ON GALERKIN METHODS FOR THE WIDE–ANGLE PARABOLIC EQUATION

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Abstract. We consider the third–order, wide–angle, parabolic approximation of underwater acoustics in a medium with depth– and range–dependent speed of sound in the presence of dissipation and horizontal interfaces. We first discuss the theory of existence and uniqueness of solutions to the problem and derive an energy estimate. We then discretize the problem in the depth variable using two types of Galerkin/finite element formulations that take into account the interface conditions, and in the range variable by the Crank–Nicolson and also a fourth–order accurate, implicit Runge–Kutta method. The resulting high–order numerical schemes are stable and convergent and are also shown to compare favorably with classical, implicit finite difference schemes in terms of computational effectiveness when applied to standard benchmark problems.

1. Introduction

In this paper we consider a model initial– and boundary–value problem for the third–order, wide–angle parabolic approximation of underwater acoustics, [1], [2], [3], in a layered medium with speed of sound and dissipation properties depending on the range (r) and depth (z) variables. We are given constants $0 < z_* < z_b < z_{\max}$ that define a medium with three horizontal layers in z, namely water of constant density ρ_1 occupying the strip $I_1 = (0, z_*)$, $r \geq 0$, one layer of sediment of constant density ρ_2 in $I_2 = (z_*, z_b)$, and an artificial bottom layer of density ρ_2 in $I_3 = (z_b, z_{\max})$; we let $I = (0, z_{\max})$. Given $R > 0$ we seek a complex–valued function $u = u(z, r)$ defined for $(z, r) \in \bar{I} \times [0, R]$ and satisfying

$$\big[1 + \sigma(\beta(z,r) + \mathrm{i}\nu(z,r))\big]u_r + \alpha\sigma u_{zzr} = \mathrm{i}\alpha u_{zz} + \mathrm{i}\big[\beta(z,r) + \mathrm{i}\nu(z,r)\big]u, \tag{1.1}$$

for $z \in I_1 \cup I_2 \cup I_3$ and $0 \leq r \leq R$, such that

$$u(z_*-, r) = u(z_*+, r), \quad 0 \leq r \leq R, \tag{1.2}$$

$$u_z(z_*-, r) = \rho u_z(z_*+, r), \quad 0 \leq r \leq R, \tag{1.3}$$

$$u(\cdot, r) \text{ is } C^1 \text{ across } z = z_b, \quad 0 \leq r \leq R, \tag{1.4}$$

$$u(0, r) = u(z_{\max}, r) = 0, \quad 0 \leq r \leq R, \tag{1.5}$$

$$u(z, 0) = u^0(z), \quad z \in I. \tag{1.6}$$

In (1.1) α and σ are real constants, β is a real–valued function, smooth on $[0, z_*]$ and on $[z_*, z_{\max}]$ for $0 \leq r \leq R$, with a possible jump discontinuity at $\{z_*\} \times [0, R]$,

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and ν is a nonnegative function, smooth on $\bar{I}_i \times [0, R]$, $1 \leq i \leq 3$, with possible jump discontinuities at $\{z_*\} \times [0, R]$ and $\{z_b\} \times [0, R]$; in (1.3) $\rho = \rho_1/\rho_2$. In the applications that we have in mind, u is the acoustic field generated by a harmonic point source in the water radiating at frequency f Hz; $u^0(z)$, $z \in I$, is a given complex–valued function modelling the initial field. We let c_0 denote a reference sound speed and put $k_0 = 2\pi f/c_0$.

The coefficients in (1.1) are defined in terms of a rational approximation to $\sqrt{1+x}$ near $x = 0$ with linear numerator and denominator of the form

$$\sqrt{1+x} \cong \frac{1+px}{1+qx}, \quad p, q \text{ real}, \quad p \neq q.$$

For example, the choice $p = 3/4$, $q = 1/4$ (Claerbout, [4]) corresponds to the (1,1) Padé approximation of $\sqrt{1+x}$, whereas setting $p = 1/2$, $q = 0$ yields the linear Taylor polynomial of $\sqrt{1+x}$ around $x = 0$ and gives the usual (Tappert, [5]) parabolic approximation. In (1.1) α and σ are then given by the formulas $\alpha = \frac{p-q}{k_0}$, $\sigma = \frac{q}{(p-q)k_0}$. We set $\beta = k_0(p-q)(\eta^2(z,r) - 1)$, $\eta = c_0/c(z,r)$, $c(z,r)$ being the sound speed of the medium. Finally $\nu \geq 0$ is an empirically determined dissipative term of the form $\nu = k_0(p-q)\theta(z,r)$, where in θ one usually incorporates dissipative mechanisms such as volume absorption in the artificial layer and attenuation (loss) coefficients in the various layers. We shall refer the reader to [1], [2] and [3] for discussions of the justification of (1.1) as a wide–angle modification of the usual parabolic equation. (The latter corresponds to the value $\sigma = 0$; we shall assume in the sequel that $\sigma \neq 0$.)

In section 2 below we briefly discuss issues of existence and uniqueness of solutions of the initial– and boundary–value interface problem (1.1)–(1.6) and derive an *a priori* L^2 bound of its solution by the energy method. In section 3 we consider Galerkin/finite element discretizations of (1.1)–(1.6) that use two different finite element formulations in the depth variable and are coupled with the Crank–Nicolson scheme, and also with a fourth–order, implicit Runge–Kutta method, for the purposes of range–stepping. (For work on *finite difference* approximations to (1.1)–(1.6) we refer the reader e.g. to [1]–[3], [6]–[8].) Finally, in section 4, we present the results of the application of these finite element methods, in comparison with classical, implicit finite–difference schemes, to some standard benchmark problems.

