections 3 and 4 to the general mesh case considered here. The proofs are straightforward analogs of the proofs of Theorems 4.1 and 5.1 and, consequently, will be omitted. The result is:

THEOREM 6.1. *Let $\lambda\gamma \geq 0$. Then the solution of the finite difference scheme (6.1) exists uniquely and satisfies $\|U^{n+1}\|_H \leq \|U^n\|_H$, for $0 \leq n \leq N-1$, where $\|\cdot\|_H$ is the norm induced on \mathbb{C}_0^{J+1} by the inner product $(\cdot, \cdot)_H$. If the solution u of (P) is sufficiently smooth in $[0, z_*] \times [0, R]$ and in $[z_*, 1] \times [0, R]$, there exists a constant $C = C(u, R)$ such that*

$$\max_{0 \leq n \leq N} \|u^n - U^n\|_H \leq C(k^2 + h^2).$$

If, in addition, $\gamma = 0$ or $q > 0$, there also holds

$$\max_{0 \leq n \leq N} (\|u^n - U^n\|_{1,H} + \max_{0 \leq j \leq J} |u_j^n - U_j^n|) \leq C(k^2 + h^2). \quad \square$$

7. The forward Euler method. In this section we shall briefly discuss the discretization of (P) by the *forward Euler scheme* which is of first order accuracy in the range and of second order in the depth variable.

Using for simplicity the uniform mesh and the notation introduced in section 1, we write the method as follows: For $0 \leq n \leq N$, we seek $U^n \in \mathbb{C}_0^{J+1}$ approximating $u^n \in \mathbb{C}_0^{J+1}$, $u_j^n := u(z_j, r^n)$, and satisfying

$$(E) \quad \begin{aligned} &U^0 = u^0 \\ &\text{for } n = 0, \dots, N-1: \\ &\partial U^n + q[\hat{\beta}(r^n) + i\hat{\gamma}(r^n)] \otimes \partial U^n - 2\alpha q \frac{\mu}{h_+} \delta \otimes \partial U^n + \alpha q \partial \Delta_h U^n = -i \frac{\lambda}{q} U^n. \end{aligned}$$

Although the forward Euler method is ‘explicit’, determining U^{n+1} in terms of U^n in (E) requires solving a $J \times J$ tridiagonal system of equations due to the ‘implicit’ character of the wide-angle equation. Hence, being roughly as expensive as the Crank–Nicolson scheme and only of first order of accuracy in range, the Euler method should not be used in practice. Nevertheless, for theoretical purposes, it is worthwhile to point out that the scheme (E) is *stable* in the weighted ℓ_2 norm $\|\cdot\|_h$, under no mesh

conditions, provided either $\gamma(z, r)$ is bounded away from zero in $[0, 1] \times [0, R]$, or a condition of the form

$$(7.1) \quad \begin{aligned} \frac{\alpha q}{C_\rho} &> 1 + q \max_{z,r} \beta(z, r), \quad \text{if } q > 0, \\ \frac{\alpha|q|}{C_\rho} &> -1 + |q| \max_{z,r} \beta(z, r), \quad \text{if } q < 0, \end{aligned}$$

is imposed, where C_ρ as in (2.8).

This result is somewhat interesting on the one hand because it is well known that the forward Euler scheme is unconditionally unstable for the standard PE. On the other hand, note that the conditions for its stability (either the one on γ or (7.1)) are sufficient for the well-posedness of (P) in that they guarantee (as the analysis in section 2 shows) the invertibility of the operator in the left-hand side of the wide-angle p.d.e. in (P) . Hence, it appears that under these conditions (and probably just under the hypothesis that (P) is well-posed), continuous-in-range discretizations of (P) are not stiff and can be stably solved by ‘explicit’ range-stepping schemes, such as explicit Runge–Kutta methods. In this respect the wide-angle equation resembles its real-valued Sobolev p.d.e. counterparts, cf., e.g., [?]. It is questionable though whether such ‘explicit’ schemes will have any computational advantages over implicit methods, given that they too require the solution of linear systems and are not conservative.