2. Mathematical Preliminaries

The wide–angle p.d.e. (1.1) may be written in the form $(1 - \gamma\mathcal{M})u_r = \mathcal{L}u$, where $\gamma = -\alpha\sigma$, and $\mathcal{M} = \mathcal{M}(r)$ and $\mathcal{L} = \mathcal{L}(r)$ are second–order linear differential operators in z with complex–valued variable coefficients; they are given by $\mathcal{M}v = v_{zz} + \alpha^{-1}[\beta(z,r) + \mathrm{i}\nu(z,r)]v$ and $\mathcal{L}v = \mathrm{i}\alpha v_{zz} + \mathrm{i}[\beta(z,r) + \mathrm{i}\nu(z,r)]v \equiv \mathrm{i}\alpha\mathcal{M}v$. As such it is then a (complex) *Sobolev–type* p.d.e.. The existence, uniqueness and regularity of solutions of initial– and boundary–value problems for such equations (indeed in multidimensional domains and with more general operators and boundary conditions, but in the presence of *smooth* coefficients), have been investigated by Lagnese, [9]. It is shown in [9] that if for each $r \in [0, R]$, $1/\gamma$ is not an eigenvalue of the differential operator $\mathcal{M}(r)$, then, existence and uniqueness of solutions follow, under standard hypotheses such as sufficient smoothness of u^0 etc.. In the case of range–independent $\mathcal{M}$ and $\mathcal{L}$, if $1/\gamma$ is an eigenvalue of $\mathcal{M}$, it is further shown in [9] that existence of solutions is guaranteed only for special u^0 that satisfy conditions involving the spectrum of $\mathcal{M}$. Although the analysis of [9] properly holds only for smooth β and ν, it is reasonable to expect that an analogous theory is valid for problems like (1.1)–(1.6), i.e. in the case of discontinuous coefficients, provided the solution and the initial data satisfy interface transmission conditions.

For our specific $\mathcal{M}$ and $\mathcal{L}$ let us first consider the *conservative* case, i.e. when $\nu = 0$ on I. For special cases, e.g. when β is piecewise constant on I, it is possible to find explicitly the eigenvalues of the operator $\mathcal{M}$ under the interface and boundary conditions (1.2)–(1.5). In general, for variable $\beta = \beta(z,r)$ one may derive *sufficient conditions* that guarantee the invertibility of the operator $1 + \alpha\sigma\mathcal{M}$. Given $r \in [0,R]$, suppose that $(1 + \alpha\sigma\mathcal{M})\varphi = 0$, on $I_1 \cup I_2 \cup I_3$, for some piecewise smooth function $\varphi = \varphi(z)$, $z \in \bar{I}$, satisfying (1.2)–(1.5). Letting

$$(u,v)_\rho = \int_0^{z_*} u\bar{v}dz + \rho \int_{z_*}^{z_{\max}} u\bar{v}dz$$

be the (natural to our problem) weighted inner product on $L^2(I)$ inducing the norm $\|\cdot\|_\rho := (\cdot,\cdot)_\rho^{1/2}$, we see that the equation

$$([1 + \alpha\sigma\mathcal{M}(r)]\varphi, \varphi)_\rho = 0 \tag{2.1}$$

implies, for $0 \le r \le R$, that

$$\|\varphi\|_\rho^2 + \sigma\left(\beta(r)\varphi,\varphi\right)_\rho - \alpha\sigma\|\varphi'\|_\rho^2 = 0, \tag{2.2}$$

where we sometimes write $\beta(r) = \beta(\cdot,r)$ etc., suppressing the z–dependence. Using now the Poincaré inequality

$$\int_0^{z_{\max}} |\varphi|^2 dz \le \left(\frac{z_{\max}}{\pi}\right)^2 \int_0^{z_{\max}} |\varphi'|^2 dz,$$

valid for φ in the Sobolev space $H_0^1(I)$ (which is the case, in view of our hypotheses on φ), and assuming $\alpha\sigma > 0$ (a similar analysis holds for $\alpha\sigma < 0$), and that $\rho = \rho_1/\rho_2 \le 1$ (the physically interesting case), we see that (2.2) yields

$$\left(\left[1 + \sigma\beta(\cdot,r) - \alpha\sigma\rho\left(\frac{\pi}{z_{\max}}\right)^2\right]\varphi,\varphi\right)_\rho \ge 0.$$

Hence, if we suppose that

$$1 + \sigma\beta(z,r) < \alpha\sigma\rho\left(\frac{\pi}{z_{\max}}\right)^2, \quad \forall (z,r) \in \bar{I}\times[0,R], \tag{2.3}$$

there follows that $\varphi = 0$ on $\bar{I}$, i.e. that $-1/\alpha\sigma$ is *not* an eigenvalue of $\mathcal{M}(r)$. It is worthwhile to note that, if $p = 3/4$ and $q = 1/4$, (2.3) will hold if the data are such that

$$\frac{fz_{\max}}{c_0} < \frac{1}{2}\sqrt{\frac{\rho}{3+\eta_{\max}^2}}\,, \qquad \eta_{\max} = \max_{\bar{I}\times[0,R]} \frac{c_0}{c(z,r)}. \tag{2.4}$$

Suppose now that $\nu \neq 0$. Then (2.1) implies that (2.2) becomes

$$\|\varphi\|_\rho^2 + \sigma(\beta(r)\varphi,\varphi)_\rho - \alpha\sigma\|\varphi'\|_\rho^2 + i\sigma(\nu(r)\varphi,\varphi)_\rho = 0.$$

Taking imaginary parts in the above equation yields

$$(\nu(\cdot,r)\varphi,\varphi)_\rho = 0, \tag{2.5}$$

from which, if $\nu > 0$ on $\bar{I} \times [0,R]$, we see that $\varphi = 0$ on $\bar{I}$. Even if for each r the dissipation coefficient $\nu(z,r)$ is positive only on a nonempty subinterval (γ,δ) of I (for example, if $\nu > 0$ on I_2 or on I_3), we may again argue that $\varphi = 0$ on $\bar{I}$: For in that case, (2.5) implies that $\varphi \equiv 0$ on (γ,δ), whence $\varphi(\gamma) = \varphi'(\gamma+) = \varphi(\delta) = \varphi'(\delta-) = 0$. Since $(1 + \alpha\sigma\mathcal{M}(r))\varphi = 0$ on $I_1 \cup I_2 \cup I_3$, i.e. since φ satisfies homogeneous, regular, second–order linear o.d.e's on each I_i, we may argue that φ is identically zero outside $[\gamma,\delta]$, i.e. on the whole interval $\bar{I}$, in view of the uniqueness of solutions of the initial–value problem for such o.d.e's (continue e.g. φ to the left of γ with initial conditions $\varphi(\gamma) = \varphi'(\gamma) = 0$ etc.), the continuity of φ at z_* and z_b, and the interface conditions $\varphi'(z_*-) = \rho\varphi'(z_*+)$, $\varphi'(z_b-) = \varphi'(z_b+)$. Hence, *adding any amount of dissipation will render the initial– and boundary–value problem* (1.1)–(1.6) *well–posed.* In the absence of dissipation a hypothesis of the type (2.3) will certainly ensure existence of solutions. While this type of condition is probably too restrictive for realistic applications, nevertheless one must somehow avoid hitting eigenvalues of $\mathcal{M}$.

Using *energy techniques* we may derive L^2 *a priori* estimates for the solution of (1.1)–(1.6). For example taking the $(\cdot,\cdot)_\rho$ inner product of both sides of (1.1) with u, using (1.2)–(1.5) and then taking real parts, yields

$$\operatorname{Re}\left((1+\sigma\beta(r))u_r,u\right)_\rho = \frac{\alpha\sigma}{2}\frac{d}{dr}\|u_z\|_\rho^2 + \sigma\operatorname{Im}\left(\nu(r)u_r,u\right)_\rho - \left(\nu(r)u,u\right)_\rho.$$

On the other hand, taking the imaginary part of the $(\cdot,\cdot)_\rho$ inner product of both sides of (1.1) with u_r, using again (1.2)–(1.5), and taking imaginary parts gives

$$\operatorname{Re}\left(\beta(r)u_r,u\right)_\rho = \frac{\alpha}{2}\frac{d}{dr}\|u_z\|_\rho^2 + \sigma\left(\nu(r)u_r,u_r\right)_\rho - \operatorname{Im}\left(\nu(r)u_r,u\right)_\rho.$$

Combining these two identities we obtain

$$\operatorname{Re}(u_r,u)_\rho = -\sigma^2(\nu(r)u_r,u_r)_\rho - (\nu(r)u,u)_\rho + 2\sigma\operatorname{Im}(\nu(r)u_r,u)_\rho \le 0. \tag{2.6}$$

Hence $(d/dr)\|u\|_\rho^2 \equiv 2\operatorname{Re}(u_r,u)_\rho \le 0$ for $0 \le r \le R$, implying that

$$\|u(r)\|_\rho \le \|u(s)\|_\rho, \qquad 0 \le s \le r \le R, \tag{2.7}$$

i.e. that the solution of (1.1)–(1.6) is nonincreasing in the $\|\cdot\|_\rho$ norm in the presence of nonzero dissipation. If $\nu = 0$ and a solution exists, then, by (2.6),

$$\|u(r)\|_\rho = \|u^0\|_\rho\,, \qquad 0 \le r \le R. \tag{2.8}$$

In both cases these *a priori* L^2 bounds imply uniqueness of solutions.

3. The Finite Element Methods

In this section we shall briefly consider some *finite element methods* for the numerical solution of (1.1)–(1.6). Specifically, we shall discretize the problem in the depth variable using two Galerkin/ finite element methods that take special account of the interface conditions, and then in range, using range–stepping techniques.

The first finite element method, which we shall refer to as the *standard Galerkin method* for this interface problem, uses piecewise polynomials in z that are *continuous* across the interfaces. Given an integer M we let $\{z_0, z_1, \dots, z_M\}$ be a (not necessarily uniform) partition of $\bar{I}$ such that $z_0 = 0$, $z_M = z_{\max}$ and for some $1 \le \kappa < \lambda < M$, $z_\kappa = z_*$, $z_\lambda = z_b$. We denote $e_i = (z_{i-1}, z_i)$, $h_i = z_i - z_{i-1}$, $h = \max_i h_i$, and, for an integer $s \ge 2$, consider Q_h, the finite dimensional subspace of $H_0^1(I)$ consisting of complex–valued functions, continuous on $\bar{I}$ and vanishing at $z = 0$ and $z = z_{\max}$, such that $\chi\big|_{e_i} \in \boldsymbol{P}_{s-1}(e_i)$, $\;1 \le i \le M$, where $\boldsymbol{P}_m(e_i)$ are the polynomials on e_i of degree $\le m$. On Q_h we consider the sesquilinear form

$$a(\varphi, \chi) := (\varphi', \chi')_\rho\,, \qquad \varphi, \chi \in Q_h. \tag{3.1}$$

Q_h has the standard approximation properties; in the numerical tests we shall use $s = 4$, i.e. C^0 cubics.

The second method employs a non–standard variational formulation due to Baker, [10], in which no continuity of the elements of the finite element space at the interfaces is required. For some integer $s \ge 2$, using the same notation as above, we let for $i = 1, 2, 3$, $S_{h,i}$ stand for the complex–valued functions $\chi \in C^{s-2}(\bar{I}_i)$ such that $\chi\big|_{e_j} \in \boldsymbol{P}_{s-1}(e_j)$ for each e_j in I_i. In addition, we let $\chi(0) = 0$ for $\chi \in S_{h,1}$ and $\chi(z_{\max}) = 0$ for $\chi \in S_{h,3}$. Our finite element space S_h will simply be $S_{h,1} \times S_{h,2} \times S_{h,3}$. On S_h define for $\gamma > 0$ the sesquilinear form

$$\begin{aligned} a_\gamma(\varphi, \chi) :=& (\varphi', \chi')_\rho + \varphi'(z_*-)[\bar{\chi}(z_*)] + \Big(\bar{\chi}'(z_*-) + \frac{\gamma}{h}[\bar{\chi}(z_*)]\Big)\; [\varphi(z_*)] \\ &+ \rho\varphi'(z_b-)[\bar{\chi}(z_b)] + \rho\,\Big(\bar{\chi}'(z_b-) + \frac{\gamma}{h}\;[\bar{\chi}(z_b)]\Big)\; [\varphi(z_b)], \end{aligned} \tag{3.2}$$

where by $[\psi(z)]$ we denote the jump $\psi(z+) - \psi(z-)$ for $z = z_*, z_b$. In addition, we shall require that the partition is quasi–uniform, i.e. $\min_j (h_j/h) \ge \mu$ for some positive constant μ (so that certain inverse inequalities hold) and that $\gamma > 0$ is sufficiently large. For details, cf. [10] or [11]. Again, in the applications we have in mind, we shall use $s = 4$, i.e. cubic splines on each I_i. Accordingly, this method will be referred to in the sequel as the *spline (nonstandard) method*.