To prove the stability of the Euler scheme under the stated conditions, we take the $(\cdot, \cdot)_h$ inner product of both sides of the difference equation in (E) with ∂U^n obtaining

$$(7.2) \quad \begin{aligned} \|\partial U^n\|_h^2 + q(\hat{\beta}(r^n) \otimes \partial U^n, \partial U^n)_h + iq(\hat{\gamma}(r^n) \otimes \partial U^n, \partial U^n)_h \\ - \alpha q[|\partial U^n|_{1,h}^2 + \rho\mu|\partial U_J^n|^2] = -i\frac{\lambda}{q}(U^n, \partial U^n)_h. \end{aligned}$$

Assume first that $\gamma(z, r)$ is of one sign on $[0, 1] \times [0, R]$. Taking imaginary parts in (7.2) yields

$$(\hat{\gamma}(r^n) \otimes \partial U^n, \partial U^n)_h = -\frac{\lambda}{q^2} \text{Re}(U^n, \partial U^n)_h.$$

Therefore, use of our hypothesis and the Cauchy–Schwarz inequality imply that for some positive constant c

$$(7.3) \quad \|\partial U^n\|_h^2 \leq c\|U^n\|_h\|\partial U^n\|_h.$$

Hence, $\|\partial U^n\| \leq c\|U^n\|_h$, implying that

$$\|U^{n+1}\|_h \leq (1 + ck)\|U^n\|_h,$$

i.e.,

$$(7.4) \quad \max_{0 \leq n \leq N} \|U^n\|_h \leq C\|U^0\|_h$$

for some $C = C(R)$.

Assume now that (7.1) holds. Taking real parts in (7.2) yields

$$\alpha q[|\partial U^n|_{1,h}^2 + \rho\mu|\partial U_J^n|^2] - [\|\partial U^n\|_h^2 + q(\hat{\beta}(r^n) \otimes \partial U^n, \partial U^n)_h] = -\frac{\lambda}{q} \text{Im}(U^n, \partial U^n)_h.$$

Using (7.1) in this equation in conjunction with the easily established discrete counterpart of the Poincaré inequality (2.8), which is precisely

$$\|v\|_h^2 \leq C_\rho |v|_{1,h}^2, \quad \forall v \in \mathbb{C}_0^{J+1},$$

where $C_\rho \leq \max(1, \rho/2)$ as in the continuous case, yields an inequality of the type (7.3). Hence (7.4) holds again. One may go on to prove an $O(k + h^2)$ error estimate for (E) in the $\|\cdot\|_h$ norm by similar techniques to those of section 3.

The inequality (7.4), weaker of course than the analogous inequality (4.5) that is valid for the Crank–Nicolson scheme, is, nevertheless, a boundedness result (derived under no mesh conditions between k and h) on U^n expressing the stability of (E) in the $\|\cdot\|_h$ norm, and at the same time guaranteeing the existence of the solution of the tridiagonal linear system represented by (E). It is worthwhile to note that whereas the linear system in (Δ) always has a unique solution because of the artificial absorption inherent in the Crank–Nicolson method, the system in (E) seems to be solvable under conditions very close to ones guaranteeing well-posedness of (P).

In [27] St.Mary and Lee apply the von Neumann stability criterion to the forward Euler scheme for a constant-coefficient, one layer version of the wide-angle equation (1.1) with $\gamma = 0$, and argue that the scheme is unconditionally unstable since the amplification factor is larger than one in modulus. This result, seemingly contradictory to the one derived above, may be explained as follows. A closer inspection of the amplification factor reveals that it is of the form $1 + ikA$, wherein $A = A(\xi, h)$ (ξ is the Fourier variable) may become unbounded for large ξ as $h \rightarrow h_0$ for some $h_0 > 0$ sufficiently small. The reason is essentially that the difference scheme, the stability of which is studied by the von Neumann method, properly mimicks the constant-coefficient wide-angle equation posed as an initial-value problem either with periodic boundary conditions at the endpoints of a finite interval or on the whole line. For such problems it is not possible to derive sufficient conditions for the well-posedness of (P) (and the stability of the Euler scheme) of the type (7.1). It is easily seen however, that adding a constant absorption coefficient $\gamma \neq 0$ to the equation fixes matters and renders the Euler scheme stable for problems for which it is permissible to apply the von Neumann criterion.

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