Semidiscrete finite element approximations to the solution of the wide–angle equation may now be defined as follows. We let (X_h, B) be any one of the pairs of spaces and sesquilinear forms already defined, i.e., let either $(X_h, B) = (Q_h, a)$ or $(X_h, B) = (S_h, a_\gamma)$. We seek $u_h\colon [0, R] \to X_h$, such that for all $\chi \in X_h$

$$\begin{aligned} &([1 + \sigma(\beta(r) + \mathrm{i}\nu(r))]u_{hr}, \chi)_\rho - \alpha\sigma B(u_{hr}, \chi) \\ &= -\mathrm{i}\alpha B(u_h, \chi) + \mathrm{i}\,([\beta(r) + \mathrm{i}\nu(r)]u_h, \chi)_\rho\,, \quad 0 \le r \le R, \end{aligned} \tag{3.3}$$

where $u_h(\cdot,0) = u_h^0 \in X_h$ is an approximation to u^0 (L^2 projection, interpolant etc.) such that

$$\|u^0 - u_h^0\|_\rho \leq c(u^0)h^s. \tag{3.4}$$

The equation (3.4) represents a system of o.d.e's for the coefficients of $u_h(r)$ with respect to a basis of X_h. We may write it compactly as

$$(1 + \alpha\sigma\mathcal{M}_h(r))u_{hr} = \mathrm{i}\alpha\mathcal{M}_h(r)u_h, \quad 0 \leq r \leq R, \qquad u_h(0) = u_h^0, \tag{3.5}$$

where $\mathcal{M}_h(r)\colon X_h \to X_h$, $0 \leq r \leq R$, approximates $\mathcal{M}(r)$ of section 2 and is defined by (3.3).

The operator $1 + \alpha\sigma\mathcal{M}_h(r)$ will be assumed to be invertible. This follows (indeed with an upper bound on the L^2–norm of the inverse independent of h) e.g. if we impose sufficient conditions such as (2.3), modulo a multiplicative constant in the right–hand side of (2.3) if $B = a_\gamma$. One may then prove, using the ideas of the energy proof that led to (2.7), that $\max_{0\leq r\leq R}\|u(r) - u_h(r)\|_\rho \leq ch^s$, for some constant c independent of h, provided (3.4) holds and u is sufficiently (piecewise) smooth, cf. [11].

Going on now to *full discretizations* of (3.5), let $R = Nk$, $r^n = nk$, $n = 0, 1, 2, \ldots, N$. We seek approximations $U^n \in X_h$ to $u(\cdot, r^n)$ obtained by discretizing in range the o.d.e. system (3.5). A straightforward scheme to consider is, of course, the Crank–Nicolson method in which the functions U^n are defined by $U^0 = u_h^0$ and for $0 \leq n \leq N-1$ by

$$\big(1 + \alpha\sigma\mathcal{M}_h(r^{n+1/2})\big)\,\partial U^n = \mathrm{i}\alpha\mathcal{M}_h(r^{n+1/2})U^{n+1/2}, \tag{3.6}$$

where $\partial U^n = (U^{n+1} - U^n)/k$, $U^{n+1/2} = (U^n + U^{n+1})/2$, and the functions $\beta(\cdot, r)$ and $\nu(\cdot, r)$ in $\mathcal{M}_h(r)$ are evaluated at $r^{n+1/2} = r^n + k/2$. It is not hard to see that for each n, a unique solution of (3.6) always exists, satisfies $\|U^{n+1}\|_\rho \leq \|U^n\|_\rho$ (with equality if $\nu = 0$) and the optimal–order L^2 error estimate

$$\max_n \|U^n - u^n\|_\rho \leq c(k^2 + h^s),$$

provided (3.4) holds. We refer the reader to [11] for details.

One may also consider discretizing the o.d.e. system representing (3.5) by a higher–order accurate range discretization to match the potential high–order of accuracy of the discretization in depth. With this aim in mind, and only for purposes of compact notation, write (3.5) as

$$u_{hr} = \mathcal{F}_h(r)u_h\ , \quad 0 \leq r \leq R, \qquad u_h(0) = u_h^0, \tag{3.7}$$

where $\mathcal{F}_h(r) = i\alpha(1 + \alpha\sigma\mathcal{M}_h(r))^{-1}\mathcal{M}_h(r)$. Consider, as an example, the two–stage Gauss–Legendre implicit Runge–Kutta scheme [11], [13], given by the constants $a_{11} = a_{22} = 1/4$, $a_{12} = (1/4) - \mu$, $a_{21} = (1/4) + \mu$, $\tau_1 = (1/2) - \mu$, $\tau_2 = (1/2) + \mu$, $\mu = \sqrt{3}/6$, $w_1 = w_2 = 1/2$. The corresponding fully discrete scheme for (3.7) is then: $U^0 = u_h^0$, and for $n = 0, 1, 2, \ldots, N-1$ (with $r^{n,j} = r^n + \tau_j k$)

$$
\begin{aligned}
U^{n,i} &= U^n + k\sum_{j=1}^{2} a_{ij}\mathcal{F}_h(r^{n,j})U^{n,j}, \quad i = 1, 2, \\
U^{n+1} &= U^n + k\sum_{i=1}^{2} w_i\mathcal{F}_h(r^{n,i})U^{n,i}.
\end{aligned}
\tag{3.8}
$$

For this scheme one may prove that its solution exists uniquely and satisfies $\|U^n\|_\rho \leq \|U^0\|_\rho$, (as an equality if $\nu = 0$) unconditionally, as well as the optimal order error estimate $\max_n \|U^n - u(r^n)\|_\rho \leq c(k^4 + h^s)$, in case the coefficients β and ν are range–independent. If β and ν are range– and depth–dependent, then the numerical evidence suggests that the optimal–order L^2 error bound $O(k^4 + h^s)$ apparently still persists. *Proving* such an error estimate runs into difficulties though due to certain incompatibilities of the variable coefficients at the interfaces; this may be viewed as yet another manifestation of 'order reduction due to stiffness'. These matters have some theoretical significance for high–order range (or time)–stepping and are currently the object of study by the authors.

The implementation of (3.7) in matrix–vector form, (after a finite element basis for X_h has been selected) is done along the lines of the similar in spirit but considerably simpler algorithm presented in [13] for the parabolic equation without interfaces. The idea is to use an inner iteration to decouple the intermediate equations defining the $U^{n,i}$, $i = 1, 2$, in (3.8) while preserving the basic sparsity structure of the various subproblems involved. What complicates matters is the presence of (the matrix representation of) the operator $I + \alpha\sigma M_h(r)$ at different points $r^{n,j}$ in the left–hand sides of the intermediate stage equations. This problem may also be resolved in a satisfactory manner, cf. [11]; the overall algorithm requires solving a number of sparse linear systems at each range step. The scheme simplifies considerably in the case of range–independent coefficients.

The resulting algorithm may then be compared in terms of its accuracy vs. computational cost with, say, a classical Crank–Nicolson finite difference scheme. This was done in the case of an artificial test problem for which the exact solution was known. For each of three methods (the two finite element methods with $s = 4$ coupled with the range–stepping technique (3.8), and the finite difference scheme) L^2–error levels at some R were fixed and 'optimal' values of h and k that minimize the computational work at a particular error level were calculated. The work required for each method with the optimal values of h and k was then measured for each error level. For small accuracies (up to 10^{-2} in our problem) the finite difference method was more economical. However, at higher accuracies the two finite element methods were much more efficient, with the cubic spline scheme having a slight edge; cf. [11] for details.

4. Numerical Experiments

In this section we consider two test problems of underwater acoustic propagation frequently used in the literature as benchmarks for testing numerical schemes for the wide–angle equation. In the sequel we shall use the $p = 3/4$, $q = 1/4$ equation and let $\nu = k_0\theta/2$, where, cf. [14], $\theta = \Theta(z) + n^2(z,r)\,\frac{b(z)}{27.287527}$, and Θ is a volume absorption function taken equal to zero in $[0, z_b]$ and equal to $0.01\exp[-9(z - z_{\max})^2/(z_{\max} - z_b)^2]$ in $[z_b, z_{\max}]$. We further let $b(z)$ be the piecewise constant attenuation function equal to $b_1 = 0$ in the water and having a nonzero value b_2 in the two bottom layers.

The first problem to be considered is posed in a range–independent environment and was initially stated in [1]; cf. also [2, p. 378], [7], [15]. For this problem, called Problem I in the sequel, $z_* = 100$ m, $z_b = 200$ m, $z_{\max} = 250$ m, $\rho_1 = 1$ g/cm^3, $\rho_2 = 1.2$ g/cm^3, $b_1 = 0$, $b_2 = 0.5$db/wavelength, and the speed of sound is piecewise constant, given by $c_1 = 1500$ m/sec in $\bar{I}_1$ and $c_2 = 1590$ m/sec in $[z_*, z_{\max}]$. We take $c_0 = c_1$. A point harmonic source of $f = 250$ Hz is placed at a depth $z_S = 99.5$ m and a receiver at the same depth. The solution is computed up to $R = 10$ km. A Gaussian starting field, [2], models the source. It is well–known that this problem is hard to integrate, develops a null of the acoustic field in the vicinity of 7 km, and is a good test for the phase accuracy of a numerical method. We used two finite element codes based on the two variational formulations of section 3, namely two programs called CCUB and SPLN, corresponding to the standard Galerkin method with C^0 cubics, and to the cubic spline nonstandard scheme, respectively. Both were coupled with the range–stepping method based on (3.8); the latter scheme simplifies considerably when β and ν are functions of z only as in this example. We also used the standard IFD WIDE Crank–Nicolson finite difference code, [6], for comparison purposes.

In Figures 1–4 we present propagation loss vs. range graphs for all three methods. In Figure 1 we give the SNAP normal mode profile which does not give the correct behavior near the null, while in Figures 2–4 we give, for various M and N, the finite element and the IFD WIDE results. (Recall that M is the number of depth mesh intervals and N the total number of range steps required to reach $R = 10$ km.) We observe that both SPLN and CCUB give results that are quite close to those of IFD WIDE (SPLN seems to be better than CCUB); we also see that as the meshes become finer, the finite element results near the null improve. It is evident that the phase accuracy of these schemes is comparable to that of IFD WIDE.

As an additional check for this problem we show in Figures 5 and 6 graphs of the *amplitude* of the acoustic field at the depth $z_R = 99.5$ m computed by SPLN and CCUB vs. the analogous graphs from IFD WIDE. (The parameters M and N are the same as in Figures 2 and 4, respectively.) All three methods give similar results and the existence of a shadow zone in the vicinity of 7 km range is confirmed.

Hence, all three methods seem to give accurate results for the values of M, N shown. However (with computations done on a CONVEX C–120 using Convex Fortran at FORTH, Heraklion) the finite element methods are considerably more economical. For the runs shown in the figures 2–6, the IFD WIDE code (with $M = 800, N = 5000$, an

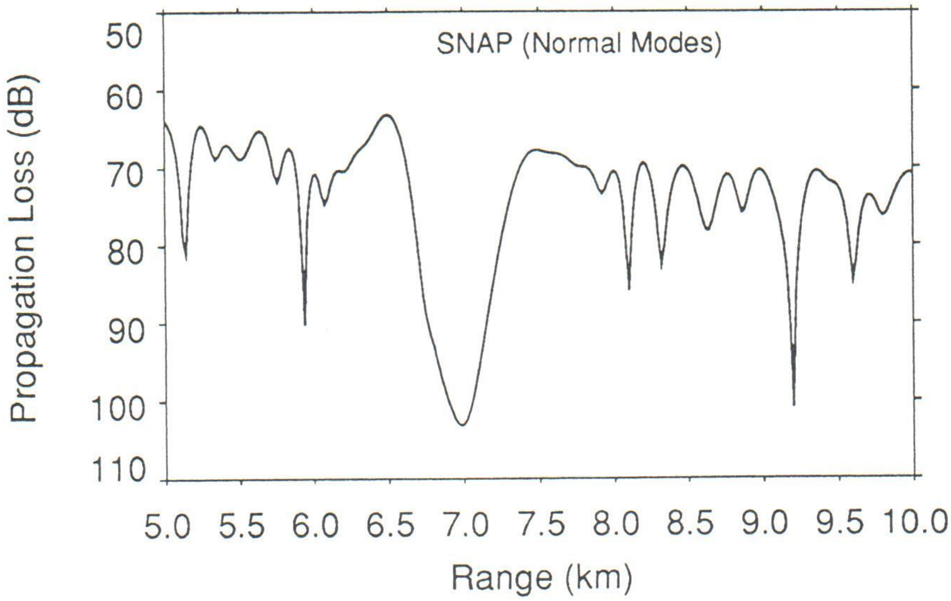

FIGURE 1. Propagation loss as a function of range for Problem I: SNAP normal mode.

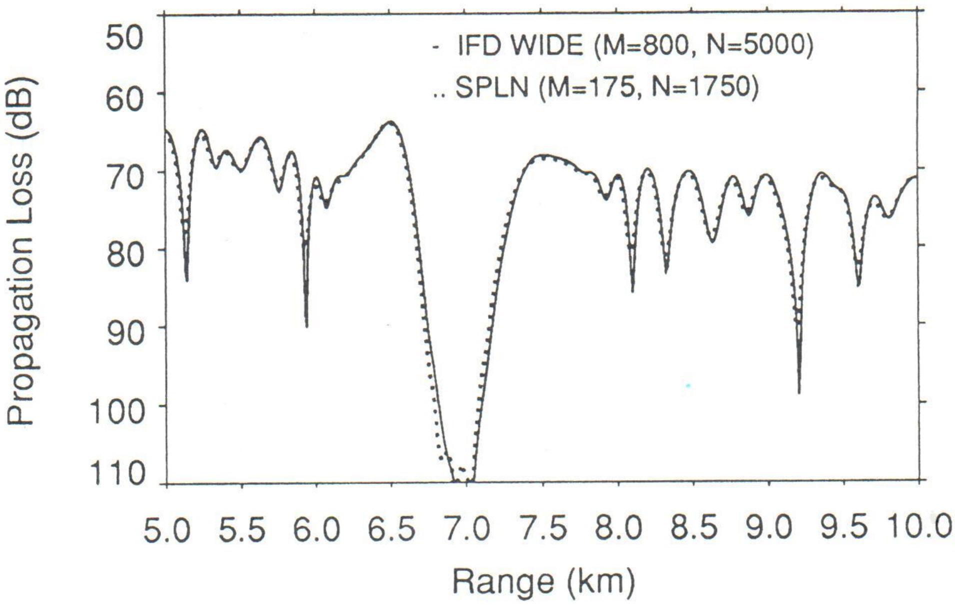

FIGURE 2. Propagation loss as a function of range for Problem I: IFD WIDE ($M = 800, N = 5000$) vs. SPLN ($M = 175, N = 1750$).

optimized pair) took 186 CPU secs, the CCUB run ($M = 200, N = 1000$, Figures 4 or 6) 135 CPU secs, while SPLN with $M = 175, N = 2400$ (Figure 3) required 96 CPU secs, going down to 70 secs when $M = 175, N = 1750$ (Figures 2 or 5). The SPLN code seems to have the advantage then, taking less than one–half the time to produce similar results to IFD WIDE.

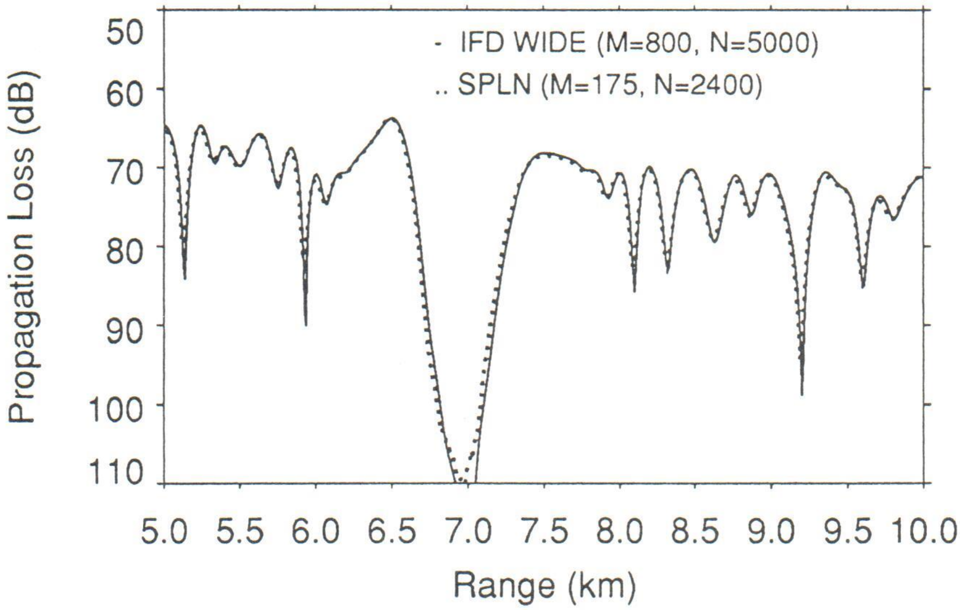

FIGURE 3. Propagation loss as a function of range for Problem I: IFD WIDE ($M = 800, N = 5000$) vs. SPLN ($M = 175, N = 2400$).

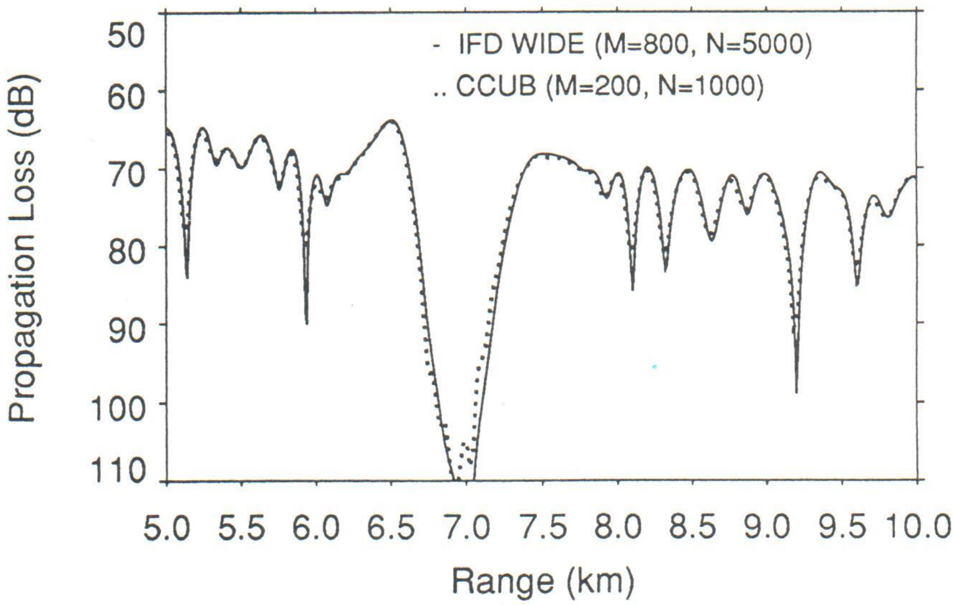

FIGURE 4. Propagation loss as a function of range for Problem I: IFD WIDE ($M = 800, N = 5000$) vs. CCUB ($M = 200, N = 1000$).

The advantage of the high–order methods is more pronounced in the case of problems with coefficients depending on range as well. The results of one such test case (referred to as Problem II and proposed in [16]) are shown in the sequel. In this problem $z_* = L = 500$ m, $z_b = 1000$ m, $z_{\max} = 1250$ m, $\rho_1 = 1$ g/cm^3, $\rho_2 = 10^5$ g/cm^3

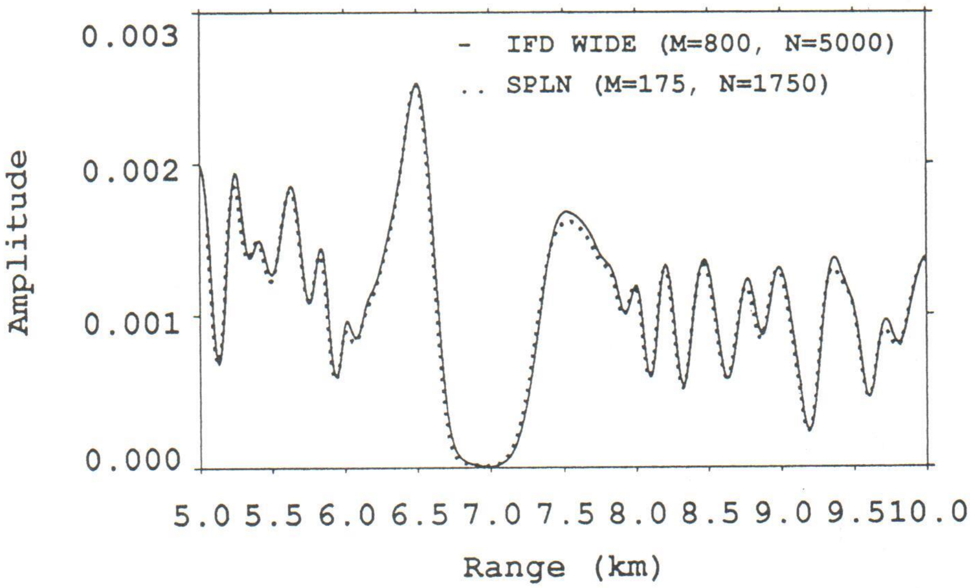

FIGURE 5. Amplitude as a function of range for Problem I: IFD WIDE ($M = 800$, $N = 5000$) vs. SPLN ($M = 175, N = 1750$).

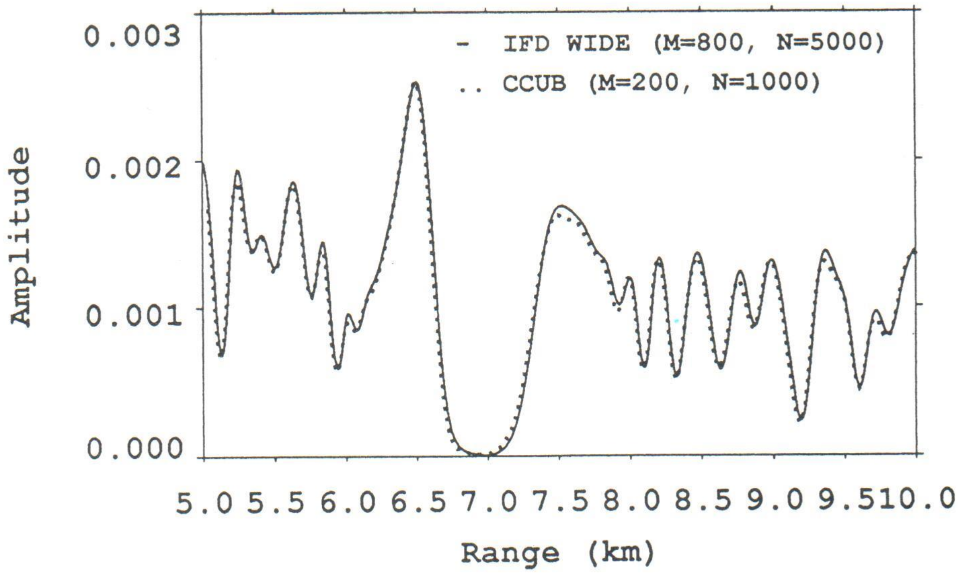

FIGURE 6. Amplitude as a function of range for Problem I: IFD WIDE ($M = 800$, $N = 5000$) vs. CCUB ($M = 200, N = 1000$).

(modelling the *rigid* bottom), and the sound speed in $[0, z_*]$ is given by (in m/sec)

$$\begin{aligned} c_1(z,r) =& 1500\big\{1 + (\pi\ell_1/L)^2 e^{-2\pi r/L} + (2\pi\ell_2/L)^2 e^{-4\pi r/L} \\ & - (2\pi\ell_1/L)[1 - (2\pi\ell_2/L)e^{-2\pi r/L}]\cos(\pi z/L)e^{-\pi r/L} \\ & - (4\pi\ell_2/L)\cos(2\pi z/L)e^{-2\pi r/L}\big\}^{-1/2}, \end{aligned}$$

where $\ell_1 = 0.032\, L$, $\ell_2 = 0.016\, L$. In $[z_*, z_{\max}]$ we take $c_2 = 10^8$ m/sec. The attenuation coefficients are again $b_1 = 0$, $b_2 = 0.5$ db/wavelength. The reference sound speed c_0 was taken equal to 1700 m/sec. The field, generated by a point harmonic source with $f = 25$ Hz at $z_s = 250$ m (initially modelled by a Gaussian) was computed up to $R = 4$ km with receiver at $z_R = 250$ m. Whilst the problem is not very realistic, [16], it has been widely used for model comparisons.

In Figures 7 and 8 we present propagation loss vs. range graphs of the output of the IFD WIDE and the SPLN programs for this problem. Both methods (and also the CCUB, not shown here) give comparable results. The IFD WIDE method was used with $M = 1000$, $N = 2000$ (for values less than these, its accuracy deteriorates) while the SPLN program with $M = 65$ and $N = 300$. The difference between the SPLN results is that, in Figure 7, the range–dependent matrices that occur in the fully discrete high–order SPLN method are updated at every range step, while in Figure 8 at every two range steps. The time of the IFD WIDE run (on the Convex) was 299 CPU secs, while the SPLN runs of figures 7 and 8 required 74 and 48 secs, respectively. (If the update is done every three steps SPLN takes 35 secs without much deterioration in the profile. Let us also mention that the CCUB code with $M = 61$ and $N = 300$ took 124 secs with update at each step.) It is thus seen that for this problem the high–order SPLN method is at least four times faster than lower–order finite difference methods even when the matrices are updated at every step.

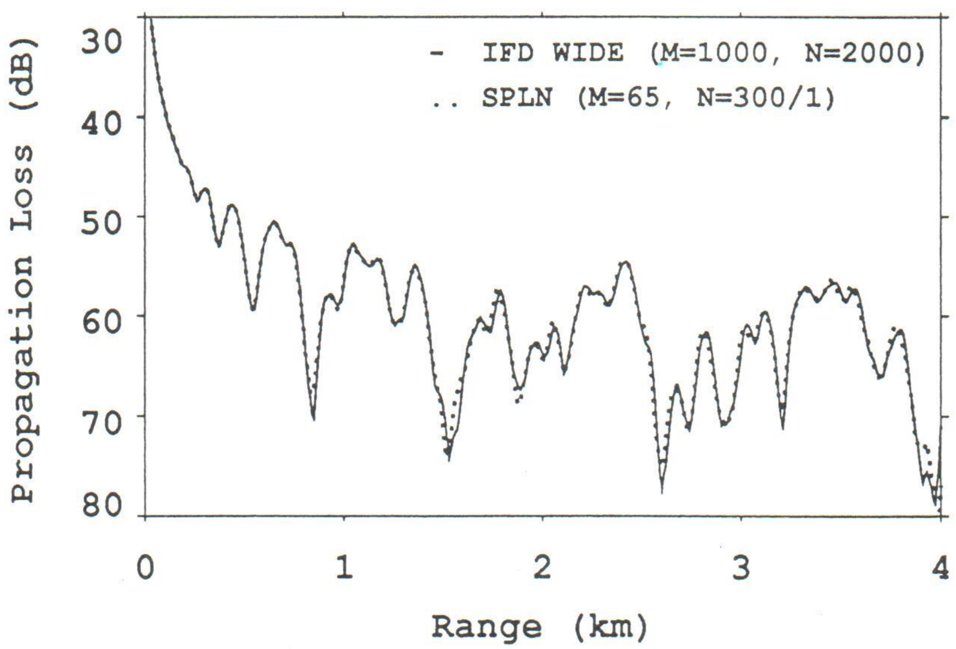

FIGURE 7. Propagation loss as a function of range, Problem II: IFD WIDE ($M = 1000$, $N = 2000$) vs. SPLN ($M = 65, N = 300/1$).

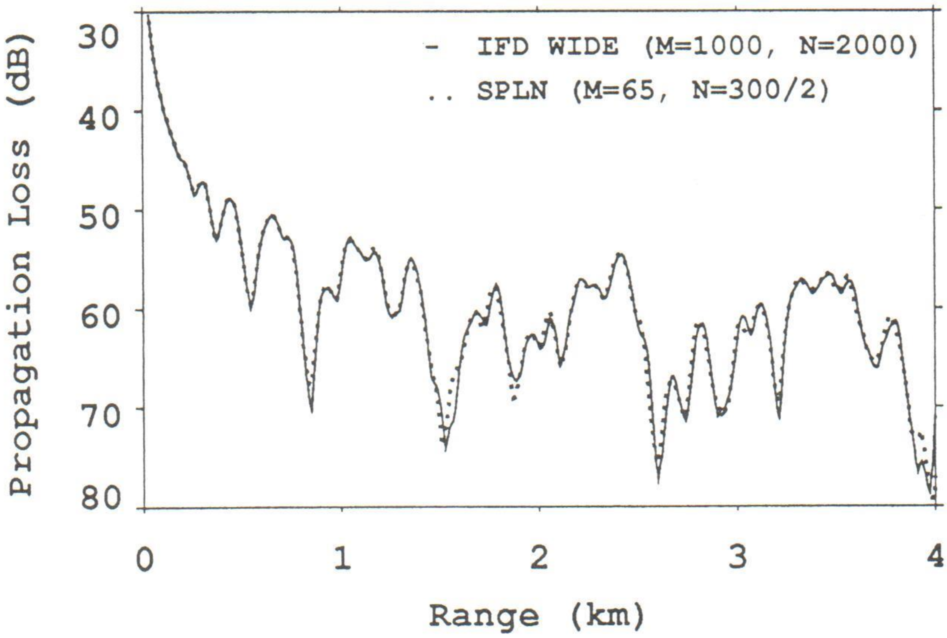

Figure 8. Propagation loss as a function of range, Problem II: IFD WIDE ($M = 1000$, $N = 2000$) vs. SPLN ($M = 65$, $N = 300/2$).

5. Conclusion

On the basis of these theoretical and computational results we may conclude that the high–order numerical methods proposed here approximate well the solution of the wide–angle equation and seem to be computationally more efficient than classical implicit finite difference schemes for the same problems. The authors feel that such schemes should be seriously considered as a basis for the development of very fast wide–angle PE codes in the future.

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